ON SUPERSIMPLICITY AND LOVELY PAIRS OF CATS

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Abstract. We prove that the definition of supersimplicity in metric structures from [Ben05b] is equivalent to an *a priori* stronger variant. This stronger variant is then used to prove that if T is a supersimple Hausdorff cat then so is its theory of lovely pairs.

Introduction

A superstable first order theory is one which is stable in every large enough cardinality, or equivalently, one which is stable (in some cardinality), and in which the type of every finite tuple over arbitrary sets does not divide over a finite subset. In more modern terms we would say that a first order theory is superstable if and only if it is stable and supersimple.

Stability and simplicity were extended to various non-first-order settings by various people. Stability in the setting of large homogeneous structures goes back a long time (see [She75]), and some aspects of simplicity theory were also shown to hold in this setting in [BL03]. The setting of compact abstract theories, or cats, was introduced in [Ben03b] with the intention, among others, to provide a better non-first-order setting for the development of simplicity theory, which was done in [Ben03c], and under the additional assumption of thickness (with better results) in [Ben03d].

Hausdorff cats are ones whose type spaces are Hausdorff. Many classes of metric structures arising in analysis can be viewed as Hausdorff cats (e.g., the class of probability measure algebras [Ben], elementary classes of Banach space structures in the sense of Henson's logic, etc.) Conversely, a Hausdorff cat in a countable language admits a definable metric on its home sort which is unique up to uniform equivalence of metrics [Ben05b] (and even if the language is uncountable this result remains essentially true). Thus Hausdorff cats form a natural setting for the study of metric structures.

There is little doubt about the definitions of stability and simplicity in the case of (metric) Hausdorff cats: all the approaches mentioned above, and others, agree and give essentially the same theory as in first order logic. Many natural examples are indeed stable. Unfortunately, no metric structure can be superstable or supersimple according

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to the classical definition, unless it is essentially discrete: Indeed, in most cases that $b_n \to a$ in the metric we have $a \not\perp_{b_{<n}} b_{<\omega}$ for all n.

A very illustrative example is the following. Let T be a first order theory. For every $M \vDash T$, we can view M^ω as a metric structure, with

$$d(a_{<\omega}, b_{<\omega}) = \inf\{2^{-n} \colon a_{<n} = b_{<n}\}.$$

The class of metric structure $\{M^\omega \colon M \vDash T\}$ is the class of complete models (in the sense of [Ben05b]) of a compact abstract theory naturally called T^ω. If T is stable (simple), then so is T^ω, but T^ω is never supersimple: this can be seen using the argument in the preceding paragraph, or directly from the fact that finite tuples in the sense of T^ω are in fact infinite tuples in the sense of T (these two arguments eventually boil down to the same thing).

In an arbitrary metric cat define that $a^\varepsilon \perp_C B$ if there is a' such that $d(a, a') \leq \varepsilon$ and $a' \perp_C B$ (later on we will slightly modify this). Then for every simple first order theory T and the corresponding T^ω we have:

$$a_{<\omega}^{(2^{-n})} \underset{C}{\perp} B \text{ (in the sense of } T^\omega) \iff a_{<n} \underset{C}{\perp} B \text{ (in the sense of } T).$$

(Knowing $a_{<\omega} \in M^\omega$ up to distance 2^{-n} is the same as knowing $a_{<n}$.) It follows that T is supersimple if and only if, in T^ω, for every $\bar{a} \in M^\omega$, $\varepsilon > 0$, and set B, there is $B_0 \subseteq B$ finite such that $\bar{a}^\varepsilon \perp_{B_0} B$.

Generalising from this example, we suggested in [Ben05b] that:
– Finite tuples in metric structures behave in some sense like infinite tuples in classical first order structures, and the right way to extract a "truly finite" part of them is to consider them only up to some positive distance.
– As a consequence, the above characterisation of the supersimplicity of T by properties of T^ω should be taken as the *definition* of the supersimplicity of a metric theory (so T^ω would be supersimple if and only if T is).

One can now define that a Hausdorff cat is *superstable* if it is stable and supersimple. An alternative approach to superstability was suggested by Iovino in the case of Banach space structures through the re-definition of λ-stability in a manner that takes the metric into account [Iov99]. The two definitions agree: T is λ-stable for all big enough λ (by Iovino) if and only if it is stable and supersimple by the definition above. (This follows from the metric stability spectrum theorem [Ben05b, Theorem 4.13] as in the classical case.) We find this a fairly reassuring evidence that the definitions are indeed "correct".

The present paper attempts to address some questions these definitions raise:

First, the definition of supersimplicity above is somewhat disturbing, as it translates the two occurrences of "finite" in the original definition differently. We would prefer something of the form: "T is supersimple if for every a, $\varepsilon > 0$ and B, there is a distance $\delta > 0$ such that $a^\varepsilon \perp_{B^\delta} B$." Of course, in order to do that we would first have to give meaning to $a^\varepsilon \perp_{B^\delta} B$. This is addressed in Section 1.

A second issue arises from the theory T^P of beautiful (or lovely) pairs of models of a stable (or simple) theory T [Poi83, BPV03, Ben04]. It was shown (by Buechler [Bue91], later extended by Vassiliev [Vas02]) that such theories of pairs of models of T can be used as means for obtaining information on T itself. More precisely, for a rank one theory T, the rank of T^P yields information about the geometry of T. While metric structures can never have "rank one", it is natural to seek to compare the rank of T^P with that of T, when T is superstable or supersimple. In order for such a course of action to be feasible, one would first have to show that in that case T^P is supersimple as well.

In [BPV03] and in [Ben04] two distinct proofs are given to the effect that if T is a supersimple first order theory, or more generally, a "supersimple" cat in the wrong sense that does not take into account the metric, then so is the theory of its lovely pairs T^P. Due to the nature of independence in T^P, both proofs inevitably use the fact that T is supersimple at least twice. These proofs do not extend to the corrected definition of supersimplicity: without entering into details, on the first application of supersimplicity we see that $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} B$ for some finite $B_0 \subseteq B$, but then we cannot apply supersimplicity to the type of B_0 over something else. On the other hand, if we did have $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\mid}_{B^\delta} B$ for some $\delta > 0$, we could apply supersimplicity to types of B^δ over another set, and the proof may be salvaged. This is addressed in Section 2.

Thus the novelty of this paper is a notion of independence over virtual tuples, i.e., over objects of the form B^δ. This is a venture into difficult and unsound terrain (for example, the results of [Ben03a], while dealing with ultraimaginary elements rather than virtual ones, suggest that independence over objects which are not "at least" hyperimaginary should be approached with extreme caution and without too many hopes). While one can come up with many definitions for such a notion of independence it is not at all obvious to come up with one which satisfies the usual axioms, or even any "large" subset thereof. Our notion of independence is merely shown to satisfy some partial transitivity properties (Proposition 1.15), and at the same time to yield an equivalent characterisation of supersimplicity (Theorem 1.18). We content ourselves with such a modest achievement as it does suffice to close the gap in the proof that if T is supersimple then so is T^P (Theorem 2.4).

Other properties, such as symmetry, are lost (when considering independence over virtual tuples). For example, in a Hilbert space, if u and v are unit vectors and $\varepsilon \leq \sqrt{2}$, then one can show that:

$$u \mathop{\smile\!\!\!\!\!\!\mid}_{v^\varepsilon} v \iff \|u - v\| \geq \varepsilon$$

$$v \mathop{\smile\!\!\!\!\!\!\mid}_{v^\varepsilon} u \iff \|u - v\| = \sqrt{2} \; (\iff u \mathop{\smile\!\!\!\!\!\!\mid} v).$$

The question of finding a notion of independence which has more of the usual properties (e.g., symmetry, full transitivity, extension) without losing those we need for the results presented in this paper, remains an open (and difficult) one.

We assume basic familiarity with the setting of compact abstract theory and simplicity theory in this setting (see [Ben05a]).

We use $a, b, c, \dots$ to denote possibly infinite tuples of elements in the universal domain of the theory under consideration. When we want them to stand for a single element, we say so explicitly. Similarly, $x, y, z, \dots$ denote possibly infinite tuples of variables. Greek letters $\varepsilon, \delta, \dots$ denote values in the interval $[0, \infty]$, or possibly infinite tuples thereof.

1. Supersimplicity

Convention 1.1. We work in a Hausdorff cat T.

We recall from [Ben05b] that every sort admits a definable metric (i.e., a metric d such that for every $r \in \mathbb{R}^+$, the properties $d(x, y) \leq r$ and $d(x, y) \geq r$ are type-definable), or, if not, can be decomposed into uncountably many imaginary sorts each of which does admit such a metric. Therefore, at the price of possibly working with a multi sorted language, we may assume that all sorts admit a definable metric. For convenience we will proceed as if there is a single home sort, but the generalisation to many sorts should be obvious.

Let us fix, once and for all, a definable metric on the home sort. By [Ben05b] we know that any two such metrics are uniformly equivalent, so notions such as supersimplicity and superstability are not affected by our choice of metric. If the reader wishes nevertheless to avoid such an arbitrary choice, she or he may use the notion of abstract distances from [Ben05b] instead of real-valued distances whose interpretation depends on a metric function.

Distances on tuples will be viewed as tuples of distances of singletons:

Definition 1.2. Let I be a set of indices.

(i) If $\bar{a}$ and $\bar{b}$ are I-tuples then we consider $d(\bar{a}, \bar{b})$ to be the I-tuple $(d(a_i, b_i) \colon i \in I) \in [0, \infty]^I$ (In fact, a definable metric is necessarily bounded, so we can replace ∞ with some real number here, but keeping ∞ as a special distance is convenient).

(ii) If $\bar{\varepsilon}, \bar{\varepsilon}' \in [0, \infty]^I$, we say that $\bar{\varepsilon} \leq \bar{\varepsilon}'$ if $\varepsilon_i \leq \varepsilon'_i$ for all $i \in I$.

(iii) If $\bar{\varepsilon}, \bar{\varepsilon}' \in [0, \infty]^I$, we say that $\bar{\varepsilon} < \bar{\varepsilon}'$ if $\varepsilon_i < \varepsilon'_i$ for all $i \in I$, *and* $\varepsilon'_i = \infty$ for all but finitely many $i \in I$.

For the purpose of this definition we use the convention that $\infty < \infty$.

(iv) Given $\bar{\varepsilon} \in [0, \infty]^I$ and $\varepsilon' \in [0, \infty]$, we understand statements such as $\bar{\varepsilon} < \varepsilon'$, $\bar{\varepsilon} \leq \varepsilon'$, etc., by replacing ε' with the I-tuple all of whose coordinates are ε'.

From our convention that $\infty < \infty$ it follows that $\bar{\varepsilon} < \infty$ for all $\bar{\varepsilon} \in [0, \infty]^I$, and $\bar{\varepsilon} > 0 \Longrightarrow \bar{\varepsilon} > \frac{1}{2}\bar{\varepsilon}$ (where $\frac{1}{2}(\varepsilon_i)_{i \in I} = (\frac{\varepsilon_i}{2})_{i \in I}$, and $\frac{\infty}{2} = \infty < \infty$).

Superstability and supersimplicity in the first order context deal with properties of independence of finite tuples of elements. In the metric setting we replace "finite tuple" with a "virtually finite" one:

Definition 1.3. (i) A *virtual element* is formally a pair (a, ε), where a is a singleton and $\varepsilon \in [0, \infty]$. Usually a virtual element (a, ε) will be denoted by a^ε, and we think of this conceptually as "the element a up to distance ε".

(ii) A *virtual tuple* is a tuple of virtual elements, i.e., an object of the form $(a_i^{\varepsilon_i} : i \in I)$. This can also be denoted by $\bar{a}^{\bar{\varepsilon}}$, or simply a^ε, as single lowercase letters may denote arbitrary tuples.

(iii) As in Definition 1.2, if $\bar{a}$ is an I-tuple, and $\varepsilon \in [0, \infty]$ a single distance, we understand $\bar{a}^\varepsilon$ as $\bar{a}^{\bar{\varepsilon}}$, where $\bar{\varepsilon}$ is a tuple consisting of I repetitions of ε.

(iv) A *virtually finite tuple* is a virtual tuple $\bar{a}^{\bar{\varepsilon}}$ such that $\bar{\varepsilon} > 0$. We remind the reader that according to Definition 1.2, this means that $\varepsilon_i > 0$ for all i, and $\varepsilon_i = \infty$ for all but finitely many $i \in I$.

Notation 1.4. Unless explicitly said otherwise, a, b, etc., denote possibly infinite tuples of elements in a model. Similarly, ε, δ, etc., denote possibly infinite tuples in $[0, \infty]$. Thus a^ε denotes a virtual tuple, possibly infinite, with the implicit understanding that a and ε are of the same length.

If we wish to render explicit the fact that these are tuples we may use notation such as $\bar{a}^{\bar{\varepsilon}}$ etc.

We identify a tuple a with the virtual tuple a^0: knowing a up to distance 0 means knowing a precisely. More generally, if the relation $d(x, y) \leq \varepsilon$ is transitive (e.g., in the rare case where the metric is an ultrametric) then it is an equivalence relation, and we can identify the virtual tuple a^ε with the hyperimaginary $a/[d(x, y) \leq \varepsilon]$.

Similarly, we identify a virtual tuple $\bar{a}^{\bar{\varepsilon}}$ with any virtual tuple obtained by omitting or adding virtual elements of the form a_i^∞: knowing a_i up to distance ∞ means not knowing a_i at all. (The reader will see that these identifications are consistent with the way we use virtual tuples later on.)

Note that every virtually finite I-tuple $\bar{a}^{\bar{\varepsilon}}$ can be thus identified with the sub-tuple corresponding to $J = \{i \in I : \varepsilon_i < \infty\}$, which is finite as $\bar{\varepsilon} > 0$.

This identification allows a convenient re-definition of the notion of a sub-tuple:

Definition 1.5. A *virtual sub-tuple* of a virtual tuple a^ε is a virtual tuple (which can be identified with) $a^{\varepsilon'}$ for some $\varepsilon' \geq \varepsilon$.

Remark 1.6. The notion of a virtual sub-tuple extends the "ordinary" notion of sub-tuple. Indeed, let $\bar{a}^{\bar{\varepsilon}}$ be a virtual I-tuple, and $\bar{b}^{\bar{\delta}}$ a sub-tuple in the ordinary sense, i.e., given by restricting so a subset of indices $J \subseteq I$. For $i \in I$ define $\varepsilon'_i = \varepsilon_i$ if $i \in J$, and $\varepsilon'_i = \infty$ otherwise. Then $\bar{\varepsilon}' \geq \bar{\varepsilon}$, and $\bar{b}^{\bar{\delta}}$ can be identified with the virtual sub-tuple $\bar{a}^{\bar{\varepsilon}'}$.

We define types of virtual tuples:

Definition 1.7. As we defined in [Ben05b], if $p(x)$ is a partial type in a tuple of variables x, and ε is a tuple of distances of the same length, then $p(x^\varepsilon)$ is defined as the partial type $\exists y \, (p(y) \wedge d(x, y) \leq \varepsilon)$. Since $d(x, y) \leq \varepsilon$ is a type-definable property, this is indeed expressible by a partial type.

We define $\mathrm{tp}(a^\varepsilon)$ as $p(x^\varepsilon)$ where $p = \mathrm{tp}(a)$. Similarly, if $p(x, y) = \mathrm{tp}(a, b)$ then $\mathrm{tp}(a^\varepsilon/b^\delta)$ is $p(x^\varepsilon, b^\delta)$.

Remark 1.8. If $a' \vDash \mathrm{tp}(a^\varepsilon/b^\delta)$ then we say that a^ε and a'^ε have the same type over b^δ, in symbols $a^\varepsilon \equiv_{b^\delta} a'^\varepsilon$. This is a symmetric relation.

Proof. Assume that $a' \vDash \mathrm{tp}(a^\varepsilon/b^\delta)$. Then there are $a''b' \equiv ab$ such that $d(a'b, a''b') \leq \varepsilon\delta$. Let a''', b'' be such that $a''b'a'b \equiv aba'''b''$. Then $d(ab, a'''b'') \leq \varepsilon\delta$ and $a'''b'' \equiv a'b$, whereby $a \vDash \mathrm{tp}(a'^\varepsilon/b^\delta)$. $\mathrm{QED}_{1.8}$

We recall from [Ben05b]:

Definition 1.9. *T is supersimple if for every virtually finite tuple a^ε and set A, there is a finite subset $A_0 \subseteq A$ such that $\mathrm{tp}(a^\varepsilon/A)$ does not divide over A_0.*

If we replace "virtually finite tuple" with "virtually finite singleton" (i.e., a is a singleton) we obtain an equivalent definition.

We now turn to the principal new definition in this paper, of independence *over* virtual tuples.

Definition 1.10. We say that an indiscernible sequence $(b_i : i < \omega)$ *could be in* $\mathrm{tp}(b/c^\rho)$ if there are (c_i) such that:

(i) $(b_ic_i : i < \omega)$ is an indiscernible sequence in $\mathrm{tp}(bc)$.
(ii) For all $i, j \leq \omega$: $d(c_i, c_j) \leq \rho$.

Remark 1.11. Let $E_\rho(x, y)$ be the relation $d(x, y) \leq \rho$. Assume that E_ρ happens to be transitive, and therefore an equivalence relation (this would happen in the rare case that the metric is an ultrametric, and also if $\rho = 0$). Then a sequence $(b_i : i < \omega)$ could be in $\mathrm{tp}(b/c^\rho)$ if and only if it has an automorphic image in $\mathrm{tp}(b/(c/E_\rho))$.

In particular, if $\rho = 0$ then E is equality, and (b_i) could be in $\mathrm{tp}(b/c^0)$ if and only if it has an automorphic image in $\mathrm{tp}(b/c)$. This justifies the terminology, as well as the identification between c^0 and c.

Definition 1.12. We say that $a^\varepsilon \mathop{\smile\!\!\!\!\!\vert}_{c^\rho} b$ if every indiscernible sequence that could be in $\mathrm{tp}(b/c^\rho)$ could also be in $\mathrm{tp}(b/a^\varepsilon c^\rho)$.

As explained in the introduction, this notion of independence has very few "nice" properties, although these suffice for the application we seek. We will not dare to extend it to, say, independence of the form $a^\varepsilon \mathop{\smile\!\!\!\!\!\vert}_{c^\rho} b^\delta$ without being able to show that such extension has useful properties.

When restricting to independence over non-virtual (real or even hyperimaginary) tuples, it is not true that $a^\varepsilon \mathop{\smile\!\!\!\!\!\vert}_{b} c$ if and only of $\mathrm{tp}(a^\varepsilon/bc)$ does not divide over c. These notions are close enough to being equivalent, though:

Lemma 1.13. *Assume that T is simple. For a^ε, b and c, the following conditions imply one another from top to bottom:*

(i) $a^\varepsilon \downarrow_c b$ *(remember to identify* c *with* c^0*).*
(ii) *There is a Morley sequence for* b *over* c *which could be in* $\mathrm{tp}(b/a^\varepsilon c)$*.*
(iii) $\mathrm{tp}(a^\varepsilon/bc)$ *does not divide over* c*.*
(iv) $a^{2\varepsilon} \downarrow_c b$*.*

Proof. (i) $\Longrightarrow$ (ii). By definition.

(ii) $\Longrightarrow$ (iii). We recall the $D(-,\Xi)$ ranks from [Ben03d]: Fixing the tuple x, we define $\Xi = \Xi(x)$ as the set of all pairs $(\varphi(x,y), \psi(y_{<k}))$ (y and k may vary) such that φ and ψ are positive formulae and ψ is a k-inconsistency witness for φ, i.e.,

$$T \vdash \neg\exists x y_{<k} \left(\psi(y_{<k}) \wedge \bigwedge_{i<k} \varphi(x, y_i) \right).$$

If $p(x)$ is a partial type with parameters in A, $D(p,\Xi)$ is a subset of $\Xi^{<|T|^+}$ such that for $\xi_{<\alpha} \in \Xi^\alpha$:

- $\alpha = 0$: $\xi_{<\alpha} \in D(p,\Xi)$ if and only if p is consistent.
- α limit: $\xi_{<\alpha} \in D(p,\Xi)$ if and only if $\xi_{<\beta} \in D(p,\Xi)$ for all $\beta < \alpha$.
- $\alpha = \beta + 1$, $\xi_\beta = (\varphi(x,y), \psi(y_{<k}))$: $\xi_{<\alpha} \in D(p,\Xi)$ if and only if there exists an A-indiscernible sequence $(b_i : i < \omega)$ in the sort of y such that $\vDash \psi(b_{<k})$ and $\xi_{<\beta} \in D(p \cup \{\varphi(x,b_0)\}, \Xi)$.

We recall that this rank characterises independence (for T simple and thick, and thus in particular simple and Hausdorff): If $A \subseteq B$ then $p \in \mathrm{S}(B)$ does not divide over A if and only if $D(p,\Xi) = D(p{\restriction}_A, \Xi)$.

So let $(b_i : i < \omega)$ be a Morley sequence for b over c which could be in $\mathrm{tp}(b/a^\varepsilon c)$. Then there are $(a_i : i < \omega)$ and c' such that $(a_i b_i c' : i < \omega)$ is an indiscernible sequence in $\mathrm{tp}(abc)$ and $d(a_i, a_j) \leq \varepsilon$ for all $i, j < \omega$. Extend the sequence to length $\omega + 1$. Then by standard arguments we have $b_\omega \downarrow_{b_{<\omega}} a_{<\omega} c'$, so:

$$D(b_\omega/b_{<\omega}, \Xi) = D(b_\omega/b_{<\omega} a_{<\omega} c', \Xi) \subseteq D(b_\omega/a_0 c', \Xi) \subseteq D(b_\omega/c', \Xi).$$

On the other hand, since (b_i) is a Morley sequence over c:

$$D(b_\omega/b_{<\omega}, \Xi) = D(b/c, \Xi) = D(b_\omega/c', \Xi)$$

Therefore equality holds all the way and we have $b_\omega \downarrow_{c'} a_0$. Since $d(a_0, a_\omega) \leq \varepsilon$, it follows that $\mathrm{tp}(a_\omega^\varepsilon/b_\omega c')$ does not divide over c', and by invariance $\mathrm{tp}(a^\varepsilon/bc)$ does not divide over c.

(iii) $\Longrightarrow$ (iv). We assume that $\mathrm{tp}(a^\varepsilon/bc)$ does not divide over c. Then there exists a' such that $a' \downarrow_c b$ and $d(a, a') \leq \varepsilon$. Let (b_i) be any indiscernible sequence that could be in $\mathrm{tp}(b/c)$. Then we might as well assume that it is in $\mathrm{tp}(b/c)$ and since $a' \downarrow_c b$ it can even be in $\mathrm{tp}(b/a'c)$. Find now (a_i) such that $a_i b_i \equiv_{a'c} ab$. Then we may always choose them such that $(a_i b_i)$ is c-indiscernible, and $d(a_i, a') \leq \varepsilon$ for all i yields $d(a_i, a_j) \leq 2\varepsilon$ for all i, j, as required. $\mathrm{QED}_{1.13}$

This means that $a \mathop{\smile\!\!\!\!\!\!\!\;\mid}_c b$ if and only if $\mathrm{tp}(a/bc)$ does not divide over c (since $2 \cdot 0 = 0$), so this definition agrees with the usual definition of independence of ordinary (i.e., non-virtual) elements.

We can continue Remark 1.11 to show that if the tuple of distances ρ defines an equivalence relation E_ρ then $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$ if and only if $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c/E_\rho} b$ (here c/E_ρ is viewed as hyperimaginary, rather than virtual). If ε also defines an equivalence relation E_ε, then $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$ if and only if $a/E_\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c/E_\rho} b$.

Also, Lemma 1.13 and the fact that $\varepsilon > 0 \Longrightarrow \frac{1}{2}\varepsilon > 0$ give:

Proposition 1.14. *T is supersimple if and only if for every virtually finite tuple (singleton) a^ε and set A there is a finite subset $A_0 \subseteq A$ such that $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{A_0} A$.*

Proposition 1.15. *Independence satisfies right downward transitivity, left upward transitivity, and two-sided monotonicity:*

(i) *If $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$ and δ is any tuple of distances of the length of b, then $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{b^\delta c^\rho} b$.*
(ii) *If $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$ and $d^\upsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{a^\varepsilon c^\rho} b$ then $a^\varepsilon d^\upsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$.*
(iii) *If $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} bd$ and $\varepsilon' \geq \varepsilon$ (i.e., if $a^{\varepsilon'}$ is a virtual sub-tuple of a^ε) then $a^{\varepsilon'} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$.*

Proof. (i) Let (b_i) be an indiscernible sequence that could be in $\mathrm{tp}(b/b^\delta c^\rho)$. This is the same as saying that (b_i) could be in $\mathrm{tp}(b/c^\rho)$ and $d(b_i, b_j) \leq \delta$ for all $i, j < \omega$. As we assume that $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$, the sequence (b_i) could be in $\mathrm{tp}(b/a^\varepsilon c^\rho)$; since $d(b_i, b_j) \leq \delta$ it could also be in $\mathrm{tp}(b/a^\varepsilon b^\delta c^\rho)$, as required.

(ii) If (b_i) is indiscernible and could be in $\mathrm{tp}(b/c^\rho)$ then it could also be in $\mathrm{tp}(b/a^\varepsilon c^\rho)$ and therefore in $\mathrm{tp}(b/a^\varepsilon c^\rho d^\upsilon)$.

(iii) Let (b_i) be an indiscernible sequence that could be in $\mathrm{tp}(b/c^\rho)$. By standard arguments we can find (d_i) such that $(b_i d_i)$ is indiscernible and could be in $\mathrm{tp}(bd/c^\rho)$. As we assume that $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} bd$, it could also be in $\mathrm{tp}(bd/a^\varepsilon c^\rho)$. Therefore (b_i) could be in $\mathrm{tp}(b/a^\varepsilon c^\rho)$ and *a fortiori* in $\mathrm{tp}(b/a^{\varepsilon'} c^\rho)$.

$\mathrm{QED}_{1.15}$

We obtain a more general form, in this context, of the finite character of independence:

Proposition 1.16. *For all a^ε, b and c^ρ: $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$ if and only if $a^{\varepsilon'} \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b'$ for all $\varepsilon' > \varepsilon$ and finite $b' \subseteq b$. (By our approach, $a^{\varepsilon'}$ should be viewed as a finite sub-tuple of a^ε, since $\varepsilon' > \varepsilon$.)*

In particular, $a \mathop{\smile\!\!\!\!\!\!\!\;\mid}_c b$ if and only if $a^\varepsilon \mathop{\smile\!\!\!\!\!\!\!\;\mid}_c b$ for all $\varepsilon > 0$.

Proof. Left to right is by monotonicity. For right to left, assume that $a^\varepsilon \not\mathop{\smile\!\!\!\!\!\!\!\;\mid}_{c^\rho} b$. Then there is an indiscernible sequence (b_i) that could be in $\mathrm{tp}(b/c^\rho)$ but not in $\mathrm{tp}(b/a^\varepsilon c^\rho)$. Letting $p(x, y, z) = \mathrm{tp}(a, b, c)$, the latter means that the following is inconsistent:

$$\bigwedge_{i<\omega} p(x_i, b_i, z_i) \wedge \bigwedge_{i,j<\omega} [d(x_i, x_j) \leq \varepsilon \wedge d(z_i, z_j) \leq \rho].$$

Since $d(x_i, x_j) \leq \varepsilon$ is logically equivalent to $\bigwedge_{\varepsilon' > \varepsilon} d(x_i, x_j) \leq \varepsilon'$, and the family of all $\varepsilon' > \varepsilon$ is closed for finite infima, we obtain by compactness some $\varepsilon' > \varepsilon$ such that the above is still inconsistent with ε' instead of ε. Therefore $a^{\varepsilon'} \not\mathop{\smile\!\!\!\!\!\mid}_{c^\rho} b$. Replacing b with a finite sub-tuple is similar (and fairly standard). $\mathrm{QED}_{1.16}$

Let us recall from [Ben03c, Lemma 1.2] the following useful fact about "extraction" of indiscernible sequence from long sequences:

Fact 1.17. *Let A be a set of parameters, and $\lambda \geq \beth_{|\mathrm{S}_\kappa(A)|^+}$. Then for any sequence $(a_i : i < \lambda)$ of κ-tuples there is an A-indiscernible sequence $(b_i : i < \omega)$ such that for all $n < \omega$ there are $i_0 < \ldots < i_{n-1} < \lambda$ for which* $\mathrm{tp}(b_0 \ldots b_{n-1}/A) = \mathrm{tp}(a_{i_0} \ldots a_{i_{n-1}}/A)$.

Theorem 1.18. *T is supersimple if and only if for every virtually finite tuple (singleton) a^ε, and any tuple b, there is a virtually finite sub-tuple b^δ of b (i.e., there exists $\delta > 0$ of the appropriate length) such that $a^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_{b^\delta} b$.*

Proof. In order to prove right to left, it would suffice to show that for every virtually finite singleton a^ε and set B there is $B_0 \subseteq B$ finite such that $a^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_{B_0} B$.

Let $b = \bar{b}$ be an I-tuple enumerating B. Then by assumption there is $\delta = \bar{\delta} > 0$ such that $a^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_{b^\delta} b$. Let $B_0 = \{b_i \colon \delta_i < \infty\}$. Then B_0 is finite, and by right downward transitivity: $a^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_{B_0} B$.

We now prove left to right: aiming for a contradiction, we assume that T is supersimple, and yet there is no δ as in the statement. Let $p(x, y) = \mathrm{tp}(a, b)$, $q(y) = \mathrm{tp}(b)$. We will construct by induction a sequence of tuples $(b_n \colon n < \omega)$ in q, and a sequence of tuples of distances $(\delta_n \colon n < \omega)$. These will satisfy, among other things, that $\delta_n \geq 2\delta_{n+1} > 0$ and $d(b_n, b_{n+1}) \leq \delta_n$.

For convenience, let $\delta_{-1} = \infty$.

At the nth step, assume we already have $b_{<n}$ satisfying q and $\delta_{n-1} > 0$. By assumption $a^\varepsilon \not\mathop{\smile\!\!\!\!\!\mid}_{b^{\delta_{n-1}}} b$. Therefore there is an indiscernible sequence $(b_n^i : i < \omega)$ such that $d(b_n^i, b_n^j) \leq \delta_{n-1}$ for all $i, j < \omega$ and yet the following is inconsistent:

$$(*_n) \qquad \bigwedge_{i<\omega} p(x^i, b_n^i) \wedge \bigwedge_{i,j<\omega} d(x^i, x^j) \leq \varepsilon$$

By a compactness argument as in the proof of Proposition 1.16, there exists $\delta_n > 0$ such that the following weakening of $(*_n)$ is still inconsistent:

$$(**_n) \qquad \bigwedge_{i<\omega} p(x^i, {b_n^i}^{2\delta_n}) \wedge \bigwedge_{i,j<\omega} d(x^i, x^j) \leq \varepsilon$$

We may always assume that $\delta_n \leq \frac{1}{2}\delta_{n-1}$.

If $n = 0$, the sequence $(b_n^i \colon i < \omega)$ is $b_{<n}$-indiscernible, and we skip the following paragraph. If $n > 0$, note that all that matters for $(**_n)$ is the type of the sequence $(b_n^i \colon i < \omega)$: we may therefore replace it with another sequence which has the same type, such that in addition $(b_n^i \colon i < \omega)$ is $b_{<n}$-indiscernible and satisfy $d(b_n^i, b_{n-1}) \leq \delta_{n-1}$.

In order to see this, extend this sequence to arbitrary length $\lambda+1$ $(b_n^i : i \leq \lambda)$. Since $b_n^\lambda \equiv b \equiv b_{n-1}$, we may assume (up to replacing $(b_n^i : i \leq \lambda)$ with an automorphic image) that $b_n^\lambda = b_{n-1}$. Applying Fact 1.17 to the sequence $(b_n^i : i < \lambda)$ over $b_{<n}$, we can find a sequence $(c_n^i : i < \omega)$ which is $b_{<n}$-indiscernible, and such that for all $m < \omega$ there are $i_0 < \cdots < i_{m-1} < \lambda$ such that

$$c_n^0, \ldots, c_n^{m-1} \equiv_{b_{<n}} b_n^{i_0}, \ldots, b_n^{i_{m-1}}.$$

Therefore $(c_n^i : i < \omega) \equiv (b_n^i : i < \omega)$, and $d(c_n^i, b_{n-1}) \leq \delta_{n-1}$, so the sequence $(c_n^i : i < \omega)$ has the required properties.

Let $b_n = b_n^0$, so in particular $d(b_n, b_{n-1}) \leq \delta_{n-1}$, and continue the construction.

Once the construction is done, let $\delta_\omega = \inf \delta_n$. If b is an I-tuple then so are $\delta_n = \delta_{n,\in I}$ and $\delta_\omega = \delta_{\omega,\in I}$. Since $\delta_n \geq 2\delta_{n+1}$ for all n, we must have $\delta_{\omega,i} \in \{0, \infty\}$ for all $i \in I$. But if $i \in I$ is such that $\delta_{\omega,i} = \infty$, then $\delta_{n,i} = \infty$ for all n, which means that the ith coordinate of b and the b_n played absolutely no role throughout the construction, and may be entirely dropped. Therefore, replacing b, b_n, δ_n, etc., with sub-tuples we may assume that $\delta_\omega = 0$.

The fact that $\delta_n \geq 2\delta_{n+1}$ implies that for every $n \leq m$: $d(b_n, b_m) \leq 2\delta_n$. Thus the partial type $q(y) \wedge \bigwedge_n d(b_n, y) \leq 2\delta_n$ is consistent, and has a realisation b_ω. Since $\inf \delta_n = 0$, b_ω is the unique realisation of this type, so $b_\omega \in \operatorname{dcl}(b_{<\omega})$ (we say that b_ω is the limit of the Cauchy sequence $(b_n : n < \omega)$). Since $b_\omega \equiv b$, there is a_ω such that $\models p(a_\omega, b_\omega)$. By supersimplicity there is n such that $a_\omega^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_{b_{<n}} b_{<\omega}$, whereby $a_\omega^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_{b_{<n}} b_\omega$.

Let us go back to our $b_{<n}$-indiscernible sequence $(b_n^i : i < \omega)$, and we recall that $b_n = b_n^0$. Find $(b_\omega^i : i < \omega)$ such that $b_\omega^0 = b_\omega$ and $(b_n^i b_\omega^i : i < \omega)$ is $b_{<n}$-indiscernible. Since $a_\omega^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_{b_{<n}} b_\omega$, there are $(a_\omega^i : i < \omega)$ realising

$$\bigwedge_{i<\omega} p(a_\omega^i, b_\omega^i) \wedge \bigwedge_{i,j<\omega} d(a_\omega^i, a_\omega^j) \leq \varepsilon$$

But $d(b_n, b_\omega) \leq 2\delta_n \Longrightarrow d(b_n^i, b_\omega^i) \leq 2\delta_n$. This shows that:

$$\bigwedge_{i<\omega} p(a_\omega^i, {b_n^i}^{2\delta_n}) \wedge \bigwedge_{i,j<\omega} d(a_\omega^i, a_\omega^j) \leq \varepsilon,$$

so $(**_n)$ was consistent after all. $\text{QED}_{1.18}$

Having given meaning to $a^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_c b$, it is natural to define the corresponding SU-ranks:

Definition 1.19. $\mathrm{SU}(a^\varepsilon/b)$ is the minimal rank taking ordinal values or ∞ satisfying:

> $\mathrm{SU}(a^\varepsilon/b) \geq \alpha+1$ if and only if there is c such that $a^\varepsilon \not\mathop{\smile\!\!\!\!\!\mid}_b c$ and $\mathrm{SU}(a^\varepsilon/bc) \geq \alpha$.

Proposition 1.20. (i) *T is supersimple if and only if $\mathrm{SU}(a^\varepsilon/b) < \infty$ for all b and virtually finite a^ε.*

(ii) *Assuming that T is supersimple, $a \mathop{\smile\!\!\!\!\!\mid}_b c \Longleftrightarrow \mathrm{SU}(a^\varepsilon/bc) = \mathrm{SU}(a^\varepsilon/b)$ for all $\varepsilon > 0$.*

Proof. (i) Standard argument, using Proposition 1.14.

(ii) If $\mathrm{SU}(a^\varepsilon/bc) = \mathrm{SU}(a^\varepsilon/b)$ for all $\varepsilon > 0$, then $a^\varepsilon \mathop{\smile\!\!\!\!\!\mid}_b c$ for all $\varepsilon > 0$, whereby $a \mathop{\smile\!\!\!\!\!\mid}_b c$ by the finite character (Proposition 1.16).

Conversely, assume that $a \mathop{\smile\!\!\!\!\!\mid}_b c$. Clearly, $\mathrm{SU}(a^\varepsilon/b) \geq \mathrm{SU}(a^\varepsilon/bc)$. We prove by induction on α that $\mathrm{SU}(a^\varepsilon/b) \geq \alpha \Longrightarrow \mathrm{SU}(a^\varepsilon/bc) \geq \alpha$. For $\alpha = 0$ and limit this is clear, so we need to prove for $\alpha = \beta + 1$.

Since $\mathrm{SU}(a^\varepsilon/b) \geq \beta + 1$, there is d such that $\mathrm{SU}(a^\varepsilon/bd) \geq \beta$ and $a^\varepsilon \not\mathop{\smile\!\!\!\!\!\mid}_b d$. We may assume that $d \mathop{\smile\!\!\!\!\!\mid}_{ab} c$. We assumed that $a \mathop{\smile\!\!\!\!\!\mid}_b c$, whereby $ad \mathop{\smile\!\!\!\!\!\mid}_b c$ and $a \mathop{\smile\!\!\!\!\!\mid}_{bd} c$. Therefore, by the induction hypothesis, $\mathrm{SU}(a^\varepsilon/bcd) \geq \beta$. On the other hand, $a^\varepsilon \not\mathop{\smile\!\!\!\!\!\mid}_{bc} d$: otherwise we'd get $a^\varepsilon c \mathop{\smile\!\!\!\!\!\mid}_b d$ by Proposition 1.15, contradicting prior assumptions. This shows that $\mathrm{SU}(a^\varepsilon/bc) \geq \beta + 1 = \alpha$.

$\mathrm{QED}_{1.20}$

Question 1.21. It is fairly easy to prove that for all virtually finite a^ε and b^δ, and all tuples c:

$$\mathrm{SU}(a^\varepsilon/bc) + \mathrm{SU}(b^\delta/c) \leq \mathrm{SU}(a^\varepsilon b^\delta/c).$$

Note, however, that we use $\mathrm{SU}(a^\varepsilon/bc)$ rather than $\mathrm{SU}(a^\varepsilon/b^\delta c)$, to which we haven't given a meaning. This is a serious problem, since the converse inequality may easily be false (for example, if $a = b$ and $\delta = \infty$.):

$$\mathrm{SU}(a^\varepsilon b^\delta/c) \not\leq \mathrm{SU}(a^\varepsilon/bc) \oplus \mathrm{SU}(b^\delta/c).$$

Is there a way to give meaning to $\mathrm{SU}(a^\varepsilon/b^\delta c)$ such that the standard Lascar inequalities (or reasonable variants thereof) hold?

2. Lovely pairs

We assume familiarity at least with the basics of lovely pairs as exposed in [BPV03], where for every simple first order theory T we constructed its theory of lovely pairs T^P, and proved that if T has the weak non-finite-cover-property then T^P is simple and independence in T^P was characterised. In [Ben04] we generalised the latter result to the case where T is any thick simple cat. Namely, for each such T we constructed a cat $T^{\mathfrak{P}}$ whose $|T|^+$-saturated models are precisely the lovely pairs of models of T, and proved it is simple with the same characterisation of independence. If T is a first order theory then $T^{\mathfrak{P}}$ is first order if and only if T has the weak non-finite-cover-property, in which case $T^{\mathfrak{P}}$ coincides with T^P.

Convention 2.1. If (M, P) is a lovely pair of models of T and $a \in M$ then $\mathrm{tp}(a)$ denotes the type of a in M (in the sense of T) while $\mathrm{tp}^{\mathfrak{P}}(a)$ denotes its type in (M, P) (i.e., in the sense of $T^{\mathfrak{P}}$).

We are going to use a few results from [Ben04] which do not appear explicitly in [BPV03].

If (M, P) is a lovely pair and $a \in M$, then $\mathrm{tp}^{\mathfrak{P}}(a)$ determines the set of all possible types of Morley sequences (both in the sense of T) for a over $P(M)$. Conversely, any of these types determines $\mathrm{tp}^{\mathfrak{P}}(a)$. Also, the property "the sequence $(a_i : i < \omega)$ has the type of a Morley sequence for a over P" is definable by a partial type in $x_{<\omega}$, which is denoted by $\mathrm{mcl}(a)$ (the *Morley class* of a).

Finally, let $a, b, c \in M$, and $(a_i b_i c_i : i < \omega) \vDash \mathrm{mcl}(abc)$ be a sequence in some model of T. Then a is independent from b over c in the sense of $T^{\mathfrak{P}}$, in symbols $a \mathop{\smile\!\!\!\!\!|}^{\mathfrak{P}}_c b$, if and only if $a_{<\omega} \mathop{\smile\!\!\!\!\!|}_{c_{<\omega}} b_{<\omega}$ (here in the sense of T).

Convention 2.2. T is a simple Hausdorff cat, and in particular thick. Therefore $T^{\mathfrak{P}}$ exists and the properties mentioned above hold.

Since T is Hausdorff, so is $T^{\mathfrak{P}}$. Also, as T is a reduct of $T^{\mathfrak{P}}$, any definable metric we might have fixed for T is also a definable metric in the sense of $T^{\mathfrak{P}}$. (Since $T^{\mathfrak{P}}$ is richer, there may be new definable metrics: however, as all definable metrics are uniformly equivalent, this makes no difference.)

Lemma 2.3. *Let a, b and c be tuples in a lovely pair (M, P), and let $(a_i b_i c_i : i < \omega) \vDash \mathrm{mcl}(abc)$ in a universal domain for T. Let a^ε be a virtual sub-tuple of a. Then $\mathrm{tp}^{\mathfrak{P}}(a^\varepsilon/bc)$ divides over c (in the sense of $T^{\mathfrak{P}}$) if and only if $\mathrm{tp}(a^\varepsilon_{<\omega}/b_{<\omega}c_{<\omega})$ divides over $c_{<\omega}$ (in the sense of T).*

As we said earlier, the notation $a^\varepsilon_{<\omega}$ here means the virtual tuple $(a^\varepsilon_i : i < \omega)$, and similarly for tuples of other lengths. As ε is a tuple of distances of the length of a, there should be no ambiguity about this.

Proof. Assume that $\mathrm{tp}^{\mathfrak{P}}(a^\varepsilon/bc)$ does not divide over c. Then there is $a' \in M$ such that $d(a, a') \leq \varepsilon$ and $a' \mathop{\smile\!\!\!\!\!|}^{\mathfrak{P}}_c b$. There exist $(a'_i : i < \omega)$ such that $(a'_i a_i b_i c_i : i < \omega) \vDash \mathrm{mcl}(a'abc)$. Then $a' \mathop{\smile\!\!\!\!\!|}^{\mathfrak{P}}_c b \Longrightarrow a'_{<\omega} \mathop{\smile\!\!\!\!\!|}_{c_{<\omega}} b_{<\omega}$, and $d(a, a') \leq \varepsilon \Longrightarrow d(a_i, a'_i) \leq \varepsilon$ for all $i < \omega$, whereby $\mathrm{tp}(a^\varepsilon_{<\omega}/b_{<\omega}c_{<\omega})$ does not divide over $c_{<\omega}$.

Conversely, assume that $\mathrm{tp}(a^\varepsilon_{<\omega}/b_{<\omega}c_{<\omega})$ does not divide over $c_{<\omega}$. Then there exist $(a'_i : i < \omega)$ such that $d(a_i, a'_i) < \varepsilon$ for all $i < \omega$ and $a'_{<\omega} \mathop{\smile\!\!\!\!\!|}_{c_{<\omega}} b_{<\omega}$, but the sequence $(a'_i a_i b_i c_i : i < \omega)$ needs not be indiscernible.

Continue the sequence $(a_i b_i c_i)$ to an indiscernible sequence of length $\lambda > \omega$, big enough to allow us to apply Fact 1.17 later on. Let $e = \mathrm{Cb}(b_\omega c_\omega / b_{<\omega} c_{<\omega})$ and $f = \mathrm{Cb}(c_\omega / c_{<\omega})$ (in the sense of T), so $(b_i c_i : i < \lambda)$ is a Morley sequence over e, and $(c_i : i < \lambda)$ is a Morley sequence over f, and indiscernible over ef, whereby $c_{<\lambda} \mathop{\smile\!\!\!\!\!|}_f e$. It follows that for all $w \subseteq \omega$:

$$b_{\in w} c_{\in w} \mathop{\smile\!\!\!\!\!|}_{e} c_{\in \omega \smallsetminus w} \Longrightarrow b_{\in w} \mathop{\smile\!\!\!\!\!|}_{c_{\in w} e} c_{<\omega} \Longrightarrow b_{\in w} e \mathop{\smile\!\!\!\!\!|}_{c_{\in w} f} c_{<\omega}$$

Whereby:

$$b_{<\omega} \mathop{\smile\!\!\!\!\!|}_{c_{<\omega}} a'_{<\omega} \Longrightarrow b_{\in w} \mathop{\smile\!\!\!\!\!|}_{c_{<\omega}} a'_{<\omega} \Longrightarrow b_{\in w} \mathop{\smile\!\!\!\!\!|}_{f c_{\in w}} a'_{\in w}.$$

For any finite $w \subseteq \lambda$, $\mathrm{tp}(b_{\in w} c_{\in w} f)$ depends solely on $|w|$ (where the tuples $b_{\in w}$ and $c_{\in w}$ are enumerated according to the ordering induced on w from λ). Thus by the definability of independence for known complete types, for every $n < \omega$ and tuple of variables x there is a partial type $\rho_{n,x}(x, y_{<n}, z_{<n}, f)$ such that for every $w \in [\lambda]^n$ and every g in the sort of x:

$$b_{\in w} \mathop{\smile\!\!\!\!\!\!|}_{c_{\in w} f} g \iff \vDash \rho_{n,x}(g, b_{\in w}, c_{\in w}, f).$$

Putting these two facts together and applying compactness we can find a sequence $(a''_i : i < \lambda)$ such that $d(a_i, a''_i) \le \varepsilon$ for all $i < \lambda$, and $b_{\in w} \mathop{\smile\!\!\!\!\!\!|}_{c_{\in w} f} a''_{\in w}$ for every finite $w \subseteq \lambda$.

As we could have chosen λ arbitrarily big, by standard extraction arguments (i.e., Fact 1.17) there exists a sequence $(\tilde{a}_i : i < \omega)$ such that $(\tilde{a}_i a_i b_i c_i : i < \omega)$ is f-indiscernible, and in addition $d(a_i, \tilde{a}_i) \le \varepsilon$ for all $i < \omega$, and $b_{\in w} \mathop{\smile\!\!\!\!\!\!|}_{c_{\in w} f} \tilde{a}_{\in w}$ for all finite $w \subseteq \omega$. As every formula in $\mathrm{tp}(\tilde{a}_{<\omega}/b_{<\omega} c_{<\omega} f)$ only involves finitely many variables and parameters, it does not divide over $c_{\in w} f$ for some finite $w \subseteq \omega$, and *a fortiori* over $c_{<\omega} f$. Therefore $\tilde{a}_{<\omega} \mathop{\smile\!\!\!\!\!\!|}_{c_{<\omega} f} b_{<\omega}$. As $f \in \mathrm{dcl}(c_{<\omega})$, we conclude that $\tilde{a}_{<\omega} \mathop{\smile\!\!\!\!\!\!|}_{c_{<\omega}} b_{<\omega}$.

As the sequence $(\tilde{a}_i a_i b_i c_i : i < \omega)$ is indiscernible, we may extend it to length $\omega + 1$. Then one can find a lovely pair (M, P) such that $\tilde{a}_{\le\omega} a_{\le\omega} b_{\le\omega} c_{\le\omega} \in M$, $\tilde{a}_{<\omega} a_{<\omega} b_{<\omega} c_{<\omega} \in P$, and $\tilde{a}_\omega a_\omega b_\omega c_\omega \mathop{\smile\!\!\!\!\!\!|}_{\tilde{a}_{<\omega} a_{<\omega} b_{<\omega} c_{<\omega}} P$. It follows that $(\tilde{a}_i a_i b_i c_i : i < \omega) \vDash \mathrm{mcl}^{(M,P)}(\tilde{a}_\omega a_\omega b_\omega c_\omega)$. Thus $d(a_\omega, \tilde{a}_\omega) \le \varepsilon$ and $\tilde{a}_\omega \mathop{\smile\!\!\!\!\!\!|}^{\mathfrak{P}}_{c_\omega} b_\omega$. In particular, $\mathrm{tp}^{\mathfrak{P}}(a_\omega{}^\varepsilon / b_\omega c_\omega)$ does not divide over c_ω.

As we assumed that $(a_i b_i c_i : i < \omega) \vDash \mathrm{mcl}(abc)$, we have $a_\omega b_\omega c_\omega \equiv^{\mathfrak{P}} abc$, so $\mathrm{tp}^{\mathfrak{P}}(a^\varepsilon/bc)$ does not divide over c. $\mathrm{QED}_{2.3}$

Theorem 2.4. *If T is supersimple, then so is $T^{\mathfrak{P}}$.*

Proof. We need to show that for every virtually finite element a^ε and every tuple $B = b_{\in I}$ in a model of $T^{\mathfrak{P}}$, there exists a finite sub-tuple $B' \subseteq B$ such that $\mathrm{tp}^{\mathfrak{P}}(a^\varepsilon/B)$ does not divide over B'.

We follow the path of [Ben04, Corollary 3.6]. Choose $(a_j B_j : j < 2\omega) \vDash \mathrm{mcl}(aB)$ in some model of T. Let $b_{\in I,j}$ be the enumeration of each B_j corresponding to $B = b_{\in I}$.

By supersimplicity of T there are tuples of distances $\upsilon = \upsilon_{<\omega} > 0$ and $\rho = \rho_{\in I,<2\omega} > 0$ such that:

$$a_\omega^\varepsilon \mathop{\smile\!\!\!\!\!\!|}_{(a_{<\omega})^\upsilon (B_{<2\omega})^\rho} a_{<\omega} B_{<2\omega}. \tag{1}$$

Then by definition, there are only finitely many $j < \omega$ such that $\upsilon_j \neq \infty$, and only finitely many pairs $(i, j) \in I \times 2\omega$ such that $\rho_{i,j} \neq \infty$, so we can define:

$$n = 1 + \max\{j < \omega : \upsilon_j \neq \infty \text{ or there exists } i \in I \text{ such that } \rho_{i,j} \neq \infty\},$$
$$\delta = \min\{\varepsilon, \upsilon_j : j < n\},$$
$$J_0 = \{i \in I : \rho_{i,j} \neq \infty \text{ for some } j < 2\omega\}.$$

In particular, $\varepsilon \geq \delta > 0$ and $J_0 \subseteq I$ is finite.

By right downward transitivity, (1) becomes:

$$a^\varepsilon_\omega \underset{a^\delta_{<n}, b_{\in J_0, \in [0,n) \cup [\omega, 2\omega)}}{\mathop{\smile\!\!\!\!\!\!|}} a_{<\omega} B_{<2\omega}.$$

Applying supersimplicity again, there is $J_1 \subseteq I$ finite such that:

$$a^\delta_{<n} \underset{b_{\in J_1, <\omega}}{\mathop{\smile\!\!\!\!\!\!|}} B_{<\omega}.$$

Let $J = J_0 \cup J_1 \subseteq I$, and let $B' = b_{\in J}$, $B'_j = b_{\in J, j}$. These are finite sub-tuples of B and B_j, respectively, and:

$$a^\varepsilon_\omega \underset{a^\delta_{<n}, B'_{\in [0,n) \cup [\omega, 2\omega)}}{\mathop{\smile\!\!\!\!\!\!|}} a_{<\omega} B_{<2\omega}, \tag{2}$$

$$a^\delta_{<n} \underset{B'_{<\omega}}{\mathop{\smile\!\!\!\!\!\!|}} B_{<\omega}. \tag{3}$$

We now prove by induction on $n \leq m < \omega$ that:

$$a^\delta_{<n} a^\varepsilon_{<m} \underset{B'_{<\omega}}{\mathop{\smile\!\!\!\!\!\!|}} B_{<\omega}. \tag{4}$$

For $m = n$, this follows from (3) since $\varepsilon \geq \delta$. Assume now (3) for some $n \leq m < \omega$. From (2) we obtain by monotonicity and right downward transitivity:

$$a^\varepsilon_\omega \underset{a^\delta_{<n} a^\varepsilon_{<m}, B'_{\in [0,m) \cup [\omega, 2\omega)}}{\mathop{\smile\!\!\!\!\!\!|}} B_{\in [0,m) \cup [\omega, 2\omega)}. \tag{5}$$

Since $(a_j, b_{\in I, j} : j < 2\omega)$ is an indiscernible sequence, there is an automorphism sending $a_{\omega+j} B_{\omega+j}$ to $a_{m+j} B_{m+j}$ for every $j < \omega$, while keeping $a_{<m} B_{<m}$ in place. Applying such an automorphism to (5) we get:

$$a^\varepsilon_m \underset{a^\delta_{<n} a^\varepsilon_{<m}, B'_{<\omega}}{\mathop{\smile\!\!\!\!\!\!|}} B_{<\omega}.$$

By left upward transitivity and the induction assumption (4) we obtain:

$$a^\delta_{<n} a^\varepsilon_{<m+1} \underset{B'_{<\omega}}{\mathop{\smile\!\!\!\!\!\!|}} B_{<\omega}.$$

This concludes the proof of (4). By the finite character of independence (Proposition 1.16) we conclude that:

$$a^\delta_{<n} a^\varepsilon_{<\omega} \underset{B'_{<\omega}}{\mathop{\smile\!\!\!\!\!\!|}} B_{<\omega}.$$

In particular, $\mathrm{tp}(a^\varepsilon_{<\omega}/B_{<\omega})$ does not divide over $B'_{<\omega}$. By Lemma 2.3 $\mathrm{tp}^{\mathfrak{P}}(a^\varepsilon/B)$ does not divide over B', which is finite, as required. $\text{QED}_{2.4}$

Corollary 2.5. *If T is superstable, then so is $T^{\mathfrak{P}}$.*

Proof. T is superstable if and only if it is stable and supersimple. In that case $T^{\mathfrak{P}}$ is stable by [Ben04, Theorem 3.10] and supersimple by Theorem 2.4. Therefore $T^{\mathfrak{P}}$ is superstable. $\text{QED}_{2.5}$

Question 2.6. Assume that T is ω-stable. Is $T^{\mathfrak{P}}$ ω-stable as well?

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