

# Lovely pairs of models

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## Abstract

We introduce the notion of a lovely pair of models of a simple theory  $T$ , generalizing Poizat's "belles paires" of models of a stable theory and the third author's "generic pairs" of models of an  $SU$ -rank 1 theory. We characterise when a saturated model of the theory  $T_P$  of lovely pairs is a lovely pair (that is when the notion of a lovely pair is "axiomatizable"), finding an analogue of the non finite cover property for simple theories. We show that, under these hypotheses,  $T_P$  is also simple, and we study forking and canonical bases in  $T_P$ . We also prove that assuming only that  $T$  is low, the existentially universal models of the universal part of a natural expansion  $T_P^+$  of  $T_P$ , are lovely pairs, and "simple Robinson universal domains".

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# 1 Introduction

In this paper we study a certain class (lovely pairs) of elementary pairs  $M \subseteq N$  of models of a complete simple theory  $T$ . The language  $L_P$  for pairs consists of the language  $L$  of  $T$  together with a predicate  $P$  for the smaller model. The work here can be seen as a common generalization of the third author's work [18] on generic pairs of models of a theory of  $SU$ -rank 1, and Poizat's theory [14] of "belles paires" of models of a stable theory. One of our motivations is to find new ways of constructing simple theories. Another motivation is to find the right analogue of the "non finite cover property" for simple theories. The work here also has connections with so-called *Robinson theories* from [8].

Roughly speaking, an elementary pair  $M \subset N$  of models of  $T$  will be called a *lovely pair* if (i) any complete  $L$ -type over a small subset  $A$  of  $N$  has some nonforking extension over  $M \cup A$  realized in  $N$ , and (ii) if the complete  $L$ -type  $p$  over the small subset  $A$  of  $N$  does not fork over  $M$ , then  $p$  is realized in  $M$ . If  $T$  happens to be stable, this essentially agrees with Poizat's notion of a belle paire. Let  $\mathcal{C}_{T,P}$  be the class of lovely pairs (as  $L_P$ -structures), and  $T_P$  the theory of this class. The general idea is that  $\mathcal{C}_{T,P}$  should be more or less the class of big models or "universal domains" of a possibly non first order simple theory. The "best" case is when  $\mathcal{C}_{T,P}$  is first order, that is when a saturated model of  $T_P$  is again a lovely pair, and we prove in this case that  $T_P$  is an outright simple first order theory. When  $T$  is stable, Poizat proved that  $\mathcal{C}_{T,P}$  is first order if and only if  $T$  does not have the *fcp*. Our necessary and sufficient condition for  $\mathcal{C}_{T,P}$  to be first order is that each of the ranks  $D(-, \phi)$  be finite valued and definable. Bearing in mind Shelah's "fcp theorem" [16], this gives what we believe to be the right analogue of the "non fcp" for simple theories. But we leave open the issue of finding a nice combinatorial equivalent. A next best case is when  $\mathcal{C}_{T,P}$  is more or less the class of existentially universal models of  $T_P$  (or rather of the universal part of  $T_P$ ). (Here, and even before, it is convenient to work in the definitional expansion of  $T_P$  obtained by adding relation symbols for formulas  $\exists x \in P(\phi(x, y))$  where  $\phi \in L$ ). In any case, we prove that this second-best case holds just if  $T$  is low (in the sense of Buechler [2]). As the first author has observed, in general case the category of lovely pairs can be viewed profitably as a "compact abstract theory" [1], and this will be discussed in a future paper.

When  $T$  is a simple one-sorted theory whose universe has  $SU$ -rank 1, the class of lovely pairs (which turns out to be first order) was studied in detail by the third author. (He called them generic pairs, but as the word “generic” is becoming rather overused we changed to lovely pairs.) He recognized the importance of condition (ii) in the definition. For example, if  $T$  is the theory of the random graph, and  $M \subseteq N$  is an elementary pair of models such that  $M$  is saturated and  $N$  is  $|M|^+$ -saturated, then the  $L_P$ -theory of this pair satisfies condition (i) but also has the strict order property (see [18]).

This paper assumes some knowledge and familiarity with simple theories. Frank Wagner’s book [19] is a good source, as well as original papers such as [9], [10], [7], [3]. Notation is standard.

In section 2, we study various properties of formulas (lowness, definability of  $D_\phi$ -rank etc.) which subsequently turn out to be important for the analysis of lovely pairs, but are also possibly important for their own sake. In section 3, we begin our study of lovely pairs, showing existence and examining types. In section 4, we give necessary and sufficient conditions for  $\mathcal{C}_{T,P}$  to be first order. In section 5, we make the connection with Poizat’s belles paires. In section 6 we prove that, if  $\mathcal{C}_{T,P}$  is first order then  $T_P$  is a simple first order theory. In section 7, under the same assumptions, we describe forking in models of  $T_P$ , study and characterize canonical bases in  $T_P$ , and prove some related results (such as preservation of 1-basedness). In section 8, we show that assuming just lowness of  $T$ , a lovely pair is a Robinson universal domain.

The second author would like to thank Enrique Casanovas for clarifying some issues regarding the “lowness” property.

## 2 Properties of formulas in simple theories

$T$  denotes a complete first order theory (not necessarily simple) in a language  $L$ . We work as usual in a very saturated model  $M$  of  $T$ , and for now we work in  $T^{eq}$ . Recall that a formula  $\phi(x, b)$  is said to divide over a set  $A$  if there is an infinite  $A$ -indiscernible sequence  $(b_i : i < \omega)$  of realizations of  $tp(b/A)$  such that  $\{\phi(x, b_i) : i < \omega\}$  is inconsistent. If we also demand  $k$ -inconsistency ( $\{\phi(x, b_i) : i < k\}$  is inconsistent) we say that  $\phi(x, b)$   $k$ -divides over  $A$ . (The same definition can be made for a partial type  $p(x, b)$  with  $b$  in general an infinite tuple, in place of the formula  $\phi(x, b)$ .)  $T$  is simple if and only if every

complete type  $p(x, b)$  does not divide over some  $b_0 \subseteq b$  of cardinality at most  $|T|$ .) We recall also the  $D(-, \phi)$ -rank, which we denote  $D_\phi(-)$ : Let  $\phi(x, y)$  be an  $L$ -formula, and  $\psi(x)$  a formula (possibly with parameters).  $D_\phi(\psi(x)) \geq 0$  if  $\psi(x)$  is consistent.  $D_\phi(\psi(x)) \geq \alpha + 1$  if there is some  $b$  such that  $\phi(x, b)$  divides over the parameters in  $\psi$ , and  $D_\phi(\psi(x) \wedge \phi(x, b)) \geq \alpha$ . For  $\delta$  limit,  $D_\phi(\psi(x)) \geq \delta$  if it is  $\geq \alpha$  for all  $\alpha < \delta$ . If we demand  $k$ -dividing in place of dividing ( $k$  a positive integer), we obtain the rank  $D(-, \phi, k)$ .  $T$  is simple if and only if  $D(\psi, \phi, k) < \omega$  for all  $\phi, \psi, k$ . On the other hand, for  $T$  a simple theory,  $D_\phi(-)$  need not even be ordinal valued ([6], [4]).

Next we want to define when a formula  $\phi(x, y)$  is low. The notion of a low (simple) theory was introduced by Buechler [2]. Other, notably Shami [15] and Casanovas and Kim [5], talk about low formulas (also in simple theories). The equivalence of various definitions of lowness is well-known, but we will take the liberty to reprove them below in the context of some new observations. It is convenient to take our definition of lowness from [3].

**Definition 2.1** *Let  $\phi(x, y)$  be an  $L$ -formula. We say that  $\phi(x, y)$  is low if there is some  $k < \omega$  such that whenever  $(b_i : i < \omega)$  is an indiscernible sequence in some model of  $T$ , and  $\{\phi(x, b_i) : i < \omega\}$  is inconsistent, then it is  $k$ -inconsistent.  $T$  is said to be low if every formula  $\phi(x, y)$  is low.*

**Remark 2.2** (i) *Note the similarity with the the non finite cover property (nfcf).  $\phi(x, y)$  has the nfcf if there is  $k < \omega$  such that for any set  $\{b_i : i \in I\}$  in a model of  $T$ ,  $\{\phi(x, b_i) : i \in I\}$  is consistent if it is  $k$ -consistent (that is if every  $\{\phi(x, b_{i_1}), \dots, \phi(x, b_{i_k})\}$  is consistent).*

(ii) *If  $\phi(x, y)$  is stable (namely there do not exist  $a_i, b_i$  for  $i < \omega$  such that  $\models \phi(a_i, b_j)$  iff  $i \leq j$ ), then  $\phi(x, y)$  is low.*

*Proof.* (ii): It is well-known ([16]) that if  $\phi(x, y)$  is stable then there is  $k < \omega$ , such that for any indiscernible sequence  $\{b_i : i < \omega\}$ , and  $a \in \bar{M}$ , the set of  $b_i$  such that  $\models \phi(a, b_i)$ , has cardinality at most  $k$ , or the set of  $b_i$  such that  $\models \neg\phi(a, b_i)$ , has cardinality at most  $k$ . So if  $\{\phi(x, b_i) : i \leq k\}$  is consistent, then  $\{\phi(x, b_i) : i < \omega\}$  is consistent.

Our definition of a low formula makes sense in any theory, simple or not. In the next lemma we assume simplicity. The implication (v)  $\rightarrow$  (i) is new.

**Lemma 2.3** *Let  $T$  be simple, and  $\phi(x, y) \in L$ . Then the following are equivalent:*

- (i)  $\phi(x, y)$  is low,
- (ii) There is  $k < \omega$  such that  $D_\phi(-) = D(-, \phi, k)$ .
- (iii)  $D_\phi(x = x) < \omega$
- (iv) There is a uniform bound (depending on  $\phi$ ) on  $m < \omega$  for which there exist  $(b_i : i < m)$  such that  $\{\phi(x, b_i) : i < m\}$  is consistent but  $\phi(x, b_i)$  divides over  $(b_0, \dots, b_{i-1})$  for all  $i < m$ .
- (v) The condition on  $(b, c)$ :  $\phi(x, b)$  divides over  $c$ , is type-definable. That is, for any (possibly infinite) tuple  $z$  of variables there is a partial type  $\Sigma(y, z)$  such that for any  $b$ , and  $c$  of length that of  $z$ ,  $\models \Sigma(b, c)$  iff  $\phi(x, b)$  divides over  $c$ .

*Proof.* (i) implies (ii): Choose  $k$  as in Definition 2.1, and then (ii) is immediate.

(ii) implies (iii), because as  $T$  is simple,  $D(x = x, \phi, k) < \omega$  for all  $k$ .

(iii) implies (iv): Suppose  $\phi(x, b_i)$  divides over  $\{b_0, \dots, b_{i-1}\}$  for all  $i < m$ , and  $\{\phi(x, b_i) : i < m\}$  is consistent. Then  $D_\phi(\bigwedge_{j < i} \phi(x, b_j)) > D_\phi(\bigwedge_{j \leq i} \phi(x, b_j))$  for all  $i < m$ , whereby  $D_\phi(x = x) \geq m$ .

(iv) implies (i): If  $(b_i : i < \omega)$  is an indiscernible sequence such that  $\{\phi(x, b_i) : i < \omega\}$  is inconsistent, then  $\phi(x, b_i)$  divides over  $\{b_0, b_1, \dots, b_{i-1}\}$  for all  $i < \omega$ . So if  $m < \omega$  is the bound given by (iv), it follows that  $\{\phi(x, b_i) : i < m + 1\}$  must be inconsistent.

(i) implies (v): There is a partial type in  $(b, c)$  expressing that there is an indiscernible over  $c$  sequence  $(b_i : i < \omega)$  of realizations of  $tp(b/c)$  such that  $\{\phi(x, b_i) : i < k\}$  is inconsistent.

(v) implies (i): Suppose that  $\phi(x, y)$  is not low. For each  $k < \omega$ , let  $(b_i^k : i < \omega + \omega)$  be an indiscernible sequence such that that  $\{\phi(x, b_i^k) : i < \omega + \omega\}$  is inconsistent but  $k$ -consistent. Let  $(c_i : i < \omega + \omega)$  realize some limit point (ultraproduct) of the  $\{tp((b_i^k)_{i < \omega + \omega}) : k < \omega\}$ . Then  $(c_i : i < \omega + \omega)$  is indiscernible, and  $\{\phi(x, c_i) : i < \omega + \omega\}$  is consistent. Note that  $(c_{\omega+n} : n < \omega)$  is a Morley sequence in  $tp(c_\omega / \{c_i : i < \omega\})$ . We conclude, by Kim's lemma ([9]) that  $\phi(x, c_\omega)$  does not divide over  $\{c_i : i < \omega\}$ . On the other hand, clearly  $\phi(x, b_\omega^k)$  divides over  $\{b_i^k : i < \omega\}$  for each  $k < \omega$ . So " $\phi(x, y)$  divides over  $z$ " is not type-definable (in  $(y, z)$ ).

From now on we assume  $T$  to be simple (so we use "forking" and "dividing")

interchangeably). Let us introduce another property.

**Definition 2.4** Let  $\psi(y, z)$  and  $\phi(x, y)$  be  $L$ -formulas.  $Q_{\phi, \psi}$  is the predicate which is defined to hold of a tuple  $c$  (in  $\bar{M}$ ) if for all  $b$  satisfying  $\psi(y, c)$ ,  $\phi(x, b)$  does not divide over  $c$ .

**Remark 2.5** Note that  $\neg Q_{\phi, \psi}(c)$  holds just if there is a complete type  $p(y, c)$  over  $c$ , containing  $\psi(y, c)$  such no nonforking extension of  $p(y, c)$  contains a formula of the form  $\phi(d, y)$ . Note also that by Lemma 2.3, if  $\phi(x, y)$  is low, then  $\neg Q_{\phi, \psi}$  is type-definable for all  $\psi$ .

**Definition 2.6** Let  $\phi(x, y) \in L$ . We say that the  $D_\phi$ -rank is finite and definable if for each  $\psi(x, y) \in L$ ,  
(i)  $D_\phi(\psi(x, b)) < \omega$  for all  $b$ , and  
(ii) for each  $n < \omega$  there is  $\chi_n(y) \in L$  such that for all  $b$ ,  $D_\phi(\psi(x, b)) = n$  iff  $\models \chi_n(b)$ .

**Remark 2.7** Similarly we can define when the  $D_\phi$ -rank is ordinal valued and definable, and by an easy compactness argument, this is equivalent to  $D_\phi$ -rank being finite and definable.

**Proposition 2.8** The following are equivalent:

- (i)  $T$  is low and  $Q_{\phi, \psi}(z)$  is type-definable for all  $\phi(x, y), \psi(y, z) \in L$ .
- (ii)  $T$  is low and  $Q_{\phi, \psi}(z)$  is definable for all  $\phi(x, y), \psi(y, z) \in L$ ,
- (iii) for all  $\phi(x, y) \in L$ , the  $D_\phi$ -rank is finite and definable.

*Proof.* (i) implies (ii): Lowness implies already type-definability of  $\neg Q_{\phi, \psi}(z)$ , so together with type-definability of  $Q_{\phi, \psi}$  we get definability of  $Q_{\phi, \psi}$ .

(ii) implies (iii): As  $T$  is low,  $D_\phi(\psi(x, c)) < \omega$  for all  $\psi(x, z) \in L$  and  $c$ .

Let us now prove by induction on  $n$  that for all formulas  $\theta(x, w) \in L$ , there is  $\chi(w) \in L$  such that  $D_\phi(\theta(x, d)) \geq n$  iff  $\models \chi(d)$  for all  $d$ . Assume true for  $n$ . Now  $D_\phi(\theta(x, d)) \geq n + 1$  iff there is  $b$  such that  $D_\phi(\theta(x, d) \wedge \phi(x, b)) \geq n$  and  $\phi(x, b)$  divides over  $d$ . Let  $\psi(y, z)$  be the  $L$ -formula defining “ $D_\phi(\theta(x, z) \wedge \phi(x, y)) \geq n$ ”. Let  $\chi(z)$  be the formula defining  $\neg Q_{\phi, \psi}(z)$ . Then clearly  $\models \chi(d)$  iff  $D_\phi(\theta(x, d)) \geq n + 1$ .

(iii) implies (i). Fix  $\phi(x, y)$ . First, as  $D_\phi(x = x) < \omega$ ,  $\phi$  is low (by 2.3). We will now prove that  $\neg Q_{\phi, \psi}(z)$  is definable for all  $\psi(y, z)$ . Now  $\neg Q_{\phi, \psi}(c)$  holds if and only if there is  $b$  such that  $\psi(b, c)$  holds and  $\phi(x, b)$  divides over  $c$ , if

and only if either

- (I) there is  $b$  such that  $\psi(b, c)$  holds and  $\phi(x, b)$  is inconsistent, or
- (II) there is  $b$  such that  $\psi(b, c)$  holds,  $\phi(x, b)$  is consistent, and  $\phi(x, b)$  divides over  $c$ .

Now (I) is clearly definable (as a property of  $c$ ). Let  $\phi'(x, z, y)$  the formula  $\phi(x, y) \wedge \psi(y, z)$ , where the variables are divided into  $(x, z)$  and  $y$ . We claim that (II) holds iff

$$(*) D_{\phi'}(x = x \wedge z = c) \geq 1.$$

This is because, for any  $b$ ,

- (a)  $\models \psi(b, c)$  and  $\phi(x, b)$  is consistent if and only if  $(x = x \wedge z = c \wedge \phi(x, b) \wedge \psi(b, z))$  is consistent iff  $D_{\phi'}(x = x \wedge z = c \wedge \phi(x, b) \wedge \psi(b, z)) \geq 0$ , and
- (b) assuming that  $\psi(b, c)$  holds,  $\phi(x, b)$  divides over  $c$  iff  $\phi(x, b) \wedge \psi(b, z)$  divides over  $c$ .

By our assumption (iii) (for  $\phi'(x, z, y)$ ),  $(*)$  is a definable property of  $c$ , so we finish.

**Remark 2.9** *Let  $\psi(y, z)$  be an  $L$ -formula. Then  $Q_{x=y, \psi(y, z)}(c)$  holds iff  $\psi(y, c)$  defines a finite set. Hence type-definability and definability of  $Q_{x=y, \psi}(z)$  are equivalent, and amount to there being a finite bound on the cardinalities of finite sets defined by  $\psi(x, c)$  as  $c$  varies.*

### 3 Lovely pairs: first properties

Until we say otherwise  $T$  will denote a complete simple theory in language  $L$ . We will also assume (by Morleyizing) that  $T$  has quantifier-elimination, and there is no harm in assuming that  $L$  is relational. As a rule we will work inside a big, saturated model  $\bar{M}$  of  $T$ . That is, any model of  $T$  is assumed to be a substructure (so elementary substructure) of  $\bar{M}$ . We assume for now that  $T$  is one-sorted, but we will make subsequent remarks about the many-sorted case (including  $T^{eq}$ ). Let  $L_P$  be the language  $L$  together with a new unary predicate  $P$ . An  $L_P$ -structure has the form  $(M, P(M))$  where  $M$  is an  $L$ -structure, and  $P(M)$  is the interpretation of  $P$ . For  $A$  a subset of  $M$ , we let  $P(A)$  denote  $A \cap P(M)$ . Given  $(M, P(M))$  and  $A \subset M$ , we can speak of types over  $A$  in the sense of the  $L$ -structure  $M$  and in the sense of the  $L_P$  structure  $(M, P(M))$ . We will refer to these as  $L$ -types ( $tp_L(\dots)$ ),  $L_P$ -types ( $tp_{L_P}(\dots)$ ) respectively, or will just say type if the meaning is clear from the

context. Usually, for any  $L_P$ -structure  $(A, P(A))$  we consider,  $A$  will be a substructure of a model of  $T$ , hence a substructure of  $\bar{M}$ .

**Definition 3.1** *Let  $\kappa$  be an cardinal  $\geq |T|^+$ . By a  $\kappa$ -lovely pair of models of  $T$  we mean an  $L_P$ -structure  $(M, P(M))$  such that both  $M$  and  $P(M)$  are models of  $T$ , and*

*(i) $_{\kappa}$ , whenever  $A \subset M$  has cardinality  $< \kappa$  and  $p$  is a complete finitary  $L$ -type over  $A$ , then there is  $a \in M$  realizing  $p$  such that  $tp_L(a/A \cup P(M))$  does not fork over  $A$ .*

*(ii) $_{\kappa}$ , whenever  $A \subset M$  has cardinality  $< \kappa$ , and  $p$  is a complete finitary  $L$ -type over  $A$ , which does not fork over  $P(A)$ , then  $p$  is realized in  $P(M)$ . By a lovely pair of models of  $T$  we mean a  $|T|^+$ -lovely pair.  $\mathcal{C}_{T,P}$  denotes the class of lovely pairs, and  $T_P$  the theory of this class in  $L_P$ .*

**Remark 3.2** *(a) (i) $_{\kappa}$  implies that  $M$  is  $\kappa$ -saturated as an  $L$ -structure. (ii) $_{\kappa}$  implies that  $P(M)$  is  $\kappa$ -saturated. On the other hand if  $T$  is stable, and  $P(M)$  is  $\kappa$ -saturated then (ii) $_{\kappa}$  holds.*

*(b) Suppose  $(M, P(M))$  is a lovely pair,  $A \subseteq M$  and  $p(x)$  is a complete  $L$ -type over  $A$  which is realized in  $M$ . Then  $p(x)$  does not fork over  $P(M)$  iff and only if  $p(x)$  is finitely satisfiable in  $P(M)$ . So  $L$ -independence inside a lovely pair exhibits some stability-like features.*

*(c) In the definition of a lovely pair, we need not assume to begin that  $P(M)$  is also a model of  $T$  (equivalently an elementary substructure of  $M$ ), as it follows from (ii).*

**Remark 3.3** *(i) and (ii) of Definition 2.1 are given for types of finite tuples in the home sort. But this implies (i) and (ii) for types and sets in  $M^{eq}$  and even in  $M^{heq}$  too.*

*Explanation.* Let  $(M, P(M))$  be a (elementary) pair of models of  $T$ . Note that  $P(M)^{eq}$  is precisely the set of  $e \in M^{eq}$  such that  $e \in dcl_L(P(M))$ . For each sort  $S_E$  of  $T^{eq}$ , and subset  $A$  of  $S_E(M)$ , let  $P(A)$  denote  $A \cap P(M)^{eq}$ . If  $A$  is an arbitrary subset of  $M^{eq}$ , let  $P(A)$  denote the union of the  $P(A \cap S_E)$  as  $E$  varies. Suppose now that  $(M, P(M))$  is  $\kappa$ -lovely, as in Definition 3.1. It is then completely routine to prove that if  $A \subset M^{eq}$  has cardinality  $< \kappa$  and  $p$  is a complete  $L$ -type over  $A$  in some imaginary sort, then a nonforking extension of  $p$  over  $P(M)^{eq} \cup A$  is realized in  $M^{eq}$ . Likewise if  $p$  does not fork over  $P(A)$ ,

then  $p$  is realized in  $P(M)^{eq}$ . (In the latter case for example, first find  $A' \subset M$  of cardinality  $< \kappa$  such that  $A \subseteq dcl_L(A')$  and  $P(A) \subseteq dcl_L(P(A'))$ , let  $p'$  be a nonforking extension of  $p$  over  $A'$ . Realize  $p'$  (in  $\bar{M}$ ) by some  $e$ . Suppose  $e = a/E$ , with  $a$  real. Choose  $a'$  such that  $a'/E = e$  and  $a'$  is independent from  $A'$  over  $e$ . Let  $q' = tp_L(a'/A')$ . Then  $q'$  does not fork over  $P(A')$  so is realized in  $P(M)$ , so  $p'$  (and hence  $p$ ) is realized in  $P(M)^{eq}$ .)

The same thing works for hyperimaginaries. First note that  $P(M)^{heq}$  is (up to interdefinability) precisely  $\{e \in M^{heq} : e \in dcl(P(M))\}$ . But if  $(M, P(M))$  is  $\kappa$ -lovely, we should be careful to work with hyperimaginaries which are  $< \kappa$ -ary (that is of the form  $a/E$  where  $a$  is a real tuple of length  $< \kappa$ ).

By the above remark, we will freely work in  $T^{eq}$ . We suppose that we should formally consider a separate predicate  $P_E$  for each sort  $S_E$ , but we will in practise work with notation as in the explanation above. Similarly we could formulate the whole set-up for many sorted theories. Also by the above explanation we will work freely with hyperimaginaries.

**Remark 3.4** For  $\kappa \geq |T|^+$ , axiom  $(ii)_\kappa$  in Definition 3.1 above can be restated as: whenever  $A \subseteq M$  has cardinality  $< \kappa$  and  $p(x)$  is a complete  $L$ -type over  $A$  which does not fork over  $P(M)$ , then  $p$  is realized in  $P(M)$ . For this reason we call  $(ii)_\kappa$  the “coheir property”.

*Proof.* Suppose  $(ii)_\kappa$  is satisfied by  $(M, P(M))$ . Let  $a$  and  $p$  be as given. So  $p$  has an extension  $p'$  to a complete  $L$ -type over  $A \cup P(M)$  which does not fork over  $P(M)$ . Let  $B \subseteq P(M)$  be of cardinality at most  $|T|$  such that  $p'$  does not fork over  $B$ . Let  $p''$  be the restriction of  $p'$  to  $A \cup B$ . By axiom  $(ii)_\kappa$ ,  $p''$  (and so  $p$ ) is realized in  $P(M)$ . The other direction is immediate.

**Lemma 3.5**  $\kappa$ -lovely pairs exist.

*Proof.* Fix any pair  $(M_0, P(M_0))$  of models of  $T$  of cardinality at least  $\kappa$ . We will construct a chain  $(M_i, P(M_i))$  for  $i < \kappa^+$  of pairs such that

(a) for any  $i$ , any complete  $L$ -type over  $M_i$  which does not fork over  $P(M_i)$  is realized in  $P(M_{i+1})$ .

(b)  $i < j$  implies that  $P(M_j)$  is  $L$ -independent from  $M_i$  over  $P(M_i)$

(c) For  $i$  successor,  $M_i$  is  $(\kappa + |P(M_i)|^+)$ -saturated (as a model of  $T$ ).

This is easily accomplished. Given  $(M_i, P(M_i))$ , let  $(p_j : j < \lambda)$  be a list of all complete  $L$ -types over  $M_i$  which do not fork over  $P(M_i)$ . Realize them independently (in  $\bar{M}$ ) by  $A = (a_j : j < \lambda)$ . Then  $A$  is independent from  $M_i$  over  $P(M_i)$ . Let  $N$  be a model of  $T$  (elementary substructure of  $\bar{M}$ ) containing  $P(M_i) \cup A$  and independent from  $M_i$  over  $P(M_i)$ . Let  $M$  be a model of  $T$  containing  $M_i \cup N$  which is  $(\kappa + |N|^+)$ -saturated. Define  $M_{i+1} = M$  and  $P(M_{i+1}) = N$

Having defined the chain of  $(M_i, P(M_i))$ , let  $(M, P(M))$  be the union. We claim that  $(M, P(M))$  is  $\kappa$ -lovely. If  $A$  is a subset of  $M$  of cardinality  $< \kappa$  then  $A \subseteq M_i$  for some  $i$ . If  $p$  is a complete  $L$ -type over  $A$  then by (c), some (in fact any) nonforking extension of  $p$  over  $A \cup P(M_i)$  is realized in  $M_i$  by  $d$  say. By (b),  $dA$  is independent from  $P(M)$  over  $P(M_i)$ , and thus  $d$  is independent from  $A \cup P(M)$  over  $A$ .

Now (for the same  $A$ ), suppose  $p$  is a complete  $L$ -type over  $A$  which does not fork over  $P(A)$ . Let  $q$  be a nonforking extension of  $p$  over  $M_i$ . Then  $q$  does not fork over  $P(M_i)$ , so by (a) is realised in  $P(M_{i+1}) \subset P(M)$ .

**Remark 3.6** (i) *The construction in Lemma 3.5 is a bit crude and could be refined to bound the cardinality of the  $\kappa$ -lovely pair.*

(ii) *The proof shows that any  $L_P$ -structure of the form  $(A, P(A))$  where  $A$  is a subset of a model of  $T$  and  $P(A)$  is relatively algebraically closed in  $A$  (in the sense of  $\bar{M}$ ), embeds in a  $\kappa$ -lovely pair  $(M, P(M))$  such that  $P(M)$  is  $L$ -independent from  $A$  over  $P(A)$ .*

We now want to see when two tuples in lovely pairs have the same type. Recall that we are assuming  $T$  to have quantifier-elimination. So the quantifier-free  $L_P$ -type of a tuple  $a$  in a pair of models of  $T$  consists precisely of the  $L$ -type of  $a$  together with the information about which coordinates of  $a$  are in  $P$  or not in  $P$ . The following definition is convenient:

**Definition 3.7** *Let  $A$  be a subset of a pair  $(M, P(M))$  of models of  $T$ . We say that  $A$  is  $P$ -independent if  $A$  is independent from  $P(M)$  over  $P(A)$  (in the sense of  $L$ ).*

**Lemma 3.8** *Let  $(M, P(M))$  and  $(N, P(N))$  be  $\kappa$ -lovely pairs ( $\kappa > |T|$ ). Let  $a, b$  be tuples of the same length  $< \kappa$  from  $M, N$ , respectively, which are both*

*P*-independent. Assume that  $a$  and  $b$  have the same quantifier-free  $L_P$ -type. Then  $a$  and  $b$  have the same  $L_P$ -type.

*Proof.* Let  $f : a \rightarrow b$  be the partial  $L_P$ -isomorphism given by our hypothesis. It is enough (by back-and-forth, and symmetry) to show that any  $c \in M$  is included in the domain of a partial  $L_P$ -isomorphism  $g$  extending  $f$ . So choose  $c$ . Extending  $c$  to a suitable small tuple, we may assume that  $ca$  is  $P$ -independent. Let  $c_1 = P(c)$  and let  $c_2$  be the rest of  $c$ . Let  $p$  be the  $L$ -type of  $c_1$  over  $a$ , and let  $p'$  be its copy over  $b$ . Then by  $P$ -independence of  $a$  and  $b$  in  $(M, P(M))$  and  $(N, P(N))$  respectively, and the axiom (ii) of lovely pairs,  $p'$  is realized in  $P(N)$  by some  $d_1$ . Now let  $q$  be the  $L$ -type of  $c_2$  over  $c_1a$ , and  $q'$  the copy over  $d_1b$ . Then by the axiom (i) of lovely pairs, some nonforking extension of  $q'$  over  $P(N) \cup a$  is realized in  $N$ , by say  $d_2$ . Note that all coordinates of  $d_2$  are outside  $P(N)$ . Let  $g$  extend  $f$  by taking  $c_1$  to  $d_1$  and  $c_2$  to  $d_2$ . Then  $g$  is a partial  $L_P$ -isomorphism. The proof is complete.

**Corollary 3.9** *All lovely pairs are elementarily equivalent (in fact  $(\infty, \omega)$  equivalent).  $T_P$  is a complete  $L_P$ -theory.*

Recall that if  $M$  is a model of  $T$ , and  $p(x)$  a complete type over  $M$ , then  $cl(p)$  denotes the set of  $L$ -formulas  $\phi(x, y)$  such that  $\phi(x, b)$  in  $p(x)$  for some  $b \in M$ . For  $T$  stable,  $p(x) \in S(A)$ , and nonforking extensions  $q_1, q_2$  of  $p$  to models,  $cl(q_1) = cl(q_2)$  equals the unique smallest class among extensions of  $p$  to models. For a simple theory  $T$  this is no longer necessarily the case, but in [11] it was pointed out that among nonforking extensions  $q$  of  $p(x) \in S(A)$  to models there is a unique maximal class. Let us call this class  $m\beta(p)$  (THE maximal or maximum class among nonforking extensions of  $p$  to models).

**Remark 3.10** *Let  $(M, P(M))$  be a lovely pair. Then for any  $A \subseteq P(M)$  of cardinality at most  $|T|$ ,  $p(x) \in S(A)$  and nonforking extension  $q(x) \in S(P(M))$  of  $p$  which is realized in  $M$ ,  $cl(q) = m\beta(p)$ .*

*Proof.* Let  $c \in M$  realize  $q$ . Suppose for a contradiction that  $cl(q)$  is not maximal (among classes of nonforking extensions of  $p$  to models). So there is (somewhere)  $d$  independent from  $c$  over  $A$ , and an  $L$ -formula  $\phi(x, y)$  such that  $\phi(c, d)$  but  $\phi(x, y) \notin cl(q)$ . As  $(M, P(M))$  is a lovely pair we may realize  $tp_L(d/Ac)$  by  $d'$  in  $P(M)$ . So  $\phi(x, y) \in cl(q)$ , contradiction.

It will be convenient for now and quite important for later to work in a certain definitional expansion  $L_P^+$  of  $L_P$ .  $L_P^+$  consists of  $L_P$  together with new relation symbols  $R_\phi(y)$  for each  $L$ -formula  $\phi(x, y)$ .  $T_P^+$  denotes  $T_P$  together with the sentences “ $\forall y(R_\phi(y) \leftrightarrow \exists x(P(x) \wedge \phi(x, y)))$ ”, which define the new relation symbols. So (by Corollary 3.9),  $T_P^+$  is a complete  $L_P^+$ -theory. Note that the unary predicate  $P$  is superfluous in  $T_P^+$  as  $T_P^+$  implies  $\forall y(P(y) \leftrightarrow R_{x=y}(y))$ . We consider any lovely pair  $(M, P(M))$  as an  $L_P^+$ -structure in the canonical way, that is so that it becomes a model of  $T_P^+$ .

We can now rephrase Lemma 2.7 using  $L_P^+$ , generalizing Theorem 4 of [14]

**Corollary 3.11** *Let  $(M, P(M)), (N, P(N))$  be lovely pairs. Let  $a, b$  be tuples of the same length from  $M, N$ , respectively. Then the following are equivalent:*

- (i)  $cl(tp_L(a/P(M))) = cl(tp_L(b/P(N)))$ .
- (ii)  $a$  and  $b$  have the same quantifier-free  $L_P^+$  type (in the structures  $(M, P(M)), (N, P(N))$  viewed as  $L_P^+$  structures as described above).
- (iii) The  $L_P$  (so also  $L_P^+$ )-types of  $a$  and  $b$  in  $(M, P(M)), (N, P(N))$  respectively, are equal.

*Proof.* The equivalence of (i) and (ii) is immediate (and only requires that the predicates  $R_\phi$  are interpreted according to their defining axioms). (iii) implies (ii) is immediate. So we only have to prove that (ii) implies (iii).

So assume (ii). Note that  $a$  and  $b$  have the same  $L$ -type. Let  $A \subseteq P(M)$  be of cardinality at most  $|T|$  such that  $tp_L(a/P(M))$  does not fork over  $A$ . Let  $q(z, a)$  be the  $L$ -type of  $A$  over  $a$ . As  $b$  has the same quantifier-free  $L_P^+$ -type as  $a$ ,  $q(z, b)$  is finitely satisfiable in  $P(N)$ , and so does not fork over  $P(N)$ . By Remark 3.4,  $q(z, b)$  is realized in  $P(N)$ , by  $B$  say.

We claim that  $tp_L(b/P(N))$  does not fork over  $B$ . Suppose for a contradiction that  $tp_L(b/P(N))$  forks over  $B$ . Then there are  $\phi(x, y) \in L$ ,  $k < \omega$ ,  $\psi(x, z) \in L$ , and  $c \in P(N)$  such that  $\psi(b, c)$  holds, but  $D(\psi(x, c), \phi, k) < n$  where  $n = D(tp_L(b/B), \phi, k)$ . Now there is a formula  $\chi(z) \in tp_L(c)$  such that for any  $c'$  realizing  $\chi$ ,  $D(\psi(x, c'), \phi, k) < n$ . As  $a$  and  $b$  have the same quantifier-free  $L_P^+$ -type, there is  $c' \in P(M)$  such that  $\psi(a, c')$ , implying that  $tp_L(a/P(M))$  forks over  $A$ , a contradiction.

So  $tp_L(b/P(N))$  does not fork over  $B$ . Note that  $aA$  and  $bB$  have the same quantifier-free  $L_P$ -type. So by Lemma 3.8 they have the same type (in  $(M, P(M)), (N, P(N))$  respectively). This completes the proof.

**Definition 3.12** *We will say (with some abuse of language) that the class  $\mathcal{C}_{T,P}$  of lovely pairs is first order if any  $|T|^+$ -saturated model of  $T_P$  is a lovely pair.*

In the next section we will find necessary and sufficient conditions on the theory  $T$  for  $\mathcal{C}_{T,P}$  to be first order. But for now note that it follows from 3.11 that:

**Corollary 3.13** *If  $\mathcal{C}_{T,P}$  is first order then  $T_P^+$  has quantifier-elimination.*

## 4 Characterization of when $\mathcal{C}_{T,P}$ is first order

The axioms (i) and (ii) for lovely pairs have quite different features, and we will separate our attempt to find necessary and sufficient conditions for  $\mathcal{C}_{P,T}$  to be first order into attempts to express each of (i), (ii) in a "first order manner".

It will be convenient to call property  $(i)_\kappa$  the " $\kappa$ -extension property", and  $(ii)_\kappa$  the  $\kappa$ -coheir property. When  $\kappa = |T|^+$  we will just say "extension property" and "coheir property". We will say, by abuse of language again, that the extension property is first order, if any  $|T|^+$ -saturated model of the  $L_P$ -theory of all pairs  $(M, P(M))$  of models of  $T$  satisfying the extension property, also satisfies the extension property. Likewise for "the coheir property is first order", "the  $\kappa$ -extension property is first order" etc.

**Lemma 4.1** *The following are equivalent:*

- (i)  $T$  is low,
- (ii) The coheir property is first order,
- (iii) for any  $\kappa \geq |T|^+$ , the  $\kappa$ -coheir property is first order.

*Proof.* (i) implies (ii) and (iii): Assume  $T$  to be low. Fix an  $L$ -formula  $\phi(x, y)$  and a tuple  $z$  of variables. Let  $\Sigma_{\phi,z}(y, z)$  be the partial  $L$ -type given by Lemma 2.3 which expresses that  $\phi(x, y)$  forks over  $z$ . Let  $C_{\phi,z}$  be the set of  $L_P$ -sentences:

$$\{\forall y, z((P(z) \wedge \neg \exists x(P(x) \wedge \phi(x, y))) \rightarrow \psi(y, z)) : \psi(y, z) \in \Sigma_{\phi,z}(y, z)\}.$$

It is then more or less immediate that (a) any pair  $(M, P(M))$  of models of  $T$  with the coheir property is a model of  $C_{\phi,z}$  for all  $\phi, z$ , and (b) any

$\kappa$ -saturated pair  $(M, P(M))$  which is a model of  $C_{\phi, z}$  for all  $\phi, z$  has the  $\kappa$ -coheir property. So (ii) and (iii) are proved.

Let us now prove (ii) implies (i). Assume (ii) and we have to prove that for any  $\phi(x, y) \in L$  and  $z$ , the condition (on  $(y, z)$ ) that  $\phi(x, y)$  forks over  $z$  is type-definable (in  $\bar{M} \models T$ ). Let  $b_i, c_i$  for  $i \in I$  be tuples from  $\bar{M}$  such that  $\phi(x, b_i)$  divides over  $c_i$ , and let  $(b, c)$  realize some ultraproduct of  $\{tp_L(b_i, c_i) : i \in I\}$ . We must show that  $\phi(x, b)$  forks over  $c$ . By Lemma 3.5 we may assume that all  $(b_i, c_i)$  are inside a lovely pair  $(M, P(M))$ , such that, moreover,  $c_i \in P(M)$  and  $tp_L(b_i/P(M))$  does not fork over  $c_i$  for all  $i \in I$ . We may assume that  $(N, P(N), b, c)$  is an ultraproduct of the  $(M, P(M), b_i, c_i)$  (for some pair  $(N, P(N))$ ). Now,  $\phi(x, b_i)$  is not realized in  $P(M)$  for all  $i$ . It follows that  $\phi(x, b)$  is not realized in  $P(N)$ . But, as  $(M, P(M))$  satisfies the coheir property, it follows from (ii) that  $(N, P(N))$  does too. Hence,  $\phi(x, b)$  forks over  $P(N)$  so over  $c$ .

**Remark 4.2** *The above proof shows that the following are equivalent:*

- (i)  $T$  is low,
- (ii) for any model  $(M, P(M))$  of  $T_P$ ,  $\phi(x, y) \in L$ , and  $b \in M$ ,  $\phi(x, b)$  does not fork over  $P(M)$  if and only if  $\phi(x, y)$  is realized in  $P(M)$ .

We now obtain a preliminary characterization of when  $\mathcal{C}_{T, P}$  is first order. Let us say that an extension  $(M, P(M)) \subseteq (N, P(N))$  of pairs of models of  $T$  is *free* if  $P(N)$  is  $L$ -independent from  $M$  over  $P(M)$ .

**Corollary 4.3** *The following are equivalent:*

- (i)  $\mathcal{C}_{T, P}$  is first order,
- (ii)  $T$  is low and  $T_P^+$  has quantifier-elimination,
- (iii) Any extension  $(M, P(M)) \subseteq (N, P(N))$  of models of  $T_P$  is elementary if and only if it is free.

*Proof.* (i) implies (ii) is by Remark 4.2 and Corollary 3.13.

(ii) implies (iii). Assume (ii). Suppose first that  $(M, P(M))$  is an elementary substructure of  $(N, P(N))$ . Then  $tp_L(P(N)/M)$  is finitely satisfiable in  $P(M)$ , hence  $P(N)$  is independent from  $M$  over  $P(M)$ . Now assume that  $(M, P(M)) \subseteq (N, P(N))$  is free. Let  $(M, P(M))^+, (N, P(N))^+$  be the canonical expansions of  $(M, P(M)), (N, P(N))$  to models of  $T_P^+$

*Claim.*  $(M, P(M))^+$  is a substructure of  $(N, P(N))^+$

*Proof of claim.* Suppose  $\phi(x, y) \in L$ ,  $b \in M$  and  $(N, P(N)) \models \exists x(P(x) \wedge$

$\phi(x, b)$ ). Then  $\phi(x, b)$  does not fork over  $P(N)$  (in  $\bar{M}$ ). As  $P(N)$  is independent from  $M$  over  $P(M)$ ,  $\phi(x, b)$  does not fork over  $P(M)$  in  $\bar{M}$ . By Remark 4.2,  $(M, P(M)) \models \exists x(P(x) \wedge \phi(x, b))$ . The claim is proved.

By the claim and the assumption that  $T_P^+$  has quantifier-elimination,  $(M, P(M))^+$  is an elementary substructure of  $(N, P(N))^+$ . In particular  $(M, P(M))$  is an elementary substructure of  $(N, P(N))$ .

(iii) implies (i). Let  $(M, P(M))$  be a  $|T|^+$ -saturated model of  $T_P$ . By Remark 3.6(ii), we can find a lovely pair  $(N, P(N))$  which is a free extension of  $(M, P(M))$ . By (ii)  $(N, P(N))$  is an elementary extension of  $(M, P(M))$ . It follows easily that  $(M, P(M))$  is lovely too.

**Remark 4.4** *It is clear that  $\mathcal{C}_{T,P}$  is first order iff for any (some)  $\kappa \geq |T|^+$ , any  $\kappa$ -saturated model of  $T_P$  is  $\kappa$ -lovely.*

Corollary 4.3 is somewhat unsatisfactory as we really seek conditions on the original theory  $T$  (not  $T_P$ ) which are equivalent to  $\mathcal{C}_{T,P}$  being first order. We proceed to do this now. The key issue is being able to axiomatize property (i) (the extension property) in the definition of lovely pairs.

We now bring into play the predicates  $Q_{\phi,\psi}$  introduced in section 2.

**Proposition 4.5** *The following are equivalent:*

- (i)  $Q_{\phi,\psi}$  is type-definable (in  $\bar{M}$ ) for all  $\phi(x, y), \psi(y, z) \in L$ .
- (ii) The extension property is first order.
- (iii) Any saturated model of  $T_P$  satisfies the extension property.

*Proof.* (i) implies (ii).

Assume type-definability of  $Q_{\phi,\psi}$  for all  $\phi, \psi \in L$ , and identify notationally  $Q_{\phi,\psi}(z)$  with this partial  $L$ -type.

For each  $\phi, \psi$  let  $\Sigma_{\phi,\psi}$  be the following collection of  $L_P$ -sentences:  $\{\forall z(\chi(z) \rightarrow \exists y(\psi(y, z) \wedge \neg \exists x(P(x) \wedge \phi(x, y)))) : \neg \chi(z) \in Q_{\phi,\psi}(z)\}$ .

*Claim 1.* If  $(M, P(M))$  is a pair of models of  $T$  which has the extension property, then  $(M, P(M)) \models \Sigma_{\phi,\psi}$  for all  $\phi, \psi$ .

*Proof.* This is immediate, but we go through the translation. Fix  $\phi(x, y), \psi(y, z) \in L$ ,  $\neg \chi(z) \in Q_{\phi,\psi}(z)$ , and  $c \in M$  such that  $\chi(c)$ . So there is a complete  $L$ -type  $p(y, c)$  over  $c$  such that for any  $d$ ,  $\neg \phi(d, y)$  is in all nonforking extensions

of  $p(y, c)$  over  $cd$ . As  $(M, P(M))$  satisfies the extension property,  $p(y, c)$  is realized by some  $b \in M$  such that  $tp_L(b/P(M)c)$  does not fork over  $c$ . In particular  $\neg\phi(d, b)$  holds for all  $d \in P(M)$ . This proves Claim 1.

*Claim 2.* If  $(M, P(M))$  is a  $\kappa$  saturated pair of models of  $T$ , which is also a model of all  $\Sigma_{\phi, \psi}$  then  $(M, P(M))$  satisfies the  $\kappa$ -extension property.

*Proof of claim 2.* Let  $p(y)$  be a complete  $L$ -type over a subset  $A$  of  $M$  of cardinality  $< \kappa$ . Let  $\phi(x, y) \in L$  be such that  $\phi(x, b')$  forks over  $A$  for some (any) realization  $b'$  of  $p$ . Let  $\psi(y, c)$  be any formula in  $p(y)$ . So  $\neg Q_{\phi, \psi}(c)$  holds, so  $\chi(c)$  holds for some  $\neg\chi(z) \in Q_{\phi, \psi}(z)$ . As  $\Sigma_{\phi, \psi}$  is true in  $(M, P(M))$ , there is  $b \in M$  realizing  $\psi(y, c)$  such that  $\neg\phi(d, b)$  holds for all  $d \in P(M)$ .  $\kappa$ -saturation of  $(M, P(M))$  ensures that some nonforking extension of  $p(y)$  over  $A \cup P(M)$  is realized in  $M$ .

Claims 1 and 2, show (ii).

(ii) implies (iii) is immediate.

(iii) implies (i): Let us fix  $\phi(x, y), \psi(y, z) \in L$ . Let  $c_i$  for  $i \in I$  be tuples in  $\bar{M}$  such that  $Q_{\phi, \psi}$  is true of  $c_i$  for all  $i$ . Let  $c$  be an ultraproduct of the  $c_i$ . We must show  $Q_{\phi, \psi}$  is true of  $c$  too. We may assume all  $c_i$  are in  $P(M)$  for  $(M, P(M))$  some lovely pair and that  $c \in P(N)$  where  $(N, P(N), c)$  is an ultraproduct of the  $(M, P(M), c_i)$ . Now as  $(M, P(M))$  is a lovely pair, the formula  $\rho(z) : \forall y \exists x (\psi(y, z) \rightarrow (P(x) \wedge \phi(x, y)))$  is true of  $c_i$  in  $(M, P(M))$  for all  $i$ , so true of  $c$  in  $(N, P(N))$ . On the other hand, by the assumption (iii),  $(N, P(N))$  has a saturated elementary extension  $(N', P(N'))$  say which satisfies the extension property. Choose any complete  $L$ -type  $p(y, c)$  containing  $\psi(y, c)$  and let  $b \in N'$  realize a nonforking extension of  $p(y, c)$  over  $P(N')c$ . As  $(N', P(N')) \models \rho(c)$ , there is  $d \in P(N')$  such that  $\phi(d, b)$ . But then  $\phi(x, b)$  does not fork over  $c$ . We have shown that  $Q_{\phi, \psi}$  is true of  $c$ . The proof is complete.

Let us summarize the results we have obtained in this section, making use also of 2.8.

**Corollary 4.6** *The following are equivalent:*

- (i)  $\mathcal{C}_{T, P}$  is first order,
- (ii) For any (some)  $\kappa \geq |T|^+$ , any  $\kappa$ -saturated model of  $T_P$  is a  $\kappa$ -lovely pair,
- (iii)  $T$  is low and  $T_P^+$  has quantifier-elimination,
- (iv) Every free extension of pairs of models of  $T_P$  is elementary.

- (v)  $T$  is low and  $Q_{\phi,\psi}(z)$  is definable for all  $\phi, \psi \in L$ .  
 (vi) For each  $\phi(x, y) \in L$ , the  $D_\phi$ -rank is finite and definable (in  $T$ ).

The following axiomatization is then clear (and already figured in proofs above). Let us recall notation:  $\Sigma_{\phi(x,y),z}(y, z)$  is the partial  $L$ -type expressing that  $\phi(x, y)$  divides over  $z$  (which exists assuming lowness of  $T$ ).

**Remark 4.7** *Assuming  $\mathcal{C}_{T,P}$  to be first order,  $T_P$  can be axiomatized as follows:*

- (i)  $(\forall z)(\forall y)(P(z) \wedge \neg\psi(y, z) \rightarrow \exists x(P(x) \wedge \phi(x, y)))$ , whenever  $\psi(y, z) \in \Sigma_{\phi,z}(y, z)$ .  
 (ii)  $(\forall z)(\neg Q_{\phi,\psi}(z) \rightarrow \exists y(\psi(y, z) \wedge (\forall x)(P(x) \rightarrow \neg\phi(x, y))))$ , for all  $\phi, \psi \in L$ .

## 5 The stable case and Poizat's "Belles Paires"

Let  $T$  be an arbitrary complete first order theory (in language  $L$ ). Poizat [14] defined a *belle paire* of models of  $T$  to be an elementary pair  $M \subseteq N$  of models of  $T$  such that (i) for any finite  $A \subset M$  and complete ( $L$ ) type  $p(x)$  over  $M \cup A$ ,  $p$  is realized in  $N$ , and (ii)  $M$  is  $|T|^+$ -saturated. Poizat proves, among other things:

**Fact 5.1** (i) *Any two belles paires of models of  $T$  are elementarily equivalent (in the language  $L_P$ ),*  
 (ii) *Assuming  $T$  stable,  $T$  has the nonfcp if and only if every  $|T|^+$ -saturated model of the theory of belles paires is a belle paire if and only if the theory of belles paires has quantifier elimination in the language  $L_P^+$ .*

It is pretty clear that for  $T$  stable, belles paires are essentially lovely pairs. Let us make this more precise: Define a pair  $(M, P(M))$  to be a  $\kappa$ -belle paire if (i) any complete type  $L$ -type over  $A \cup P(M)$  for  $A \subseteq M$  of cardinality  $< \kappa$ , is realized in  $M$ , and (ii)  $P(M)$  is  $\kappa$ -saturated.

**Remark 5.2** *Suppose  $\kappa \geq |T|^+$ .*

- (i) *A  $\kappa$ -belle paire is a belle paire.*  
 (ii) *For  $T$  stable,  $(M, P(M))$  is a  $\kappa$ -belle paire iff it is a  $\kappa$ -lovely pair.*

*Proof.* (ii): Axiom (i) for  $\kappa$ -belles paires immediately implies axiom (i) for  $\kappa$ -lovely pairs. On the other hand suppose  $(M, P(M))$  satisfies axiom (i) for

$\kappa$ -lovely pairs. Let  $A \subseteq M$  be of cardinality  $< \kappa$  and let  $p(x)$  be a complete  $L$ -type over  $P(M) \cup A$ . There is no harm, by stability, in assuming  $p(x)$  to be stationary. Let  $B \subset P(M) \cup A$  be of cardinality  $\leq |T|$  such that  $p$  does not fork over  $B$  and  $p|_B$  is stationary. In particular  $p$  is the unique nonforking extension of  $p|(A \cup B)$  over  $A \cup P(M)$ . Axiom (i) for  $\kappa$ -lovely pairs implies  $p$  is realized in  $M$ .

The equivalence of Axiom (ii) for  $\kappa$ -lovely pairs and  $\kappa$ -belles paires is already in Remark 3.2 (a).

Let us assume for this paragraph that  $T$  is stable. By Fact 5.1 (i) and Remark 5.2 (and our earlier results) the theory of belles paires coincides with the theory  $T_P$  of lovely pairs. As any stable theory is low, by Fact 5.2(ii) and Corollary 4.6 the “first-orderness” of the class of belles paires coincides with that of the class of lovely pairs. So we see that  $T$  has the nonfcpc if and only if the predicates  $Q_{\phi,\psi}$  are definable if and only if the  $D_\phi$ -ranks are definable. So we already obtain some “new” equivalents to the nonfcpc for stable theories.

Poizat in [14] goes on to show (for stable  $T$ ) that, assuming  $T$  has nonfcpc,  $T_P$  is stable (with the nonfcpc). We will generalize this to the simple case in the next section.

## 6 Simplicity of $T_P$ , when $\mathcal{C}_{T,P}$ is first order

Let us begin with a little lemma which enables us to check simplicity of a theory by looking at types over models:

**Lemma 6.1** ( *$T$  any complete first order theory.*)

(i)  *$T$  is simple if and only if any (finitary) type over a model  $M$  of  $T$  does not divide over some subset  $A$  of  $M$  of cardinality at most  $|T|$ .*

(ii) *Likewise  $T$  is supersimple if and only if every type over a model  $M$  does not divide over some finite subset of  $M$ .*

*Proof.* We know the left to right directions. Let us start with the right to left direction of (i). If  $T$  is not simple, then there is a formula  $\phi(x, y) \in L$ , and a sequence  $(b_i : i < |T|^+)$ , such that  $\phi(x, b_i)$  divides over  $\{b_j : j < i\}$  for all  $i$ , and  $\{\phi(x, b_i) : i < |T|^+\}$  is consistent. Let  $B_i = \{b_j : j < i\}$ .

*Claim.* There is an increasing chain of models  $M_i \supseteq B_i$  ( $i < |T|^+$ ), each of

cardinality at most  $|T|$ , such that  $\phi(x, b_i)$  divides over  $M_i$  for all  $i$ .

*Proof of claim.* We construct the chain  $M_i \supseteq B_i$  inductively, satisfying the stronger conditions: for all  $j \geq i$ ,  $\phi(x, b_j)$  divides over  $M_i \cup B_j$ . Let us show how to find  $M_0$  and leave the rest to the reader. Let  $r(z)$  be the type of an enumeration of some model of cardinality at most  $|T|$ . Let  $(b_0^j : j < \omega)$  be an indiscernible sequence such that  $b_0^0 = b_0$  and  $\{\phi(x, b_0^j) : j < \omega\}$  is inconsistent (by our assumption that  $\phi(x, b_0)$  divides over  $\emptyset$ ). By Erdos-Rado, we can find a complete type  $r_1(z)$  over  $\{b_0^j : j < \omega\}$  such that if  $m$  realizes  $r_0(z)$  then  $(b_0^j : j < \omega)$  is  $m$ -indiscernible. Let  $r'_1(z)$  be the restriction of  $r_1(z)$  to  $b_0$ . So if  $m$  realizes  $r'_1(z)$ , then  $\phi(x, b_0)$  divides over  $m$ . Now, in exactly the same way, extend  $r'_1(z)$  to a complete type  $r'_2(z)$  over  $\{b_0, b_1\}$  such that if  $m$  realizes  $r'_2(z)$ , then  $\phi(x, b_1)$  divides over  $mb_0$ . Continue, taking unions at limit stages, to find  $r'_i(z)$  over  $B_i$  for  $i < |T|$ . If  $M_0$  realizes the union of these types, then it does the job.

Let  $M$  the union of the  $M_i$ . Let  $p(x) \in S(M)$  be a completion of  $\{\phi(x, b_i) : i < |T|^+\}$ . Then by construction,  $p(x)$  divides over each  $M_i$ . But any subset of  $M$  of cardinality  $\leq |T|$  is contained in some  $M_i$ . Hence the right hand side of (i) fails.

Now we prove the right to left direction of (ii). By part (i) we may assume  $T$  to be simple. Suppose  $T$  is not supersimple, so there is some chain  $p_i(x) \in S(A_i)$  of types over finite sets such that  $p_{i+1}$  divides over  $A_i$  for  $i < \omega$ . Either use the method of (i), or choose inductively models  $M_i \supset A_i \cup M_{i-1}$  such that  $M_i$  is independent from  $\bigcup_j A_j$  over  $A_i$ . Note that then  $p_{i+1}(x)$  divides over  $M_i$ . So if  $M$  is the union of the  $M_i$  and  $p(x)$  a completion of  $\bigcup_i p_i$  over  $M$ , then  $p$  divides over each  $M_i$  so divides over every finite subset of  $M$ .

For the rest of this section we assume that  $T$  is simple and  $\mathcal{C}_{T,P}$  is first order. Let us situate ourselves in a very saturated model of  $T_P$  which we may assume to be  $(\bar{M}, P(\bar{M}))$  where  $\bar{M}$  is our big saturated model of  $T$ . So  $(\bar{M}, P(\bar{M}))$  is a  $\bar{\kappa}$ -lovely pair.

**Proposition 6.2**  *$T_P$  is simple. Moreover if  $T$  is supersimple, so is  $T_P$ .*

*Proof.* By the lemma above it is enough to consider types over models. Let  $(M, P(M))$  be an  $L_P$ -elementary substructure of the universe. and  $a$  a finite tuple. We will find a subset  $A$  of  $M$  of cardinality at most  $|T|$  such that  $tp_{L_P}(a/M)$  does not divide over  $A$ . In the case where  $T$  is supersimple, we'll

see from the proof that  $A$  could be chosen finite.

We will just talk about  $P$  rather than  $P(\bar{M})$ . When we say something like “ $tp_L(a/B)$  does not divide over  $A$ ” we actually mean that  $tp(a/B)$  does not divide over  $A$  in the  $L$ -structure  $\bar{M}$ . (We hope that there is no ambiguity here.) Note that  $tp_L(c/M)$  does not divide over  $P(M)$  for any  $c \in P$  (in fact  $tp_L(c/M)$  is finitely satisfiable in  $P(M)$ ).

By simplicity of  $T$  we can find  $C \subset P$  and  $A \subset M$  both of cardinality at most  $|T|$  such that

- (i)  $tp_L(a/M \cup P)$  does not divide over  $A \cup C$ ,
- (ii)  $tp_L(C/M)$  does not divide over  $P(A)$ .

*Claim.*  $tp_{L_P}(a/M)$  does not divide over  $A$ .

*Proof of claim.* Let  $\{M_i : i < \omega\}$  be an  $A$ -indiscernible (in the  $L_P$  sense) sequence of realizations of  $tp_{L_P}(M/A)$  with  $M_0 = M$ . Let  $p_0(y) = tp_L(C/M_0)$ , and  $p_i(y)$  its copy over  $M_i$ . By (ii) there is  $C'$  realizing  $\bigcup_i p_i(y)$  such that  $tp_L(C'/\bigcup_i M_i)$  does not divide over  $A$ , and  $(M_i : i < \omega)$  is  $AC'$ -indiscernible in the sense of  $L$ . Note that  $tp_L(C'/\bigcup_i M_i)$  does not divide over  $P(A)$ , hence by the  $\bar{\kappa}$ -loveliness of  $(\bar{M}, P(\bar{M}))$ , we may assume that  $C'$  is contained in  $P$ . As both  $M \cup C$  and  $M \cup C'$  are  $P$ -independent and have the same quantifier-free  $L_P$ -type, we conclude from 3.8 that they have the same  $L_P$ -type. So, (by changing the sequence of  $M_i$ 's) we may assume that  $C' = C$ . By (i),  $tp_L(a/M \cup C)$  does not divide over  $A \cup C$ . Let  $r(x) = tp_L(a/MC)$ . Remember that  $(M_i : i < \omega)$  is  $AC$ -indiscernible in the sense of  $L$ . So letting  $r_i(x)$  be the copy of  $r(x)$  over  $M_i \cup C$ ,  $\bigcup_i r_i(x)$  can be realized by some  $a'$  which is  $L$ -independent from  $\bigcup_i M_i \cup C$  over  $A \cup C$ . By loveliness of  $(\bar{M}, P(\bar{M}))$ , we may assume that  $a'$  is  $L$ -independent from  $P$  over  $\bigcup_i M_i \cup C$ . In particular each  $a'M_iC$  is  $P$ -independent. Finally note that all  $a'M_iC$  have the same quantifier-free  $L_P$ -type, so by 3.8, they have the same  $L_P$ -type. But for the same reason this is the  $L_P$ -type of  $aMC$ . We have shown that  $tp_{L_P}(a/M)$  does not divide over  $A$ .

## 7 Forking in $T_P$ , when $\mathcal{C}_{T,P}$ is first order

We assume throughout this section that  $\mathcal{C}_{T,P}$  is first order, unless we say otherwise. This long and central section initiates the analysis of the simple theory  $T_P$ . We characterize forking, give some information on canonical bases, show that 1-basedness is preserved (in passing from  $T$  to  $T_P$ ), and

finally examine the implications of  $T_P$  being  $\omega$ -categorical. There are many more things to be done and questions to be settled. A few such problems will be stated at the end of the paper.

We begin by characterizing forking in  $(\bar{M}, P(\bar{M}))$ . In fact this would also be another route to Proposition 6.2, but in the proofs below it is convenient to know the simplicity of  $T_P$  in advance.

Let us introduce some notation:

**Definition 7.1** *Let  $a$  be a (possibly infinite) tuple. By  $a^c$  we mean the canonical base of  $tp_L(a/P(\bar{M}))$ . By  $\hat{a}$  we mean  $(a, a^c)$ .*

**Remark 7.2** (i)  $a^c$  is contained in  $dcl_{L_P}(a)$ .  
(ii)  $a$  is  $L_P$ -independent from  $P$  over  $a^c$ .

*Proof.* (i) is clear, as any automorphism of  $(\bar{M}, P(\bar{M}))$  fixes  $P$  setwise so if it also fixes  $a$  it fixes  $a^c$ .

(ii). Let  $B \subset P$ ,  $B$  containing  $a^c$ . Let  $(B_i)_i$  be an  $a^c$ -indiscernible sequence in the sense of  $L_P$ , with  $B_0 = B$  (so all  $B_i \subset P$ ). By lovelieness we can find  $a'$  such that  $tp_L(a'B_i) = tp_L(aB)$  for all  $i$  and  $a'$  is  $L$  independent from  $P$  over  $\bigcup_i B_i$ . It follows (by 3.8) that  $tp_{L_P}(a'B_i) = tp_{L_P}(aB)$  for all  $i$ , proving (ii).

**Proposition 7.3** *Let  $A \subseteq B$  be sets and  $c$  a tuple. Then the following are equivalent:*

- (i)  $tp_{L_P}(c/B)$  does not fork over  $A$ .
- (ii)  $tp_L(c/B \cup P)$  does not fork over  $A \cup P$  and  $tp_L((cA)^c/B^c)$  does not fork over  $A^c$ ,
- (iii)  $tp_L(c/B \cup P)$  does not fork over  $A \cup P$  and  $tp_L(\widehat{cA}/\widehat{B})$  does not fork over  $\widehat{A}$ .
- (iv) There is (small)  $a \subset P$  such that  $tp_L(c/B \cup P)$  does not fork over  $A \cup a$  and  $tp_L(a/\widehat{B})$  does not fork over  $\widehat{A}$ .

*Proof.* (i) implies (ii): Assume (i). So  $tp_{L_P}(cA/B)$  does not fork over  $A$ . By Remark 7.2(i),  $tp_{L_P}(\widehat{cA}/\widehat{B})$  does not fork over  $\widehat{A}$ . In particular  $tp_{L_P}((cA)^c/\widehat{B})$  does not fork over  $\widehat{A}$ . By Remark 7.2(ii),  $\widehat{A}$  is  $L_P$ -independent from  $P$  over  $A^c$ . We conclude that  $(cA)^c$  is  $L_P$ -independent from  $B^c$  over  $A^c$ . As these sets all live in  $P$  we can replace  $L_P$ -independence by  $L$ -independence. This

gives the second part of (ii).

We now prove the first part of (ii), that is  $tp_L(c/B \cup P)$  does not fork over  $A \cup P$ . We may assume that  $B = \hat{B}$ . Suppose for a contradiction that  $tp_L(c/B \cup P)$  forks over  $A \cup P$ . By lovelieness, let  $(B_i)_{i < \lambda}$  be a very big sequence of realisations of  $tp_L(B/A \cup B^c)$  such that  $B_i$  is  $L$ -independent from  $\bigcup_{j < i} B_j$  over  $A \cup P$  and  $B_0 = B$ . Note that each  $B_i$  is  $P$ -independent, so  $tp_{L_P}(B_i) = tp_{L_P}(B)$  for all  $i$ . By Erdos-Rado we may assume that  $(B_i : i < \lambda)$  is  $A \cup B^c$ -indiscernible in the sense of  $L_P$ , so in particular  $(B_i : i < \lambda)$  is  $A$ -indiscernible in the sense of  $L_P$ . By (i) we may find  $c'$  such that  $tp_{L_P}(c'B_i) = tp_{L_P}(cB)$  for all  $i$ . Now  $\{B_i : i < \lambda\}$  is  $L$ -independent over  $A \cup P$ , and  $c'$   $L$ -forks with each  $B_i$  over  $A \cup P$ . As we chose  $\lambda \geq |T|^+$ , this contradicts simplicity of  $T$ .

(ii) implies (iii): Assuming (ii) all we have to do is prove

(\*)  $\widehat{cA}$  is  $L$ -independent from  $\hat{B}$  over  $\hat{A}$ .

We begin to make some observations:

First  $(cA)^c$  is  $L$ -independent from  $\hat{B}$  over  $B^c$ . Together with the second part of (ii), we obtain:

(I)  $(cA)^c$  is  $L$ -independent from  $\hat{B}$  over  $\hat{A}$ .

On the other hand, As  $cA$  is  $L$ -independent from  $P$  over  $(cA)^c$ , we have:

(II)  $cA$  is  $L$ -independent from  $A \cup P$  over  $A \cup (cA)^c$ .

But the first part of (ii) yields that  $cA$  is  $L$ -independent from  $B \cup P$  over  $A \cup P$ , so together with (II) this gives:

(III)  $cA$  is  $L$ -independent from  $\hat{B} \cup (cA)^c$  over  $\hat{A} \cup (cA)^c$ .

(I) and (III) give (\*).

(iii) implies (iv): Assume (iii) and take  $a$  to be  $(cA)^c$ . So by the second part of (iii)  $tp_L(a/\hat{B})$  does not fork over  $\hat{A}$ . But also as  $tp_L(cA/P)$  does not fork over  $a$ ,  $tp_L(c/AP)$  does not fork over  $Aa$ . Together with the assumption that  $tp_L(c/BP)$  does not fork over  $AP$  this implies that  $tp_L(c/BP)$  does not fork over  $Aa$ .

(iv) implies (iii): Let  $a \subset P$  be as given by (iv). So we see immediately that  $tp_L(c/BP)$  does not fork over  $AP$ .

On the other hand, we also have that  $tp_L(c/AP)$  does not fork over  $Aa$ , and together with the fact that  $tp_L(A/P)$  does not fork over  $A^c$  we conclude that:

(\*)  $tp_L(cA/P)$  does not fork over  $A^c a$ .

As  $c$  is  $L$ -independent from  $\hat{B}a$  over  $\hat{A}a$  and  $a$  is  $L$ -independent from  $\hat{B}$  over

$\hat{A}$ , we see that  $ca$  is  $L$ -independent from  $\hat{B}$  over  $\hat{A}$  and thus:

(\*\*)  $tp_L(\widehat{ca\hat{A}}/B)$  does not fork over  $\hat{A}$ .

By (\*)  $\widehat{cA}$  is contained in  $ca\hat{A}$ , and so from (\*\*) we conclude that  $tp_L(\widehat{cA}/\hat{B})$  does not fork over  $\hat{A}$ . (iii) is proved.

Finally we prove (iii) implies (i). The proof is like that of 6.2, so we are brief. First as any tuple  $d$  is  $L_P$ -interdefinable with  $\hat{d}$ , it suffices to prove:

(\*\*\*)  $\widehat{cA}$  is  $L_P$ -independent from  $\hat{B}$  over  $\hat{A}$ .

Let  $(\hat{B}_i)_i$  be  $L_P$ -indiscernible over  $\hat{A}$  with  $\hat{B}_0 = \hat{B}$ . As (by (iii))  $(cA)^c$  is  $L$ -independent from  $\hat{B}$  over  $\hat{A}$ , we can, by loveliness, assume that  $(\hat{B}_i)_i$  is  $L$ -indiscernible over  $\hat{A} \cup (cA)^c$ . Now (iii) implies that  $\widehat{cA}$  is  $L$ -independent from  $\hat{B}$  over  $\hat{A} \cup (cA)^c$ . Let  $p(x) = tp_L(\widehat{cA}/\hat{B} \cup (cA)^c)$ , and  $p_i$  the copy over  $\hat{B}_i$ . So (using also loveliness) we can realise  $\bigcup_i p_i$  by some  $d$  which is  $L$ -independent from  $P$  over  $\bigcup_i (\hat{B}_i) \cup (cA)^c$ . It follows that  $tp_{L_P}(d\hat{B}_i/(cA)^c) = tp_{L_P}(\widehat{cA}\hat{B}_i/(cA)^c)$  for all  $i$ . We have proved (\*\*\*) .

We now turn our attention to canonical bases of Lascar strong types in  $T_P$ . Understanding canonical bases in a simple (in particular stable) theory is important, as by [12] all hyperimaginaries are essentially canonical bases of types of real tuples over models.

It is worth noting to begin with that  $T_P$  may contain really new (hyper-) imaginaries. For example, let  $T$  be the theory of a vector space  $V$  over a field  $F$ , in the module language.  $T$  is strongly minimal and every element of  $V^{eq}$  is interalgebraic with a real tuple. Let  $(V, P(V))$  be a model of  $T_P$  and  $a \in V \setminus P$ . Then the coset  $a + P$  is an element of  $(V, P(V))^{eq}$  but is not interalgebraic with any real tuple. Note that  $a + P$  is the canonical base of  $tp_{L_P}(a/a + c)$  where  $c \in P$  is generic.

We first work towards proving that the only canonical bases we need consider in  $T_P$  are of the form  $Cb(tp_{L_P}(d/B))$  where  $B$  is a model,  $d$  is a hyperimaginary in the sense of  $L$ , and  $d \in bdd_L(B \cup P)$ . (Of course the base set can always be chosen to be a model.)

We start with a preliminary lemma:

**Lemma 7.4** *Suppose that  $d = Cb(Lstp_L(c/B \cup P))$ . Then*

(i)  $tp_{L_P}(c/B \cup P)$  does not fork over  $d$ .

(ii)  $d \in bdd_{L_P}(cB)$ .

*Proof.* (i) By Proposition 7.3, all we have to prove is

(\*)  $\widehat{cd}$  is  $L$ -independent from  $\widehat{BP}$  over  $\hat{d}$ .

Note that  $\widehat{BP} = BP$ , and so our assumptions imply that  $c$  is  $L$ -independent from  $\widehat{BP}$  over  $\hat{d}$ , and thus  $c\hat{d}$  is  $L$ -independent from  $\widehat{BP}$  over  $\hat{d}$ . So it is enough to prove that  $\widehat{cd} = cd$ . But the latter follows as  $c$  is  $L$ -independent from  $dP$  over  $dd^c$  and  $d$  is  $L$ -independent from  $P$  over  $d^c$  (hence  $cd$  is  $L$ -independent from  $P$  over  $d^c$ ).

(ii) Note that  $c$  is  $L$ -independent from  $BP$  over  $B \cup (cB)^c$ , and thus  $d \in bdd_L((cB)^c)$ . Now use Remark 7.2(i).

**Proposition 7.5** *Let  $B$  be an elementary substructure of  $(\bar{M}, P(\bar{M}))$ , and  $c$  a real tuple. Let  $d = Cb(Lstp_L(c/B \cup P))$ . Then  $Cb(tp_{L_P}(c/B))$  is  $L_P$ -interdefinable with  $Cb(tp_{L_P}(d/B))$ .*

*Proof.* Let  $p_B = tp_{L_P}(c/B)$  and  $r_B = tp_{L_P}(d/B)$ . Let  $e = Cb(p_B)$  and  $e' = Cb(r_B)$ . So  $e, e' \in dcl_{L_P}(B)$  are hyperimaginaries in the  $L_P$ -sense (objects which we have not considered before).

*Claim 1.*  $c$  is  $L_P$ -independent from  $B$  over  $e'$ , and  $tp_{L_P}(c/e')$  is an amalgamation base.

*Proof.* We will make repeated use of Proposition 7.3 ((i) if and only if (iii)), and Lemma 7.4. These first imply that  $c$  is  $L_P$ -independent from  $BP$  over  $d$ , and so  $c$  is  $L_P$ -independent from  $Bd$  over  $d$ , whereby  $c$  is  $L_P$ -independent from  $Bd$  over  $e'd$ . As  $d$  is  $L_P$ -independent from  $B$  over  $e'$  we conclude that  $cd$  is  $L_P$ -independent from  $B$  over  $e'$ , yielding the first part of claim I. The second part is left to the reader.

*Claim II.*  $d$  is  $L_P$ -independent from  $B$  over  $e$ .

*Proof.* It will be enough to show that if  $B'$  realizes  $tp_{L_P}(B/e)$  with  $B'$   $L_P$ -independent from  $B$  over  $e$ , then  $r_B \cup r_{B'}$  have a common nonforking extension. By the assumption on  $e$  (and the independence theorem) Let  $c'$  realize  $p_B \cup p_{B'}$ , such that  $c'$  is  $L_P$ -independent from  $BB'$  over  $e$  (and so over each of  $B, B'$ ). By 7.3,  $Cb(Lstp_L(c'/BB'P)) = Cb(Lstp_L(c'/BP)) = Cb(Lstp(c'/B'P)) = d'$  say where  $d'$  realizes  $r_B \cup r_{B'}$ . By Lemma 7.4(ii),  $d' \in bdd_{L_P}(c'B) \cap bdd_{L_P}(c'B')$ . Thus  $d'$  is  $L_P$ -independent from  $BB'$  over each of  $B, B'$ .

Claims I and II show that  $e$  and  $e'$  are interdefinable, proving the Proposition.

Now suppose that  $B$  is a model say, and  $d \in bdd_L(B \cup P)$ . We will give an explicit description of the type-definable equivalence relation  $E$  on  $tp_{L_P}(B)$

such that  $B/E = Cb(tp_{L_P}(d/B))$ . Let  $a = (dB)^c$ . So  $d \in bdd_L(Ba)$ . Let  $q_{Ba}(z) = tp_L(d/Ba)$ . Define  $E_0$  on  $tp_{L_P}(B)$  by:  
 $(B', B'') \in E_0$  iff there are  $a', a''$  in  $P$  such that  
(i)  $tp_L(B'a') = tp_L(B''a'') = tp_L(Ba)$ ,  
(ii)  $a'$  is  $L$ -independent from  $\widehat{B'B''}$  over  $B$ , and  $a''$  is  $L$ -independent from  $\widehat{B'B''}$  over  $B'$ ,  
(iii) there is  $d'$  realizing  $q_{B'a'}(z) \cup q_{B''a''}(z)$ .

**Proposition 7.6** *Let  $E$  be the 2-iterate of  $E_0$ . Then  $E$  is  $L_P$ -type-definable over  $\emptyset$ , and  $B/E = Cb(tp_{L_P}(d/B))$ .*

*Proof.* Let  $p_B(z) = tp_{L_P}(d/B)$ . We show first that if  $(B', B'') \in E_0$  then  $p_{B'}(z)$  and  $p_{B''}(z)$  have a common nonforking extension. Let  $a', a'', d'$  be as in the definition of  $E_0$ . Then  $a'$ , being in  $P$ , is  $L_P$ -independent of  $\widehat{B'B''}$  over  $B$ , hence  $d'$  is  $L_P$ -independent over  $\widehat{B'B''}$  over  $B'$ . Likewise  $d'$  is  $L_P$ -independent of  $\widehat{B'B''}$  over  $B''$ . It is easy to see that  $d'$  realizes  $p_{B'} \cup p_{B''}$ .

On the other hand, let  $e = Cb(p_B)$ . We will show that if  $B'$  is  $L_P$ -independent of  $B$  over  $e$ , with the same  $L_P$ -type as  $B$  over  $e$ , then  $(B, B') \in E_0$ . Let  $d'$  realize a common nonforking extension of  $p_B \cup p_{B'}$ . Then  $(d'B)^c \in dcl_{L_P}(d'B)$  is  $L_P$ -independent of  $BB'$  over  $B$ , and likewise  $(d'B')^c$  is  $L_P$ -independent of  $BB'$  over  $B'$ . Let  $a' = (d'B)^c$  and  $a'' = (d'B')^c$ .

Next we will apply the above results to show that 1-basedness of  $T$  implies 1-basedness of  $T_P$ . Recall that the simple theory  $T$  is said to be 1-based iff for any  $c$  and  $B$ ,  $Cb(Lstp(c/B)) \subseteq bdd(c)$ . Other equivalent conditions are:  
(a) for any  $c, b$ ,  $c$  is independent from  $b$  over  $bdd(c) \cap bdd(b)$ ,  
(b) whenever  $c'$  realises a nonforking extension of  $Lstp(c/B)$  over  $Bc$  then  $tp(c'/Bc)$  does not fork over  $c$ .  
In (b)  $B$  can be taken to be a model.

**Proposition 7.7** *Suppose that  $T$  is 1-based. Then  $T_P$  is 1-based.*

*Proof.* By 7.5 and 1-basedness of  $T$  it is enough to consider types of the form  $tp_{L_P}(d/B)$  where  $B$  is a model, and  $d \in bdd_L(B \cup P)$ . Let  $d'$  realize an  $L_P$ -nonforking extension of  $tp_{L_P}(d/B)$  over  $B \cup d$ . We must show that  $tp_{L_P}(d'/Bd)$  does not fork over  $d$ . We may assume that  $d = \hat{d}$  (and so  $d' = \hat{d}'$ ).  
*Claim 1.*  $\widehat{d'd}$  is  $L$ -independent from  $\widehat{Bd}$  over  $\hat{d}$ .

*Proof of claim 1.* By our assumption, as well as Proposition 7.3,  $d$  is  $L$ -independent from  $d'$  over  $B$ , and note that  $d$  and  $d'$  have the same  $L$ -type over  $B$  and that  $B$  is also a model in the  $L$ -sense. By 1-basedness of  $T$ ,  $d'd$  is independent from  $Bd$  over  $d$ . Thus  $bdd_L(d'd)$  is independent from  $bdd_L(Bd)$  over  $d$ . But as  $T$  is 1-based,  $Cb_L(tp(a/P)) \subseteq bdd(a)$  for any  $a$ . In particular  $\widehat{d'd} \subseteq bdd_L(d'd)$  and  $\widehat{Bd} \subseteq bdd_L(Bd)$ . So we get claim 1.

*Claim 2.*  $d' \in bdd_L(Pd)$

*Proof of Claim 2.* Let  $a$  be a tuple in  $P$  such that  $d \in bdd_L(Ba)$ , and let  $a'$  be such that  $tp_{L_P}(ad/B) = tp_{L_P}(a'd'/B)$ . So also  $a' \in P$ . Let  $b_0 = Cb(tp_L(ad/B)) = Cb(tp_L(a'd'/B))$ . Then  $d \in bdd_L(a, b_0)$  and  $d' \in bdd_L(a', b_0)$ . By 1-basedness of  $T$ ,  $b_0 \in bdd_L(ad)$ . It follows that  $d' \in bdd_L(d, a, a')$ , proving Claim 2.

It trivially follows from Claim 2 that  $tp_L(d'/BP)$  does not fork over  $dP$ . Together with Claim 1 and Proposition 7.3, we conclude that  $tp_{L_P}(d'/Bd)$  does not fork over  $d$ . The proof is complete.

Let us finally discuss the strength of the hypothesis that  $T_P$  is  $\omega$ -categorical. Let  $T$  now be a one-sorted theory in a countable language  $L$ , and we make no a priori assumptions on the first-orderness of  $\mathcal{C}_{T,P}$ . It should be remarked that it is not known whether the  $\omega$ -categoricity of  $T$  implies that  $T$  is low. On the other hand, if  $T$  is  $\omega$ -categorical and low, then clearly the  $D_\phi$ -ranks are finite and definable for all  $\phi \in L$  whereby  $\mathcal{C}_{T,P}$  is first order. Also note that by [7] any  $\omega$ -categorical 1-based simple theory is supersimple of finite  $SU$ -rank (in particular low).

**Proposition 7.8** ( *$T$  countable, simple*).

- (i) *Suppose that  $T$  is  $\omega$ -categorical and 1-based. Then  $T_P$  is  $\omega$ -categorical.*
- (ii) *Suppose that  $\mathcal{C}_{T,P}$  is first order and that  $T_P$  is  $\omega$ -categorical. Then  $T$  is 1-based and  $\omega$ -categorical.*

*Proof.* (i) Fix  $n$ . We will show that there are only finitely many  $L_P$ -types of real  $n$ -tuples in  $(\bar{M}, P(\bar{M}))$ . Fix an  $n$ -tuple  $a$ .  $tp_L(a/Cb_P(a))$  is realized in  $P$  (as  $P$  is a saturated model of  $T$ ), by  $a'$  say. By 1-basedness of  $T$ ,  $Cb_P(a) \in bdd_L(a')$ , hence  $(a, a')$  is  $P$ -independent, so its type is determined by its quantifier-free  $L_P$ -type. Clearly there are only finitely many quantifier-free  $L_P$ -types of  $2n$ -tuples.

(ii) Assume that  $\mathcal{C}_{T,P}$  is first order and that  $T_P$  is  $\omega$ -categorical. As usual we

work in a very saturated model  $(\bar{M}, P(\bar{M}))$  of  $T_P$ . We will make a series of easy claims.

*Claim I.*  $T$  is supersimple.

*Proof.* Suppose for a contradiction that there are, in  $\bar{M}$ , complete 1-types  $p_i(x) \in S(A_i)$  for  $i < \omega$ , such that  $i < j$  implies  $p_j(x)$  is a forking extension of  $p_i(x)$ . So for each  $i < j$  there is an  $L$ -formula  $\phi_{i,j}(x, y_{i,j})$  such that the  $D_{\phi_{i,j}}$ -ranks of  $p_i$  and  $p_j$  differ. We may assume that all  $A_i$  are in  $P$  and then for each  $i$  some  $L$ -nonforking extension of  $p_i$  over  $P$  is realized in  $\bar{M}$  by  $a_i$  say. There are only finitely many  $tp_{L_P}(a_i)$ . But  $tp_{L_P}(a_i) = tp_{L_P}(a_j)$  implies that the  $D_\phi$ -ranks of  $tp_L(a_i/P)$  are the same as those of  $tp_L(a_j/P)$ . This is a contradiction.

*Claim II.*  $T$  has finite  $SU$ -rank.

*Proof.* If not, we can find elements  $a \in \bar{M}$  such that  $tp_L(a/P)$  has arbitrarily large finite  $SU$ -rank. But then we obtain infinitely many 1-types in  $L_P$ .

*Claim III.* Let  $p(x) \in S(A)$  be any type of  $SU$ -rank 1 in  $T$ . Then  $p$  is pseudo-linear. That is there is a finite bound on the  $SU$  ranks of  $Cb(tp(a_1, a_2/B))$ , for  $a_1, a_2$  realizing  $p$ ,  $B \supset A$ , and with  $SU(tp(a_1, a_2/B)) = 1$ .

*Proof.* If not we can find infinitely many  $L_P$  types over  $\emptyset$  extending  $p \times p$ , as above.

Note that  $T$  is  $\omega$ -categorical. By Claims II and III, and [17], every type of  $SU$ -rank 1 in  $T$  is “locally modular” (that is 1-based), hence by [7],  $T$  is 1-based.

## 8 The existentially closed/Robinson case

We saw in section 3 that if  $\mathcal{C}_{T,P}$  is first order then  $T_P^+$  has quantifier-elimination, and so is the model completion (companion) of its universal part  $(T_P^+)_\forall$ . In particular existentially universal (see [13]) models of  $(T_P^+)_\forall$  coincide with saturated models of  $T_P^+$  and so are lovely pairs.

In this section we consider the category of existentially closed models of  $(T_P^+)_\forall$  without the assumption that  $\mathcal{C}_{T,P}$  is first order. (It is clear that in any such model,  $(\forall y)(R_\phi(y) \leftrightarrow \exists x \in P(\phi(x, y)))$  is true.) In particular we examine the condition:

(EC): Any existentially universal model of  $(T_P^+)_\forall$  is a lovely pair.

Bearing in mind the remarks in the first paragraph, this is a weaker condition

than  $\mathcal{C}_{P,T}$  being first order. We will show that (EC) holds if and only if  $T$  is low. Moreover we will show that under these conditions an existentially universal  $(M, P(M))^+$  model of  $(T_P^+)_{\forall}$  is a “Robinson universal domain”, that is it is saturated and homogeneous for quantifier-free types. The results of the previous section go through. In particular  $(M, P(M))^+$  is a simple Robinson universal domain.

The theory of existentially closed and existentially universal models is standard, but we will follow the notation of [13].

We fix a simple complete theory  $T$  (with quantifier-elimination in language  $L$ ) as usual.  $T_P^+$  is the theory of lovely pairs in the language  $L_P^+$ , as described in section 3. Namely new relations  $R_\phi(y)$  for  $\phi(x, y) \in L$  are introduced together with defining axioms:

$$A_\phi: \forall y(R_\phi(y) \leftrightarrow \exists x(P(x) \wedge \phi(x, y))).$$

We will denote an  $L_P$ -structure  $(M, P(M), R_\phi)_\phi$  by  $(M, P(M))^+$  (so the  $+$  refers to the interpretation of the  $R_\phi$ 's, which may or may not be in accordance with the axioms  $A_\phi$ ).

**Lemma 8.1** *Let  $(M, P(M))^+$  be an existentially closed model of  $(T_P^+)_{\forall}$ . Then  $(M, P(M))^+$  is an elementary pair of models of  $T$ , as well as being a model of each  $A_\phi$*

*Proof.* The model concerned is a substructure of a model of  $T_P^+$ . The sentences  $A_\phi$  as well the sentences expressing that the pair is an elementary pair of models of  $T$  are  $\forall\exists$ , hence are true in  $(M, P(M), R_\phi)_\phi$  by existential closure.

**Definition 8.2** *Let  $(M, P(M))$  be an elementary pair of models of  $T$  (in the language  $L_P$ ) and  $(M, P(M))^+$  some expansion to an  $L_P^+$ -structure. We will say that  $(M, P(M))^+$  is correct if for each  $\phi(x, y) \in L$  and  $b \in M$ ,  $R_\phi(b)$  holds in  $(M, P(M))^+$  iff  $\phi(x, b)$  does not divide over  $P(M)$  (in the model  $M$  of  $T$ ) iff  $(M, P(M)) \models \exists x \in P(\phi(x, b))$ .*

Here is our main result. The equivalence of (i) and (ii) has more or less the same content as Remark 4.2.

**Proposition 8.3** *The following are equivalent:*

(i)  $T$  is low,

(ii) Any existentially closed model of  $(T_P^+)_{\forall}$  is correct.

(iii) For some (any)  $\kappa \geq |T|^+$ , if  $(M, P(M))^+$  is a  $\kappa$ -existentially universal model of  $(T_P^+)_{\forall}$ , then  $(M, P(M))$  is a  $\kappa$ -lovely pair of models of  $T$ .

*Proof.* (i) implies (ii). By lowness of  $T$ , for each  $\phi(x, y) \in L$  and tuple  $z$  of variables, there is a partial  $L$ -type  $\Sigma_{\phi, z}(y, z)$  expressing (in a model of  $T$ ) that  $\phi(x, y)$  divides over  $z$ . As  $T$  has quantifier-elimination we may take this partial type to be quantifier-free. Let  $\Gamma_{\phi} = \{(\forall y)(\forall z)((\neg(R_{\phi}(y) \wedge P(z)) \rightarrow \psi(y, z))) : \psi(y, z) \in \Sigma_{\phi, z}\}$ . Then  $\Gamma_{\phi}$  is a set of universal  $L_P^+$ -sentences which are in  $T_P^+$  hence in  $(T_P^+)_{\forall}$ .

Now let  $(M, P(M))^+$  be an existentially closed model of  $(T_P^+)_{\forall}$ . So by Lemma 8.1,  $(M, P(M))$  is an elementary pair of models of  $T$ . Let  $\phi(x, y) \in L$  and  $b \in M$ . We will show that

$\neg R_{\phi}(b) \Rightarrow \phi(x, b)$  divides over  $P(M) \Rightarrow \phi(x, b)$  is not satisfied in  $P(M) \Rightarrow \neg R_{\phi}(b)$  (so  $(M, P(M))^+$  will be correct).

First, if  $\neg R_{\phi}(b)$ , then as  $(M, P(M))^+$  is a model of  $\Gamma_{\phi}$ , we have  $M \models \Sigma_{\phi, z}(b, c)$  for all  $z$  and  $c \in P(M)$ . As  $M$  is a model of  $T$ ,  $\phi(x, b)$  divides over  $P(M)$ . If  $\phi(x, b)$  divides over  $P(M)$  it is clearly not satisfied in  $P(M)$ . Finally, by Lemma 8.1 (that is,  $(M, P(M))^+$  is a model of  $A_{\phi}$ ), if  $\phi(x, b)$  is not satisfied in  $P(M)$  then  $\neg R_{\phi}(b)$ .

(ii) implies (iii): Let  $(M, P(M))^+$  be a  $\kappa$ -existentially universal model of  $(T_P^+)_{\forall}$ . By Remark 3.6 (ii), there is a  $\kappa$ -lovely  $(N, P(N))$  which extends  $(M, P(M))$  and such that moreover  $P(N)$  is  $L$ -independent from  $M$  over  $P(M)$ . Note that  $M$  is an elementary substructure of  $N$ . Let  $(N, P(N))^+$  be the canonical expansion of  $(M, P(M))$  to a model of  $T_P^+$ .

*Claim.*  $(M, P(M))^+$  is an  $(L_P^+)$ -substructure of  $(N, P(N))^+$ .

*Proof of claim.* Let  $\phi(x, y) \in L$  and  $b \in M$ . We have to show that  $R_{\phi}(b)$  holds in  $(M, P(M))^+$  iff it holds in  $(N, P(N))^+$ . Suppose first that  $(M, P(M))^+ \models R_{\phi}(b)$ . By (ii),  $\phi(x, b)$  is satisfied in  $P(M)$ , so in  $P(N)$ . Thus (as  $(N, P(N))^+$  is a model of  $A_{\phi}$ ),  $(N, P(N))^+ \models R_{\phi}(b)$ .

Conversely, suppose that  $(N, P(N))^+ \models R_{\phi}(b)$ . So  $\phi(x, b)$  is satisfied in  $P(N)$  by  $c$  say. But  $c$  is  $L$ -independent from  $b$  over  $P(M)$ , so  $\phi(x, b)$  does not fork over  $P(M)$ . By correctness of  $(M, P(M))^+$ , we have  $(M, P(M))^+ \models R_{\phi}(b)$ . The claim is proved.

We proceed to prove that  $(M, P(M))$  is a  $\kappa$ -lovely pair. Let  $B \subset M$  have cardinality  $< \kappa$ , and let  $p(z)$  be a complete  $L$ -type over  $B$ . We want to realize  $p(z)$  in  $M$  by some  $a$  such that  $tp_L(a/B \cup P(M))$  does not  $L$ -fork

over  $B$ . Using local  $D$ -ranks for example, we can find a set  $\Psi$  of  $L$ -formulas  $\phi(z, y, x)$ , such that an extension  $q(z)$  of  $p(z)$  to a complete  $L$ -type over a set  $C \supset B$  is a nonforking extension of  $p$  if and only if for no  $\phi(z, y, x) \in \Psi$ ,  $c \in C$  and  $b \in B$ , is  $\phi(z, b, c) \in q(z)$ . As  $(N, P(N))^+$  is  $\kappa$ -lovely, we can find  $a' \in N$  such that  $tp_L(a'/B \cup P(N))$  does not  $L$ -fork over  $B$ . Hence, for each  $\phi(z, y, x) \in \Psi$ , and  $b \in B$  we have  $(N, P(N))^+ \models \neg R_\phi(a', b)$ . As  $(M, P(M))^+$  is  $\kappa$ -existentially closed, we can find  $a \in M$  satisfying  $p(z)$  and also such that  $(M, P(M)) \models \neg R_\phi(a, b)$  for all  $\phi \in \Psi$  and  $b \in B$ . So  $a$  realises a nonforking extension of  $p(z)$  over  $B \cup P(M)$ .

We have shown that  $(M, P(M))$  satisfies the  $\kappa$ -extension property. The  $\kappa$ -coheir property follows as  $(M, P(M))^+$  is correct (and  $\kappa$ -existentially universal).

(iii) implies (i): Suppose that  $T$  is not low, and let  $\phi(x, y) \in L$  be non low. As in the proof of Lemma 4.1, we can find tuples  $b_i, c_i$  ( $i$  in some index set  $I$ ), inside a lovely pair  $(M, P(M))$  such that  $c_i \in P(M)$ ,  $b_i$  is independent from  $P(M)$  over  $c_i$  and  $\phi(x, b_i)$  divides over  $c_i$  for  $i \in I$ , but for some ultraproduct  $(b, c)$  of  $(b_i, c_i)_i$ ,  $\phi(x, b)$  does not divide over  $c$ . Let  $(M, P(M))^+$  be the canonical expansion of  $(M, P(M))$  to a model of  $T_P^+$ . We may assume that  $b, c$  live in an  $L_P^+$ -structure  $(N, P(N))^+$  such that  $(N, P(N), b, c)^+$  is an ultraproduct of the  $(M, P(M), b_i, c_i)^+$ . Note that  $(M, P(M))^+ \models \neg R_\phi(b_i)$  for all  $i \in I$ , as  $\phi(x, b_i)$  is not realized in  $P(M)$ . Thus  $(N, P(N))^+ \models \neg R_\phi(b)$ . But  $c \in P(N)$ , so  $\phi(x, b)$  does not divide over  $P(N)$ . As  $(N, P(N))^+$  is a model of  $T_P^+$  it extends (as an  $L_P^+$ -structure) to a  $\kappa$ -existentially universal model  $(N', P(N'))^+$  of  $(T_P^+)_{\forall}$ . Now  $(N, P(N))^+ \models \neg R_\phi(b)$ , so by Lemma 8.1 (or simply the fact that  $(N', P(N'))^+$  is a model of  $(T_P^+)_{\forall}$ ),  $\phi(x, b)$  is not satisfied in  $P(N')$ . But as  $N'$  is an elementary extension of  $N$ , and  $P(N) \subseteq P(N')$ ,  $\phi(x, b)$  does not divide over  $P(N')$ . Hence  $(N', P(N'))$  is not a lovely pair.

Assume now that the equivalent conditions of Proposition 8.3 are satisfied. Take large  $\kappa$  and let  $(\bar{M}, P(\bar{M}))^+$  be a  $\kappa$ -existentially universal model of  $(T_P^+)_{\forall}$ . Then as in [13],  $(\bar{M}, P(\bar{M}))^+$  is an  $e$ -universal domain of cardinality  $\kappa$ , that is,  $\kappa$ -saturated and  $\kappa$ -homogeneous for existential types. But  $(\bar{M}, P(\bar{M}))^+$  is also a lovely pair, hence by 3.11, types are determined by quantifier-free types. It follows that the structure is  $\kappa$ -saturated and  $\kappa$ -homogeneous for quantifier-free ( $L_P^+$ -) types. Namely  $(\bar{M}, P(\bar{M}))^+$  is a universal domain in the ‘‘Robinson’’ sense (see [8]). All the results of section 6:

proof of simplicity, characterization of forking, go through for this Robinson universal domain. So:

**Proposition 8.4** *Suppose  $T$  is low. Then any  $\kappa$ -existentially universal model of  $(T_P^+)_\forall$  is a simple Robinson universal domain.*

Finally, let us give axioms for  $(T_P^+)_\forall$  when  $T$  is low. (This is analogous to Remark 4.7 where we gave axioms for  $T_P$  assuming  $\mathcal{C}_{T,P}$  first order.)

**Remark 8.5** *Assume  $T$  is low. Then  $(T_P^+)_\forall$  can be axiomatized by*

- (i)  $T_\forall$ ,
- (ii) “ $\forall y_1 \dots y_n \forall z_1 \dots z_m (\bigwedge_{i=1, \dots, n} R_{\phi_i}(y_i) \wedge \bigwedge_{j=1, \dots, m} \neg R_{\psi_j}(z_j)) \rightarrow \exists x_1 \dots x_n (\bigwedge_i \phi_i(x_i, y_i) \wedge \bigwedge_j (\psi_j(t_j, z_j) \text{ divides over } (x_1, x_2, \dots, x_n)))$ ”, for all sequences  $\phi_1(x_1, y_1), \dots, \phi_n(x_n, y_n), \psi_1(t_1, z_1), \dots, \psi_m(t_m, z_m)$  of  $L$ -formulas.

*Explanation.* (ii) is of course a set of sentences (for each sequence  $\phi_i, \psi_j$ ): The expression “there exists  $x$  such that  $\chi(x, y)$  and  $\delta(t, z)$  divides over  $z$ ” is, by lowness and QE of  $T$ , equivalent (in models of  $T$ ) to a quantifier-free partial  $L$ -type in variables  $y$  and  $z$ . So (ii) is a set of universal  $L_P^+$ -sentences.

*Proof.* (i) and (ii) are clearly consequences of  $T_P^+$ . (In (ii) take  $x$  in  $P$  realising the  $\phi_i(x_i, y_i)$ .) For the converse, we will show that any existentially closed model of (i) and (ii) is correct (as in Definition 8.2). (Note that an ec model of (i) will be an elementary pair of models of  $T$ .) So let  $(M, P(M))^+$  be an ec model of (i) and (ii). Remember that by convention (or by adding a universal axiom),  $P(M)$  coincides with the interpretation of  $R_{x=y}(y)$ . Consider  $M$  as living in the big model  $\bar{M}$  of  $T$ . By axiom (ii) we can find some set  $B$  in  $\bar{M}$  such that (a)  $\phi(x, b)$  is realized in  $B$  for any  $\phi(x, y) \in L$  and  $b \in M$  such that  $(M, P(M))^+ \models R_\phi(b)$ , and (b)  $\psi(t, b)$  divides over  $B$  for every  $\psi(t, z) \in L$  and  $b \in M$  such that  $(M, P(M))^+ \models \neg R_\psi(b)$ . Let  $M' = M \cup B$  (which is equipped with an  $L$ -structure by being a substructure of  $\bar{M}$ ). Expand  $M'$  to an  $L_P^+$ -structure, by defining  $R_\phi$  to hold of  $c \in M'$  if  $\phi(x, c)$  does not divide over  $B$  (in  $\bar{M}$ ). Let us call this expansion  $(M', P(M'))^+$ .

It follows immediately from the construction that

*Claim 1.*  $(M', P(M'))^+$  is an  $L_P^+$ -extension of  $(M, P(M))^+$ .

*Claim 2.*  $(M', P(M'))^+$  is a model of axioms (i) and (ii).

*Proof of claim 2.* Consider axiom (ii) and  $\phi_i(x_i, y_i)$  ( $i = 1, \dots, n$ ),  $\psi_j(t_j, z_j)$  ( $j =$

$1, \dots, m$ ) as there, and suppose that  $(M', P(M'))^+ \models \bigwedge_i R_{\phi_i}(c_i) \wedge \bigwedge_j \neg R_{\psi_j}(d_j)$ . So each  $\phi_i(x_i, c_i)$  does not divide over  $B$  and each  $\psi_j(t_j, d_j)$  divides over  $B$ . Let  $e_i \in \bar{M}$  realize  $\phi_i(x_i, c_i)$  such that  $e_i$  is  $L$ -independent from  $M'$  over  $B$ , and  $\{e_1, \dots, e_n\}$  is  $M'$ -independent. Then each  $\psi_j(t_j, d_j)$  divides over  $e$ . Claim 2 is proved.

By Claims 1, 2 and the assumption that  $(M, P(M))^+$  is an existentially closed model of axioms (i) and (ii), we see that if  $R_\phi(b)$  in  $(M, P(M))^+$  then  $\phi(x, b)$  is realized in  $P(M)$ . It easily follows that  $(M, P(M))^+$  is correct. The proof of (ii) $\rightarrow$ (iii) in 8.3, shows that any existentially universal model of axioms (i) and (ii) is a model of  $T_P^+$ . Thus any model of (i) and (ii) embeds in a model of  $T_P^+$ , completing the proof of the remark.

Let us conclude with some questions.  $T$  still denotes a complete simple first order theory with quantifier-elimination.

*Problem 1.* Suppose the simple theory  $T$  has finiteness and definability of the  $D_\phi$ -ranks. Is the same true for  $T_P$ ?

*Problem 2.* What can be said about  $T_P$  assuming just that that  $T_P^+$  has quantifier elimination? (For example, is it simple?)

*Problem 3.* Find a combinatorial equivalent to the finiteness and definability of all  $D_\phi$ -ranks.

*Problem 4.* Suppose  $\mathcal{C}_{T,P}$  is first order and  $T$  has elimination of hyperimaginaries. Does  $T_P$  have elimination of hyperimaginaries?

*Problem 5.* Prove that finiteness and definability of all  $D_\phi$ -ranks implies that  $T$  has elimination of hyperimaginaries.

*Problem 6.* Describe the imaginaries (up to to interdefinability) in  $T_P$  when  $T$  is stable without the fcp, in particular when  $T$  is theory of algebraically closed fields of a given characteristic.

## References

- [1] Itay Ben-Yaacov, Positive model theory and compact abstract theories, preprint 2002.

- [2] S. Buechler, Lascar strong types in some simple theories, *Journal of Symbolic Logic*, 64 (1999), 817-824.
- [3] S. Buechler, A. Pillay, and F. Wagner, Supersimple theories, *Journal of AMS*, 14 (2001), 109-124.
- [4] E. Casanovas, The number of types in simple theories, *Annals of Pure and Applied Logic*, 98 (1999), 69-86.
- [5] E. Casanovas and B. Kim, A supersimple nonlow theory, *Notre Dame Journal of Formal Logic*, 39 (1998), 507-518.
- [6] E. Casanovas and F. Wagner, Local supersimplicity and related concepts, preprint, 2001.
- [7] B. Hart, B. Kim and A. Pillay, Coordinatization and canonical bases, *Journal of Symbolic Logic*, 65 (2000), 293-309.
- [8] E. Hrushovski, Simplicity and the Lascar group, preprint, 1998.
- [9] B. Kim, Forking in simple unstable theories, *Journal of London Math. Society*, 57 (1998), 257-267.
- [10] B. Kim and A. Pillay, Simple theories, *Annals of Pure and Applied Logic*, 88 (1997), 149-164.
- [11] D. Lascar and A. Pillay, Fundamental order and forking in simple theories, *Journal of Symbolic Logic*, 64 (1999), 1155-1158.
- [12] D. Lascar and A. Pillay, Hyperimaginaries and automorphism groups, *Journal of Symbolic Logic*, 66 (2001), 127-143.
- [13] A. Pillay, Forking in the category of existentially closed structures, in *Connections between Model Theory and Algebraic and Analytic Geometry*, Quaderni di matematica, vol. 6, University of Naples, 2000.
- [14] B. Poizat, Paires de structures stables, *Journal of Symbolic Logic*, 48 (1983), 234-249.
- [15] Z. Shami, Definability in low simple theories, *Journal of Symbolic Logic*, 65 (2000), 1481-1490.

- [16] S. Shelah, *Classification theory*, North-Holland, 1990.
- [17] I. Tomasic and F. Wagner, Applications of the group configuration theorem in simple theories, preprint 2001.
- [18] E. Vassiliev, Generic pairs of  $SU$ -rank 1 structures, to appear.
- [19] F. Wagner, *Simple theories*, Kluwer, 1999.