JOURNAL OF DIFFERENTIAL EQUATIONS 87, 38–61 (1990)

Linear Dispersive Equations of Airy Type*

WALTER CRAIG

Department of Mathematics, Brown University, Providence, Rhode Island 02912

AND

JONATHAN GOODMAN

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012

Received March 16, 1989; revised July 25, 1989

1. INTRODUCTION

The important role of the effects of dispersion on the description of linear and nonlinear wave motion is well known. Perhaps the most familiar problems in which dispersive effects are present come from the classical problem of water waves, and the long wave descriptions given by the Boussinesq and the Korteweg-de Vries (KdV) equations. In this paper we discuss a linear version of the latter equation

$$\partial_t u = a(x, t)\, \partial_x^3 u, \tag{1.1}$$

for which the coefficient of dispersion is allowed to vary in space and time. We will describe several mathematical results for the initial value problem for (1.1), as well as give a brief description of the application of the method of geometrical optics to describe the propagation of wave packet solutions, and its mathematical justification in several cases. The method of geometrical optics provides a heuristic description of what is perhaps the most interesting phenomenon, the effect of dispersive smoothing of solutions. Basically, for spatially localized initial data the solution operator acts as a smoothing operator. In this paper, we give the heuristics as well as the rigorous proof of this fact for the problem (1.1). In the last several years other authors have discussed this effect in related settings; for the KdV

* Research supported in part by the National Science Foundation. The first author is additionally supported by an Alfred P. Sloan Foundation Fellowship.

0022-0396/90 $3.00

equation or related generalizations we refer to the work of T. Kato [5], A. Cohen [1], R. Sachs [10], T. Kappeler [6], G. Ponce [9], and P. Constantin and J. C. Saut [2, 3].

Versions of the KdV equation or its linearization arise in the description of the slow variation of a wave front in coordinates moving with the wave. Equations of the form (1.1) arise through the linearization of problems with nonlinear dispersion, or in situations in which the wavefront travels through heterogeneous media, such as through liquid–gas or liquid–liquid interface, whose surface tension is varying due to inhomogeneities, temperature variations, or other effects. An important example is the KdV equation with a variable coefficient of surface tension which takes the form

$$\partial_t u = \left(\frac{1}{6} - \frac{\tau(x, t)}{2}\right) \partial_x^3 u + \frac{3}{2} u\, \partial_x u. \tag{1.2}$$

Problems with nonlinear dispersion, such as

$$\partial_t u = a(u)\, \partial_x^3 u + \text{lower order terms} \tag{1.3}$$

invariably arise in many other contexts; the following example was communicated to us by David Levermore. The difference equations

$$\dot{u}_j = \frac{1}{2\,\Delta x}\,(u_{j+1} - u_{j-1})\, u_j$$

model the differential equation

$$\partial_t u = u\, \partial_x u \tag{1.4}$$

with the sequence u_j serving as an approximation to $u(j\,\Delta x)$. The "modified equation" is a differential equation that describes the difference between (1.8) and (1.9); its solutions will describe the error committed when modelling (1.8) with Eq. (1.9). Letting $u_j(t) = v(j\Delta x, t)$ for some smooth function v, one finds that for small Δx,

$$\partial_t = v\, \partial_x v + \frac{(\Delta x)^2}{6}\, v\, \partial_x^3 v + \mathcal{O}(\Delta x^4) \tag{1.5}$$

Dropping the higher order terms yields the modified equation

$$\partial_t v = v\, \partial_x v + \frac{(\Delta x)^2}{6}\, v\, \partial_x^3 v, \tag{1.6}$$

a typical example of an equation with nonlinear dispersion.

We begin by presenting a formal geometrical optics expansion for solutions to Eq. (1.1), with which one describes the propagation of oscillatory wave packets. To start, we consider solutions whose oscillation will be much more rapid than the variation of the coefficient. Letting $a = a(\varepsilon x, \varepsilon t)$ and scaling the independent variables $x' = \varepsilon x$, $t'\varepsilon t$, the equation becomes

$$\partial_{t'} u = \varepsilon^2 a(x', t')\, \partial_{x'}^3 u. \tag{1.7}$$

For convenience of notation the primes are dropped.

The proper ansatz is $u(x, t) = A(x, t; \varepsilon) \exp(i/\varepsilon)\, S(x, t)$. Putting this into Eq. (1.7) and equating terms with like powers of ε, an expression for the solution is generated by solving a hierarchy of partial differential equations. This is the usual procedure:

$$\partial_t S + a(x, t)(\partial_x S)^3 = 0 \tag{1.8}$$

is the eiconal equations. Letting $A = \sum_{j=0}^{\infty} \varepsilon^j A_j$, the first transport equation is

$$L(a, S)\, A_0 \equiv \partial_t A_0 + 3a(x, t)\, \partial_x((\partial_x S)^3 A_0) = 0. \tag{1.9}$$

All higher transport equations involve inhomogeneous problems for the operator $L(a, S)$, whose right hand sides involve only previously computed quantities and their derivatives,

$$L(a, S)\, A_j = R_j(S, A_i; 0 \leqslant i < j). \tag{1.10}$$

Equation (1.8) is a Hamilton Jacobi equation, which can be solved by the method of characteristics. Define the Hamiltonian system by $H(s, t, k, \omega) = \omega + a(x, t)\, k^3$;

$$\begin{aligned} \dot{x} &= \partial_k H = 3a(x, t)\, k^2 \\ \dot{k} &= -\partial_x H = -\partial_x a k^3 \\ \dot{t} &= \partial_\omega H = 1 \\ \dot{\omega} &= -\partial_t H = -\partial_t a k^3. \end{aligned} \tag{1.11}$$

Orbits of (1.11) describe the change in oscillation and the spatial location of a wave packet, and the phase function $S(x, t)$ can be recovered by the integration of $k\, dx + \omega\, dt$ from the initial surface $t = 0$. The trajectories $x(t)$ are known as group lines, describing the spatial position of the packet. Differentiating (1.8) with respect to k gives

$$\partial_t(\partial_k S) + 3a(x, t)(\partial_x S)^2\, \partial_x(\partial_k S) = 0 \tag{1.12}$$

from which we see that points at which the phase is stationary propagate with the group velocity $3a(x, t)\, k^2$, with location described by the trajectories $x(t)$. Since Eqs. (1.9) and (1.10) involve the same hyperbolic differential operator as (1.12), this formal expansion procedure predicts that all phase and amplitude information propagate with the group velocity, along the trajectories $x(t)$ of the system (1.11).

The mathematical results of this paper are in part a justification of this asymptotic description of solutions to (1.1). In Section 2, two explicit and instructive examples are presented, for which the geometrical optics expansion leads to an exact solution. The simple case is of course the Airy equation, for which we review the construction of a fundamental solution and the justification of the method of stationary phase. The more interesting case involves a variable coefficient of dispersion. Two options differ by a choice of sign; in one distributional initial data become analytic for any positive t, while the other choice of sign (solve the problem with time reversed) leads to singularity formation in finite time even for initial data analytic, of exponential type.

Sections 3 and 4 contain the existence, uniqueness, and regularity results for solutions to Eq. (1.1), and are based on energy estimates using weighted Sobolev norms. It is shown in Lemma 3.1 that if $a(x, t)$ is sufficiently smooth, and does not change sign, then the initial value problem is well-posed in appropriate Sobolev spaces. This kind of dispersive problem exhibits the interesting phenomenon of dispersive smoothing; that is, if initial data are effectively spatially localized, then the solution at any time $t \neq 0$ is much smoother than the data. There is a simple heuristic argument for this effect based on the formal geometrical optics expansion. One decomposes the initial data into oscillatory components, for example, with a partition of unity on the Fourier transform side. The effect of the solution operator is to propagate components with different oscillation at different speeds; the more oscillation the higher the velocity. When initial data are localized to the right half-line, this effect propagates all highly oscillatory components out of any finite set, and the solution is C^∞. A quantitative statement of the effect of dispersive smoothing appears in Lemma 3.5. This is, in fact, a general phenomenon among dispersive evolution equations with the property that $\lim_{k \to \pm\infty} |\partial\omega/\partial k| = \infty$. These results for the nonlinear but constant dispersion case have been previously discussed by A. Cohen [1], R. Sachs [10], and T. Kappeler [6] for the KdV equation using methods of inverse scattering theory, and for a class of generalized KdV equations by T. Kato [5] using energy estimates and by G. Ponce [9] using commutator estimates. Our approach was devised independently of, but most closely resembles, that of Kato.

Of the remaining sections, 4 discusses briefly the periodic case, and Section 5 discusses further including the questions of well-posedness and ill-posedness when the coefficient $a(x, t)$ changes sign.

We would like to remark that this article concerns a simple linear partial differential equation, for which the machinery of Fourier integral operators is often well suited. However, for these problems, the solution operator is $S^{-\infty}$, or better in some cases, and the techniques are less useful in describing the smoothness properties of solutions. We have chosen to proceed with the quite simple weighted Sobolev estimates for these problems of variable dispersion.

A preprint of this paper was written and circulated in Spring 1985; however, it was never submitted for publication. Since then there has been a growth of interest in the field, including many new developments. In particular, the phenomenon of dispersive smoothing is being discussed is many settings, both linear and nonlinear. Using the techniques of Section 3 of this paper, fully nonlinear versions of the phenomenon of dispersive smoothing have been obtained; these will appear in an article by W. Craig, T. Kappeler, and W. Strauss.

The authors thank Russell Caflisch and Peter Lax of the Courant Institute, Joseph B. Keller of Stanford University, Gustavo Ponce of the University of Chicago, and Jean Claude Saut of the Université de Paris–Sud for many useful discussions.

2. Two Instructive Examples

There are two examples of dispersive equations of the class under consideration which are explicitly solvable, and which are instructive to present. The first is the constant coefficient Airy equation

$$\begin{aligned} \partial_t u - a\,\partial_x^3 u = 0, \qquad -\infty < x < \infty,\quad 0 \leqslant t < \infty \\ u(x, 0) = f(x) \qquad \text{initial data} \end{aligned} \tag{2.1}$$

whose solution can be represented conveniently either as a superposition of Airy functions

$$u(x, t) = \frac{1}{\sqrt[3]{at}}\,(Ai * f)\left(\frac{x}{\sqrt[3]{at}}\right), \tag{2.2}$$

or directly by the Fourier transform

$$u(x, t) = \frac{1}{2\pi} \iint e^{i(k(x-y) + ak^3 t)} f(y)\, dy\, dk. \tag{2.3}$$

Since this is a constant coefficient equation the geometrical optics expansion in this case is exact and gives rise to expression (2.3).

The second example is an equation with a variable coefficient of dispersion

$$\partial_t u \pm x\,\partial_x^3 u = 0, \qquad -\infty < x < \infty, \quad 0 \leqslant t < \infty$$
$$u(x, 0) = f(x). \tag{2.4}$$

It turns out that in this case as well the geometrical optics expansion gives the exact solution and can be applied to give an expression similar to (2.3). From this representation we will see that the difference in sign in (2.4) changes drastically the character of the solution operator; the positive sign is stable, for which the initial value problem is well-posed, and the solution operator is a smoothing operator for all positive t. On the other hand the problem with the negative sign is ill-posed, generating solutions for even analytic initial data which develop infinite oscillation in finite time.

We first give a description of the solution operator for the simplest example (2.1). The Airy function $(at)^{-1/3}Ai(x/(at)^{-1/3})$ is the fundamental solution, giving rise to the expression (2.2). The first remark is that the fundamental solution, which is a Dirac mass at $t=0$, by virtue of the effect of dispersive smoothing, for all nonzero t is a real analytic function of x. This smoothing is not, however, accompanied by a loss of "energy," or L^2 norm. If a solution possesses α many $L^2(\mathbf{R})$ derivatives, then by integration by parts for $\beta \leqslant \alpha - 3$

$$\partial_t \int_{-\infty}^{\infty} (\partial_x^\beta u)^2\,dx = 2\int_{-\infty}^{\infty} \partial_x^\beta u a\,\partial_x^{\beta+3} u\,dx$$
$$= \int_{-\infty}^{\infty} a\,\partial_z(\partial_x^{\beta+1} u)^2\,dx = 0;$$

that is, the solution operator preserves all Sobolev norms. Of course for a constant coefficient equation, it is best to do Fourier analysis. To continue our point of view make the ansatz that $u(x, t) = \exp(iS(x, t))\,A(x, t)$. The eiconal equation is

$$\partial_t S + a(\partial_x S)^3 = 0, \qquad S(x, 0) = k(x-y), \tag{2.5}$$

and the first transport equation is

$$\partial_t A + 3a\,\partial_x((\partial_x S)^2 A) = 0. \tag{2.6}$$

Hamilton Jacobi theory can be used to solve (2.5) by the method of characteristics; set $H(x, t, k, \omega) = \omega + ak^3$ and solve the system

$$\dot{x} = \partial_k H, \qquad \dot{k} = -\partial_x H, \tag{2.7}$$

whose trajectories represent "group lines," along which wave packets propagate. In the constant coefficient case, the solution of (2.5) and (2.7) is simply

$$x(t) = x(0) + 3ak^2 t, \qquad k(t) = k(0)$$
$$S(x, t) = k(x - y) - ak^3 t.$$

These group lines are exhibited in Fig. 1. Clearly $\partial_t S = -ak^3 = \omega(k)$ gives the dispersion relation. Solving the transport equation (2.6) with initial data $A(x, 0) = 1$ results in an expression for the fundamental solution, and in this roundabout manner, gives the expression (2.3).

To finish this discussion we will give a simple justification of the method of stationary phase for the solution operator, which gives the result of the propagation of wave packets along the characteristics of (2.7).

LEMMA 2.1. *Consider initial data $f(x)$ of Schwartz class, such that* $\operatorname{supp} \hat{f}(k) \subset\subset [b_1, b_2]$, *where* $0 \leqslant b_1 < b_2$. *There are constants* $C_{\beta\gamma\delta}$ *such that the solution* $u(x, t)$ *of* (2.1) *satisfies*

$$|(\partial_x)^\gamma (\partial_t)^\delta u(x, t)| \leqslant C_{\beta\gamma\delta} \begin{cases} (1 + |x - \partial_k \omega(b_1) t|)^{-\beta} & \text{for} \quad x/t < \partial_k \omega(b_1) \\ (1 + |x - \partial_k \omega(b_2) t|)^{-\beta} & \text{for} \quad \partial_k \omega(b_2) < x/t. \end{cases} \tag{2.8}$$

Furthermore, if $b_1 > 0$ *then*

$$|u(x, t)| \leqslant C_0 (1 + t)^{-1/2} \tag{2.9}$$

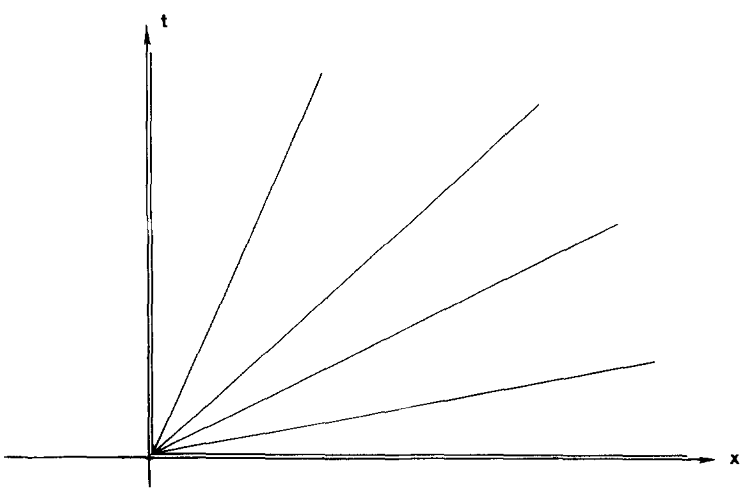

FIGURE 1

while if $b_1 = 0$ *then in general, the weaker estimate holds*

$$|u(x, t)| \leqslant C_0(1+t)^{-1/3}.$$

Proof. The results of the lemma are precisely the behavior of the leading term in the method of stationary phase, and the proof as usual consists of integration by parts. From (2.3)

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{b_1}^{b_2} e^{i(kx - \omega(k)t)} \hat{f}(k)\, dk.$$

Consider $x/t \notin [\partial\omega/\partial k](b_1), \partial\omega/\partial k(b_2)]$ so that for any $k \in (b_1, b_2)$, $\partial_k(kx - \omega(k)t) = (x - \partial\omega/\partial k t) \neq 0$. Integration by parts involves the operator $L(k) = i\partial_k((x - \partial_k \omega t)^{-1})$;

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{b_1}^{b_2} \frac{1}{i} \partial_k(e^{i(kx - \omega t)})(x - \partial_k \omega t)^{-1} \hat{f}(k)\, dk \\ &= \frac{1}{\sqrt{2\pi}} \int e^{i(kx - \omega t)}\, L(k)\, \hat{f}\, dk \\ &= \frac{1}{\sqrt{2\pi}} \int e^{i(kx - \omega t)}\, L^\beta(k) \hat{f}\, dk. \end{aligned} \tag{2.10}$$

Each application of $L(k)$ adds to the spatial decay, since

$$|L^\beta(k)\hat{f}| \leqslant C_\beta |x - \partial_k \omega t|^{-\beta} \left| \sum_j a_j(\partial_k)^j \hat{f} \right|.$$

The same computation applies to derivatives of $u(x, t)$; this proves (2.8). Statements (2.9) follow from similar simple considerations. ∎

The second example (2.4) has a variable coefficient of dispersion, and the solution operator is more interesting. With the ansatz $u(x, t) = \exp(iS)A$, we obtain the eiconal and transport equations

$$\begin{aligned} &\partial_t S \mp x(\partial_x S)^3 = 0, \qquad S(x, 0) = k(x - y), \\ &\partial_t A \mp 3x\, \partial_x((\partial_x S)^2 A) = 0. \end{aligned} \tag{2.11}$$

By Hamilton Jacobi theory one obtains the Hamiltonian $H(x, t, k, \omega) = \omega \mp xk^3$, and the system

$$\begin{aligned} \dot{x} &= \partial_k H = \mp 3xk^2 \\ \dot{k} &= -\partial_x H = \pm k^3. \end{aligned} \tag{2.12}$$

There is a closed form solution to this given by

$$k(t) = \xi(1 \mp 2\xi^2 t)^{-1/2}, \qquad k(0) = \xi$$
$$x(t) = x(0)(1 \mp 2\xi^2 t)^{3/2} \tag{2.13}$$
$$S(x, k, t) = (x - y)\, k(1 \mp 2k^2 t)^{-1/2}.$$

The cases of $\pm$ in (2.4) have clearly different behavior, in particular for $a(x) = -x$ the group lines $x(t)$ focus to zero in finite time, $t = 1/2\xi^2$ while the associated wave numbers $k(t)$ become infinite. In the stable case $a(x) = x$, the group lines diverge, while the associated wave numbers $k(t)$ all lie within the envelope $k = \pm(2t)^{-1/2}$.

It is remarkable that the geometrical optics expansion for (2.4) is exact. To demonstrate this consider the Fourier transform of (2.4),

$$\partial_t \hat{u} \mp \partial_k(k^3 \hat{u}) = 0. \tag{2.14}$$

Because of the simple nature of the coefficient, the result is a first order linear hyperbolic equation, which can be solved by the method of characteristics.

$$\hat{u}(k, t) = (1 \pm 2k^2 t)^{-3/2} \hat{f}(k(1 \pm 2k^2 t)^{-1/2}).$$

Thus

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}(1 \pm 2k^2 t)^{-3/2} \hat{f}(k(1 \pm 2k^2 t)^{-1/2})\, dk$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int e^{ix\xi(1 \mp 2\xi^2 t)^{-1/2}} e^{-iy\xi} f(y)\, dy\, d\xi \tag{2.15}$$

which verifies the geometrical optics derivation of $S(x, t, k) = xk(1 \mp 2k^2 t)^{-1/2}$, $A(x, t, k) = 1$ for the fundamental solution of the problem. The characteristics for the hyperbolic problem (2.14) are of course the trajectories $k(t)$ of the Hamiltonian system (2.12), whose orbits projected onto the x-variable give the group lines. Sketches of these trajectories in the stable and unstable cases appear as Figs. 2a and 2b. Inspection of this explicit solution provided the following lemma.

LEMMA 2.2. (i) *For $a(x) = x$, the stable case, the initial value problem is well-posed. For distributional initial data, a unique solution exists which is analytic, of exponential type $R = (2t)^{-1/2}$ for any $t > 0$.*

(ii) *On the other hand, for the unstable case $a(x) = -x$ the initial value problem is ill-posed. Even for initial data of exponential type, whose Fourier*

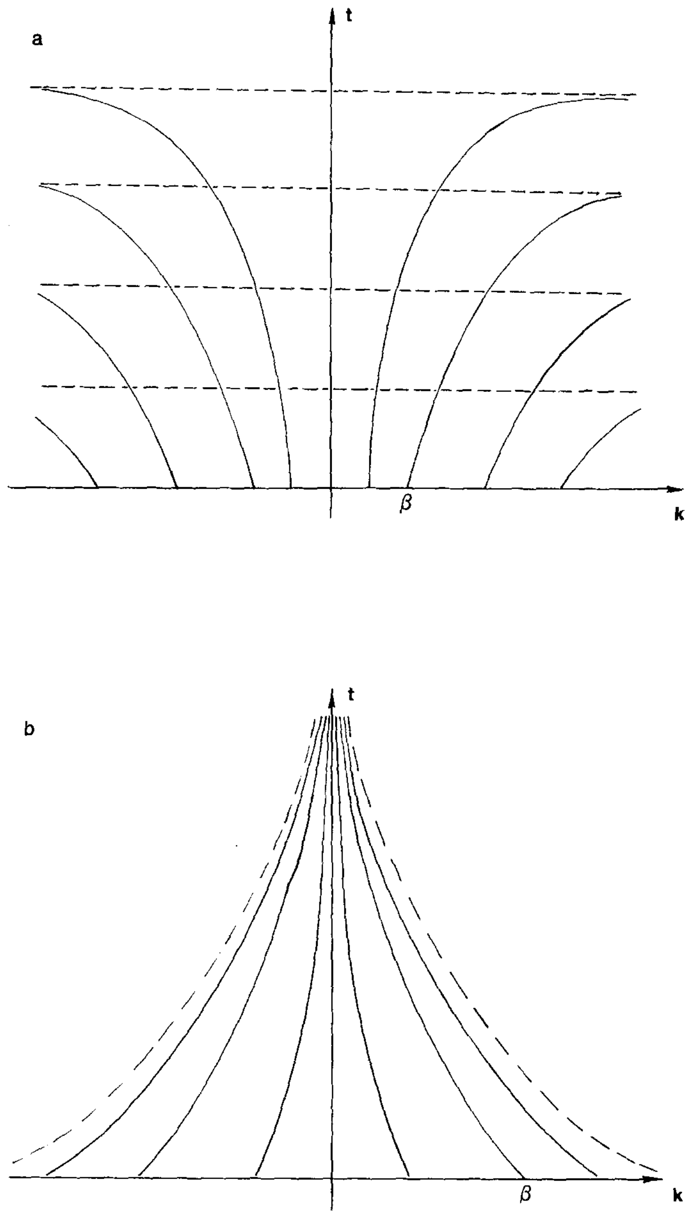

FIG. 2. Trajectories in the (a) stable and (b) unstable cases.

505/87/1-4

transform thus has bounded support, the solution becomes singular in finite time. If $R = \max\{|k|; k \in \operatorname{supp} \hat{f}\}$, *then for any* ε,

$$\|u(x, t)\|_{H, 3/2+\varepsilon} \to \infty \qquad as \quad t \to \frac{1}{2R^2}.$$

3. Sobolev Estimates on the Line

In this section we present the rigorous results of an existence and uniqueness theory for the initial value problem.

$$\begin{gathered} \partial_t u = a(x, t)\, \partial_x^3 u, \qquad -\infty < x < \infty, \quad 0 \leqslant t < \infty \\ u(x, 0) = f(x) \in H^r(\mathbf{R}). \end{gathered} \tag{3.1}$$

This will follow from Sobolev estimates for the solution and its higher derivatives with respect to x and t. The basic tool is a weighted energy inequality, with a weight constructed from the variable coefficient $a(x, t)$. In this regular case, we assume that for a constant $1 \leqslant c$, $1/c \leqslant a(x, t) \leqslant c$, and that $a(x, t) \in C^\beta(\mathbf{R})$ for some integer $3 \leqslant \beta \leqslant +\infty$. It is clear that the method is not confined to problem (3.1). In particular, it carries over to the case in which lower order terms are present as well.

In this section we will prove two theorems. The first is an existence, uniqueness, and regularity result for the class of Sobolev initial data without conditions of spatial decay.

Theorem 3.1. *For* $\alpha \leqslant \beta$ *let* $u(x, 0) = f(x) \in H^\alpha(\mathbf{R})$ *be initial data for the initial value problem* (3.1). *A unique solution* $u(x, t)$ *exists, which satisfies the estimate*

$$\|u(x, t)\|_{H,\alpha} \leqslant C_2 \exp(C_1 t)\, \|f(x)\|_{H,\alpha}. \tag{3.2}$$

When initial data are given which are appropriately spatially localized, the solution operator for problem (3.1) is smoothing; this is due to the dispersive nature of the equation. This effect of dispersive smoothing can be explained roughly as follows: if the initial data were decomposed into constituents whose Fourier transforms were supported on disjoint intervals, the solution operator would propagate their principal contributions at different velocities. In particular, high frequency components propagate very rapidly, and pass out of any bounded region after a small initial time interval. A precise statement of the effect of dispersive smoothing is the following theorem.

THEOREM 3.2. *Let $u(x, 0) = f(x) \in L^2(\mathbf{R})$ such that in addition*

$$\int_{-\infty}^{\infty} (1 + \chi_{\{x \leqslant 0\}} |x|^{\alpha}) f^2(x)\, dx < \infty. \tag{3.3}$$

Then for each $\eta(x) \in C_0^{\infty}(\mathbf{R})$ a cutoff function there are constants C_1, C_2 such that for $\gamma \leqslant \alpha$

$$\int_0^t \int_{-\infty}^{\infty} |\partial_x^{\gamma} u|\, \tau^{\gamma} \eta(x)\, dx\, d\tau \leqslant C_2(1 + t^{\gamma}) \exp(C_1 t).$$

If furthermore (3.3) holds for all α and $a(x, t) \in C^{\infty}(\mathbf{R})$ then for any time $t > 0$ the solution $u(x, t) \in C^{\infty}(\mathbf{R})$.

The results of existence, uniqueness, and regularity follow in a straightforward fashion from the Sobolev estimates that appear in Lemmas 3.1 and 3.4. Both results use weighted Sobolev estimates. The results of dispersive smoothing follow from the estimate of Lemma 3.5.

We start by deriving weighted Sobolev estimates for the $L^2(\mathbf{R})$ norm of a smooth solution. Putting in a bounded weight function $\xi(x, t) \in C^{\infty}$, a smooth solution $u(x, t)$ which vanishes with it first two derivatives as $x \to \pm\infty$ satisfies the identity

$$\begin{aligned}
0 &= \int_{\infty}^{\infty} u(\partial_t u - a(x, t)\, \partial_x^3 u)\, \xi\, dx \\
&= \tfrac{1}{2} \partial_t \int_{-\infty}^{\infty} u^2 \xi\, dx - \tfrac{1}{2} \int_{-\infty}^{\infty} u^2\, \partial_t \xi\, dx \\
&\quad + \int_{-\infty}^{\infty} (a\xi)\, \partial_x^2 u\, \partial_x u\, dx + \int_{-\infty}^{\infty} \partial_x(a\xi)\, \partial_x^2 u\, dx \\
&= \tfrac{1}{2} \partial_t \int_{-\infty}^{\infty} u^2 \xi(x)\, dx - \tfrac{3}{2} \int_{-\infty}^{\infty} \partial_x(a\xi)(\partial_x u)^2\, dx \\
&\quad - \tfrac{1}{2} \int_{-\infty}^{\infty} u^2 \partial_t \xi\, dx + \tfrac{1}{2} \int_{-\infty}^{\infty} \partial_x^3(a\xi)\, u^2\, dx.
\end{aligned} \tag{3.4}$$

With the choice of ξ such that $-3/2(a\xi) \geqslant 0$, a useful differential inequality is obtained

$$\begin{aligned}
\partial_t \int_{-\infty}^{\infty} u^2 \xi\, dx &\leqslant \int_{-\infty}^{\infty} |\partial_x^3(a\xi) - \partial_t \xi|\, u^2\, dx \\
&\leqslant |(\partial_x^3(a\xi) - \partial_t \xi)/\xi|_{L^{\infty}} \int_{-\infty}^{\infty} u^2 \xi\, dx
\end{aligned} \tag{3.5}$$

resulting in the energy estimate

$$\int_{-\infty}^{\infty} u^2(t, x)\, \xi(x)\, dx \leqslant \exp(t\,|(\partial_x^3(a\xi) - \partial_t \xi)/\xi|_{L^\infty}) \int_{-\infty}^{\infty} f^2(x)\, \xi(x)\, dx.$$

Several choices of weight function $\xi(x)$ are appropriate; the first is obviosly $\xi(x, t)$ a solution to the adjoint problem $\partial_t \xi - \partial_x^3(a\xi) = 0$, which gives the proper weight for which the L^2 norm is nonincreasing time. Here the more practical choice $\xi(x, t) = 1/a(x, t)$ results in the upper bound on the exponential growth rate by the factor $|(\partial_a/\partial_t)/a|_{L^\infty}$.

LEMMA 3.1. *Sufficiently smooth solutions to Eq.* (3.1) *satisfy the energy estimate*

$$\int_{-\infty}^{\infty} u^2(x, t)\, a(x, t)\, dx \leqslant \exp(t\, |\partial_t a/a^2|_{L^\infty}) \int_{-\infty}^{\infty} f^2(x) \frac{dx}{a(x, 0)}. \tag{3.6}$$

Solutions to Eq. (3.1) in fact are smoother for $t > 0$ than the initial data; this was discussed for the instructive examples presented in Section 2. The simplest indication of this phenomenon of smoothing due to dispersive effects from another choice of weight function. Let $\xi(x, t)$ satisfy the differential inequality

$$-\tfrac{3}{2}\, \partial_x(a\xi) \geqslant \eta(x) \geqslant 0 \tag{3.7}$$

for a localizing C^∞ cutoff function $\eta(x)$. It follows from (3.4) that

$$\int_{-\infty}^{\infty} (\partial_x u)^2\, \eta\, dx \leqslant \tfrac{1}{2} \int_{-\infty}^{\infty} |(\partial_x^3(a\xi) - \partial_t \xi)|\, u^2\, dx - \tfrac{1}{2}\, \partial_t \int_{-\infty}^{\infty} u^2 \xi\, dx.$$

Integrating twice in t and using Lemma 3.1 we obtain the estimate

$$\begin{aligned} &\int_0^t \int_{-\infty}^{\infty} (\partial_x u)^2\, \tau\eta\, dx\, d\tau \\ &\quad \leqslant \tfrac{1}{2} \int_0^t \int_{-\infty}^{\infty} (1 + \tau\, |(\partial_x^3(a\xi) - \partial_t \xi)/\xi|_{L^\infty})\, u^2 \xi\, dx\, d\tau \\ &\quad \leqslant \frac{t}{2} \exp(t\, |(\tfrac{3}{2}\, \partial_x^2 \eta - \partial_t \xi)/\xi|_{L^\infty}) \int_{-\infty}^{\infty} f^2(x) \xi\, dx. \end{aligned} \tag{3.8}$$

LEMMA 3.2. *For sufficiently smooth solutions to Eq.* (3.1), *the following energy estimate holds*:

$$\int_0^t \int_{-\infty}^{\infty} (\partial_x u)^2\, \tau\eta\, dx\, d\tau \leqslant \frac{t}{2} \exp(tC_1) \int_{-\infty}^{\infty} f^2(x) \xi\, dx. \tag{3.9}$$

This represents an increased regularity of the solution over the initial data.

To complete the proof of this lemma, we construct a satisfactory solution of the differential inequality (3.7); set $\xi(x) = (C_0 + \int_{-\infty}^{x} \eta(y)\,dy)/a(x, t)$ with $C_0 > 2\|\eta\|_{L^1}$. The constant $C_1 = |(\frac{3}{2}(\partial_x)^2\eta - \partial_t \xi)/\xi|_{L^\infty}$. The same proof gives weighted Sobolev estimates, with weight functions ξ, η satisfying (3.7) but which are not necessarily bounded. Such weight functions will be introduced in the proof of Lemma 3.5 below. Versions of this lemma on the increased regularity of solutions to dispersive problems have been discussed by others. In particular, Constantin and Saut [3] handle very general constant coefficient dispersive problems.

Higher Sobolev regularity of solutions is obtained by estimates of higher derivatives of Eq. (3.1), combined with the choice of a hierarchy of weight functions. To start, assume that a solution $u(x, t) \in C^1([0, T]; H^{\alpha+3})$ and set $v(x, t) = \partial_x^\alpha u(x, t)$. The function $v(x, t)$ satisfies

$$\partial_t v = a\,\partial_x^3 v + a\partial_x a\,\partial_x^2 v + \binom{\alpha}{2} \partial_x^2 a\,\partial_x v + \binom{\alpha}{3} \partial_x^3 a v + F_\alpha(a, u). \quad (3.10)$$

One first estimates the error terms F_α.

LEMMA 3.3. (i) $\|F_\alpha\|_{L^2} \leqslant C_0(\alpha)\,|a(x, t)|_{C,\alpha}\,\|u(x, t)\|_{H,\alpha-1}$. *A more refined estimate which involves the weight functions* ξ_α *is*

$$\text{(ii)} \quad \int_{-\infty}^{\infty} F_\alpha^2 \xi_\alpha\,dx \leqslant \sum_{\nu=4}^{\alpha} \binom{\alpha}{\nu} |\partial_x^\nu a|_{L^\infty} \int_{-\infty}^{\infty} |\partial_x^{\alpha-\nu+3} u(x, t)|^2\,\xi_a\,dx.$$

The use of (ii) *is in the effort to control the dependence of the estimate on order of differentiation* α.

Proof. The Leibniz rule describes the remainder

$$\partial_x^\alpha(aw) = \left(\sum_{\nu=0}^{3} + \sum_{\nu=4}^{\alpha}\right)\binom{\alpha}{\nu} \partial_x^\nu a\,\partial_x^{\alpha-\nu} w. \quad (3.11)$$

Thus

$$F_\alpha(a, u) = \sum_{\nu=4}^{\alpha} \binom{\alpha}{\nu} \partial^\nu{}_x a\,\partial_x^{\alpha-\nu+3} u.$$

Estimates (i) and (ii) then follow immediately. ∎

The higher Sobolev estimates are contained in this lemma.

LEMMA 3.4. *Suppose that* a, $\partial_t a \in C^\infty$. *Let* $v(x, t) = \partial_x^\alpha u(x, t)$, *where*

$u(x,t)\in C^1([0,T],H^{\alpha+3})$ is a solution of (3.1) in the time interval $[0,T]$. Then there exist constants $C_1(\alpha)$, $C_2(\alpha)$ such that $v(x,t)=\partial_x^\alpha u(x,t)$ satisfies the higher Sobolev estimate

$$\|v(x,t)\|_{L^2}\leqslant C_2(\alpha)\exp(tC_1(\alpha))\,\|f(x)\|_{H,\alpha}.$$

The constant

$$C_1(\alpha)\leqslant C_0(\alpha^3+1)\,C^{2\alpha/3-2}(|a|_{C,3}+|\partial_t a|_{L^\infty}).$$

Proof. As usual one integrates Eq. (3.10) against $v\xi_\alpha$, with an appropriate weight function. Upon integrating by parts, the identity is obtained

$$\begin{aligned}\frac{1}{2}\partial_t\int_{-\infty}^{\infty} v^2\xi_\alpha\,dx &= \int_{-\infty}^{\infty}\left(\frac{3}{2}\partial_x(\xi_\alpha a)+a\xi_\alpha\,\partial_x a\right)(\partial_x v)^2\,dx\\ &\quad+\int_{-\infty}^{\infty}\left(\frac{1}{2}\partial_t\xi_\alpha-\frac{1}{2}\partial_x^3(\xi_\alpha a)+\frac{\alpha}{2}\partial_x^2(\xi_\alpha\,\partial_x a)-\frac{1}{2}\binom{\alpha}{2}\partial_x(\xi_\alpha\,\partial_x^2 a)\right.\\ &\quad\left.+\frac{1}{2}\binom{\alpha}{3}(\xi_\alpha\,\partial_x^3 a)\right)v^2\,dx+\int_{-\infty}^{\infty} vF_\alpha(a,u)\,\xi_\alpha\,dx. \qquad (3.12)\end{aligned}$$

One constructs a weight function so that the quantity

$$-\tfrac{3}{2}\partial_x(\xi_\alpha a)+a\xi_\alpha\,\partial_x a=\eta_\alpha(s,t)\geqslant 0.$$

The choice for the present lemma will be that $\eta_\alpha=0$, leading to the definition of $\xi_\alpha=a^{2\alpha/3-1}(x,t)$. It is then straightforward to estimate

$$\begin{aligned}&|(\partial_t\xi_\alpha-\partial_x^3(\xi_\alpha a)+\alpha\,\partial_x^2(\xi_\alpha\,\partial_x a)-\alpha(\alpha-1)\,\partial_x(\xi_\alpha\,\partial_x^2 a))/\xi_\alpha|_{L^\infty}\\ &\quad\leqslant C_0(\alpha^3+1)\,C^{2\alpha/3-2}\left(|a(x,t)|_{C,3}+\left|\frac{\partial a}{\partial t}\right|_{L^\infty}\right). \qquad (3.13)\end{aligned}$$

Thus the energy estimate is obtained

$$\partial_t\int_{-\infty}^{\infty} v^2\xi_\alpha\,dx\leqslant C_1(\alpha)\int_{-\infty}^{\infty} v^2\xi_\alpha\,dx+\int_{-\infty}^{\infty} vF_\alpha\xi_\alpha\,dx \qquad (3.14)$$

for some constant $C_1(\alpha)$. Integrating this differential inequality and using Lemma 3.3,

$$\int_{-\infty}^{\infty} v^2(t)\, \xi_\alpha\, dx$$

$$\leqslant \exp(tC_1(\alpha)) \left(\int_{-\infty}^{\infty} v^2(0)\, \xi_\alpha\, dx \right.$$

$$\left. + \sum_{v=4}^{\alpha} \binom{\alpha}{v} |\partial_x^v (a\xi_\alpha)/\xi_{\alpha-v+3}|_{L^\infty} \int_0^t \int_{-\infty}^{\infty} |\partial_x^{\alpha-v+3} u|^2\, \xi_{\alpha-v+3}\, dx \right)$$

$$\leqslant C_2(\alpha) \exp(tC_1(\alpha)) \left(\int_{-\infty}^{\infty} v^2(0)\, \xi_\alpha\, dx + t\, \|f(x)\|^2_{H,\alpha-1} \right). \tag{3.15}$$

The constant $C_1(\alpha)$ bounded terms $v = 0$, 1, 2, and 3 of (3.11) as well as the supremum norms of the weight functions for $v < \alpha$, and thus can be chosen as in the statement of the lemma. ∎

When the initial data are not necessarily smooth, but instead possess certain decay properties as $x \to -\infty$, the solution for any positive time is smoother than the initial data. This phenomenon is due to the dispersive property of the differential equation, which propagates different frequency components of the initial data at different velocities. Roughly, if the data are localized, then after a short time interval all high frequency components of the solution will have traveled out of any bounded set. To state this precisely, we have the following lemma:

LEMMA 3.5. *Let* $|a(x, t)|_{C,\alpha} < \infty$, *and assume that the initial data* $u(x, 0) = f(x) \in L^2(\mathbf{R})$ *satisfy the weighted estimate*

$$\int_{-\infty}^{\infty} f^2(x)\, \eta_{\alpha-1}(x)\, dx \equiv e(f) < \infty. \tag{3.16}$$

The weight functions η_γ *are one sided; that is,*

$$|\eta_\gamma(x)| \sim \begin{cases} 1 & \text{for } x \geqslant 0 \\ (1+|x|)^\gamma & \text{for } x \leqslant 0. \end{cases} \tag{3.17}$$

Then the solution $u(x, t)$ *satisfies the estimate*

$$\int_0^t \int_{-\infty}^{\infty} |\partial_x^\alpha u(x, \tau)|^2\, \tau^\alpha \eta_{-1}(x)\, dx\, d\tau \leqslant e(f)\, C(\alpha)(1 + t^\alpha) \exp(C_0 t).$$

If (3.16) *holds for any* α, *then* $L^2(\mathbf{R})$ *data give rise to* $C^\infty(\mathbf{R})$ *solutions for any positive time* t.

Of course the equation is time reversible; to be smoothing for negative time the condition (3.16) must hold with weight $\eta_\gamma(-x)$. These smooth solutions do not decay as fast as an integer power of the left (respectively on the right) half-line; if so, running time backwards would produce contradictory smoothness properties at $t=0$.

Proof. One starts wil the higher energy estimate (3.12), proceeding in an induction that starts with Lemma 3.2. Let $v=\partial_x^\beta u$, with any $\beta<\alpha$. For any weight function ξ (one will be chosen below),

$$\int_{-\infty}^{\infty}\left(-\frac{3}{2}\partial_x(\xi a)+\beta\xi\,\partial_x a\right)(\partial_x v)^2\,dx$$
$$=-\frac{1}{2}\partial_t\int_{-\infty}^{\infty}v^2\xi\,dx+\frac{1}{2}\int_{-\infty}^{\infty}\Bigg(\partial_t\xi-\partial_x^3(\xi a)+\beta\,\partial_x^2(\xi\,\partial_x a)$$
$$-\binom{\beta}{2}\partial_x(\xi\,\partial_x^2 a)+\binom{\beta}{3}\xi\,\partial_x^3 a\Bigg)v^2\,dx+\int_{-\infty}^{\infty}vF_\beta(a,u)\xi\,dx. \tag{3.18}$$

Define $\eta_{-1}(x)=\operatorname{sech}(\rho x)$ or some other cutoff function such that $|\eta_{-1}(x)|\sim\exp(-\rho|x|)$ as $x\to\pm\infty$. Then recursively construct

$$\eta_\gamma(x,t)=\int_x^\infty \eta_{\gamma-1}(y,t)\,a^{-1}(y,t)\,dy.$$

For the induction a scale of weight functions $\xi_{\beta\gamma}$ is constructed so that

$$-\tfrac{3}{2}\partial_x(\xi_{\beta\gamma}a)+\beta\,\partial_x a\xi_{\beta\gamma}=\eta_\gamma\geqslant 0, \tag{3.19}$$

where this time the non-zero η_γ are used. By (3.19) we see that $\xi_{\beta\gamma}=a^{2\beta/3-1}(x,t)\,\eta_\gamma(x)$. Use $\xi_{\beta\gamma}$ in (3.18) and integrate $\delta+1$-many times with respect to t. The resulting estimate is

$$\int_0^t\int|\partial_x^{\beta+1}u|^2\frac{\tau^\delta}{\delta!}\eta_\gamma\,dx\,d\tau$$
$$\leqslant\int_0^t\int|\partial_x^\beta u|^2\frac{\tau^{\delta-1}}{(\delta-1)!}\,\xi_{\beta\gamma}\,dx\,d\tau+\frac{1}{2}\int_0^t\int\Bigg(\xi_{\beta\gamma t}-\partial_x^3(\xi_{\beta\gamma}a)$$
$$+\beta\,\partial_x^2(\xi_{\beta\gamma}\,\partial_x a)-\binom{\beta}{2}\partial_x(\xi_{\beta\gamma}\,\partial_x^2 a)$$
$$+\binom{\beta}{3}(\xi_{\beta\gamma}\,\partial_x^3 a)\Bigg)|\partial_x^\beta u|^2\frac{\tau^\delta}{\delta!}\,dx\,d\tau$$
$$+\int_0^t\int\partial_x^\beta uF_\beta\frac{\tau^\delta}{\delta!}\xi_{\beta\gamma}\,dx\,d\tau. \tag{3.20}$$

The weight functions have been chosen to satisfy two properties:

$$\text{(i)}\quad \xi_{\beta\gamma} \leqslant C_1(\beta)\eta_{\gamma+1}$$
$$\text{(ii)}\quad \left|\partial_t \xi_{\beta\gamma} - \partial_x^3(\xi_{\beta\gamma} a) + \beta\, \partial_x^2(\xi_{\beta\gamma}\, \partial_x a) - \binom{\beta}{2} \partial_x(\xi_{\beta\gamma}\, \partial_x^2 a) + \binom{\beta}{3} (\xi_{\beta\gamma}\, \partial_x^3 a)\right| \leqslant C_1(\beta)\, \eta_{\gamma+1}. \tag{3.21}$$

The lemma will follow from a double induction on β and γ. Let $\gamma \geqslant -1$ and $\delta \geqslant \sigma$, and assume that there exists $C_3(\sigma, \gamma, \delta)$ such that

$$\int_0^t \int |\partial_x^\sigma u|^2\, \tau^\delta \eta_\gamma\, dx\, d\tau \leqslant C_3 \int_0^t \int |u(x, \tau)|^2 (1 + \tau^\delta)\, \eta_{\gamma+\sigma}\, dx\, d\tau \tag{3.22}$$

holds for all $\sigma \leqslant \beta$. Turn to (3.20) and use properties (3.21) to estimate the right hand side. The result is that

$$\int_0^t \int |\partial_x^{\beta+1} u|^2\, \tau^\delta \eta_\gamma\, dx\, d\tau \leqslant \int_0^t \int (\delta + \tau)\, C_1(\beta)\, |\partial_x^\beta u|^2\, \tau^{\delta-1} \eta_{\gamma+1}\, dx\, d\tau + \int_0^t \int F_\beta^2 \tau^\delta \xi_{\beta\gamma}\, dx\, d\tau. \tag{3.23}$$

That is, one additional derivative is estimated in terms of lower derivatives and lower powers of τ, at the cost of a more stringent weight, $\eta_{\gamma+1}$.

Finally, use Lemma 3.3, the fact that $\delta \geqslant \beta$, and the induction hypothesis to conclude that

$$\int_0^t \int |\partial_x^{\beta+1} u|^2\, \tau^\delta \eta_\gamma\, dx\, d\tau \leqslant C_3(\beta + 1, \gamma, \delta) \int_0^t \int |u|^2 (1 + \tau^\delta)\, \eta_{\gamma+\beta+1}\, dx\, d\tau.$$

An easy double induction in β and γ, keeping $\delta \geqslant \beta$ proves the lemma. The induction argument continues through indices β, γ until $\beta = \alpha$, when either (3.16) fails to hold or $\partial_x^\alpha a$ loses smoothness. If all derivatives are able to be estimated in Sobolev norm, the Sobolev inequality provides supremum norm estimates on the solution for any time $t > 0$. ∎

For coefficient $a(x, t)$ which are analytic in x in the strip $|\mathrm{im}\, x| < p$, it can be shown that $C_3(\alpha, \gamma, \delta) \sim (\alpha!)^3 C^{(2/3)\alpha^2}$. In fact, for $a(x, t) = a$, a constant, one finds that $C_3(\alpha) \sim C^\alpha$. With Lemma 3.5 this provides a roundabout proof that the Airy function, or any solution to the Airy equation with L^2 initial data supported on the right 1/2 line, is analytic for any positive t. It remains open whether the solution $u(x, t)$ to problem (3.1) becomes analytic whenever the coefficient $a(x, t)$ is taken to be real analytic.

4. Periodic Results

Consider the variable coefficient $a(x, t)$ to be 2π-periodic in the spatial variable, and consider the initial value problem on the interval with periodic boundary conditions.

$$\begin{aligned} &\partial_t u = a(x, t)\, \partial_x^3 u, \qquad 0 \leqslant t < \infty \\ &u(x, 0) = f(x) \qquad \text{initial data} \\ &u(x, t) = u(x + 2\pi, t). \end{aligned} \tag{4.1}$$

The existence, uniqueness, and Sobolev regularity theorem, the analog of Theorem 3.1, can be demonstrated via parallel arguments to those of Lemmas 3.1, 3.3, and 3.4.

THEOREM 4.1. *Let* $|a(x, t)|_{C,\alpha}$, $|\partial_t a|_{LM^\infty}$, *for* $\alpha \geqslant 3$. *For* $f(x) \in H^\alpha(S^1)$ *there exists a unique solution to the initial value problem* (4.1), *satisfying the Sobolev estimate*

$$\|u(x, t)\|_{H,\alpha} \leqslant \exp(C_1(\alpha) t)\, \|f(x)\|_{H,\alpha},$$

if the coefficient $a(x, t)$ *and the data* $f(x)$ *are* C^∞, *then the solution is also.*

The analog of Lemma 3.2 also holds in the periodic case; that is, there is a mild increase in smoothness of the solution over the initial data. There is, however, no analog of the dispersive smoothing of Theorem 3.2. In fact, the spatially periodic nature of the solutions precludes this. Much the same regularity results hold for the nonlinear KdV equation [1, 5]. In the first reference, for distributional initial data of compact support on the line, solutions of the KdV equation become C^∞ for any positive time; this is demonstrated by the inverse spectral transform. However, within the class of periodic solutions the Sobolev regularity of the solution of the KdV is identical to the regularity of the initial data.

5. Additional Cases

The second example of Section 2, (2.4), is a special case of a dispersive problem where the coefficient of dispersion may vanish. This happens in various physical problems, including (1.3), the KdV with varying surface tension. In this section, we present well-posedness and ill-posedness results for the initial value problem

$$\begin{aligned} \partial_t u &= a(x)\, \partial_x^2 u, \qquad x \in \mathbf{R},\ t > 0 \\ u(x, 0) &= f(x), \end{aligned} \tag{5.1}$$

where the dispersion coefficient changes sign at $x = 0$. As one expects, these parallel the result of Lemma 2.2, in that the problem is ill-posed if $\partial_x a(0) > 0$, and only well-posed if $\partial_x a(0) \leqslant 0$, and $x = 0$ is the only sign change in $a(x)$. Problems for which $a(x)$ changes sign more than once are thus ill-posed.

LEMMA 5.1. *Let $a(x) \in C^\infty(\mathbf{R})$ be coefficient of dispersion for problem* (5.1), *which changes sign at $x = 0$, with $a(0) = 0$, $\partial_x a(0) = 1$. Then the initial value problem is ill-posed. That is, for any choice of indices $3/2 < \beta \leqslant \gamma$ and any interval $[0, T)$ there is no continuous functions $C_{\beta\gamma}(t) \in C([0, T))$ such that*

$$\|u(x, t)\|_{H, \beta} \leqslant C_{\beta\gamma}(t)\, \|f(x)\|_{H, \gamma}. \tag{5.2}$$

Proof. The simplest demonstration is via a contradiction argument, comparing solutions $u(x, t)$ with closed form solutions $v(x, t)$ to the problem (2.4) in which $a(x) = x$. Suppose that estimate (5.2) with $\gamma \geqslant 2$. For $f(x) \in H^\gamma(\mathbf{R})$ such that $\hat{f}(k) \in C^\infty(\mathbf{R})$, $\operatorname{supp}(\hat{f}) \subseteq [-R, R]$, let $u(x, t)$ be the solution to the initial value problem (5.1) and let $v(x, t)$ solve (2.4). The difference between them $w(x, t) = (u - v)(x, t)$ satisfies

$$\begin{aligned} \partial_t w - a(x)\, \partial_x^3 w &= (a(x) - x)\, v(x, t) \\ w(x, 0) &= 0. \end{aligned} \tag{5.3}$$

Duhamel's principle is used to express $w(x, t)$ in terms of $v(x, t)$:

$$w(x, t) = \int_0^t S(t - \tau)(a(x) - x)\, v(x, \tau)\, d\tau,$$

where $S(t)$ is the solution operator for (5.1). Using estimate (5.2),

$$\|w(x, t)\|_{H, \beta} \leqslant \int_0^t C_{\beta\gamma}(t - \tau)\, \|(a(x) - x) v\|_{H, \gamma}\, d\tau.$$

The explicit solution for $v(x, t)$ is used to analyze the right hand side. Write $a(x) - x = b(x)\, x^\sigma$, with $\sigma \geqslant 2$ and $b(x)$ bounded. Then

$$(a(x) - x)\, v(x, t) = \frac{b(x)}{\sqrt{2\pi}} \int x^\sigma e^{ix\xi/(1-2\xi^2 t)^{1/2}} \hat{f}(\xi)\, d\xi. \tag{5.4}$$

Usung that $\exp(ix\xi/(1-2\xi^2 t)^{1/2}) = ((1-2\xi^2 t)^{3/2}/ix)\, \partial_\xi \exp(ix\xi/(1-2\xi^2 t)^{1/2})$, integrate by parts σ-many times in (5.4)

$$\begin{aligned} b(x)\, x^\sigma v(x, t) &= \frac{b(x)}{\sqrt{2\pi}} \int e^{ix\xi/(1-2\xi^2 t)^{1/2}} x^\sigma \left(-\partial_\xi \frac{(1-2\xi^2 t)^{3/2}}{ix} \right)^\sigma \hat{f}(\xi)\, d\xi \\ &= \frac{b(x)}{\sqrt{2\pi}} \int e^{ix\xi/(1-2\xi^2 t)^{1/2}} p_\sigma(\xi, (1-2\xi^2 t), \partial_\xi)\, \hat{f}(\xi)\, d\xi. \end{aligned} \tag{5.5}$$

The term $p_\sigma(\xi, \eta, \partial_\xi)$ is a canonical polynomial in $(\xi, \eta, \partial_\xi)$ of at most order 3σ in ∂_ξ, containing an overall factor of $(1-2\xi^2 t)^{\sigma/2}$. Since each x-derivative brings from the exponent a factor of $i\xi/(1-2\xi^2 t)^{1/2}$, we can control $\partial_x^\sigma b(x)\, x^\sigma v(x, t)$;

$$\|b(x)\, x^\sigma v(x, t)\|_{H,\gamma} \leqslant C_\gamma \|(1 + |x|^\sigma)\, f(x)\|_{H,\gamma+3\sigma}, \tag{5.6}$$

for $0 \leqslant \gamma \leqslant \sigma$. Thus

$$\begin{aligned} \|w(x, t)\|_{H,\beta} &\leqslant \int_0^t C_{\beta\gamma}(t-\tau)\, \|b(x)\, x^\sigma v(x, \tau)\|_{H,\gamma}\, d\tau \\ &\leqslant \tilde{C}_{\beta\gamma}(t)\, \|(1 + |x|^{3\sigma})\, f(x)\|_{H,\gamma+\sigma}. \end{aligned}$$

Take $f(x)$ from among the initial data suggested above, with $\operatorname{supp}(\hat{f}) \subseteq [-R, R]$; then the right hand side is bounded. On the other hand, $\|v(x, t)\|_{H,\beta}$ blows up as $t \to 1/2R^2$, hence $\|u(x, t)\|_{H,\beta}$ does as well, violating any possible constant in (5.2).

This argument works for $3/2 < \beta \leqslant \gamma \leqslant \sigma$. For any $\gamma - \beta \leqslant \sigma$ one obtains a similar contradiction using a scaling argument as $t \to 1/2R^2$. To prove the lemma with a bigger gap in differentiability, $\gamma - \beta > 2$, a related argument is used. However, one works with weighted norms in order to control the right hand side of (5.3). ∎

We use variations of the standard energy estimate of Section 3 to prove the well-posedness results. These results are possible if the coefficient of dispersion $a(x)$ changes sign at most once, and is otherwise bounded away from zero.

LEMMA 5.2. *Consider problem* (5.1) *with coefficient of dispersion* $a(x) \in C^\infty(\mathbf{R})$ *such that* $a(0) = 0$, $\partial_x a(x) \leqslant 0$ *for* $-1 \leqslant x \leqslant 1$, $a(x) > 0$ *for*

$x<0$ (respectively $a(x)<0$ for $x>0$) and there exists a constant $C \geqslant 1$ such that $1/C \leqslant a(x) \leqslant C$ for $x \leqslant -1$, and $-C \leqslant a(x) \leqslant -1/C$ for $x \geqslant 1$. Then the initial value problem is well-posed. In particular, the estimate holds

$$\|u(x, t)\|_{L^2} \leqslant \exp(C_0 |a|_{C,3} t)\, \|f(x)\|_{L^2}. \tag{5.7}$$

Proof. We will just give an indication of the proof by deriving the basic weighted Sobolev estimates satisfied by solutions of (5.1). Let $\xi(x)$ be a bounded smooth weight function, and $u(x, t)$ a smooth solution. Then

$$\tfrac{1}{2}\, \partial_t \int u^2 \xi \, dx = \int \frac{3}{2}\, \partial_x(a\xi)(\partial_x u)^2 \, dx - \int \frac{1}{2}\, \partial_x^3(a\xi)\, u^2 \, dx. \tag{5.8}$$

The basic criterion for a successful weight function is that $\partial_x(a\xi) \leqslant 0$. Let $\eta(x) \in C_0^\infty(\mathbf{R})$, $\operatorname{supp}(\eta) \subseteq [-1, 1]$ be a cutoff function such that $0 \leqslant \eta(x) \leqslant 1$ with $\eta(0)=1$ and $\partial_x \eta(x) \geqslant 0$ for $x<0$, $\partial_x \eta(x) \leqslant 0$ for $x>0$. Define $\xi(x) = (1-\eta(x))/|a(x)| + C\eta(x)$; then $\partial_x(a\xi) = 0$ for $x \notin [-1, 1]$. For $-1 \leqslant x \leqslant 1$

$$\partial_x(a\xi) = \partial_x \eta (Ca(x) + \operatorname{sgn}(x)) + C\eta(x)\, \partial_x a(x). \tag{5.9}$$

If $C>0$ is chosen sufficiently small so that $\max_{|x| \leqslant 1} |Ca(x)| \leqslant 1$, then expression (5.9) is nonpositive, and estimate (5.7) follows from the identity (5.8).

Sobolev estimates for higher derivatives proceed similarly; the method mimics the estimates of Section 3, using weight functions $\xi_\alpha = (a(x)\, \chi(x))^{2\alpha/3 - 1} + C\eta(x)$. The function $\chi(x) = -\tanh(x)$, or some other smooth approximation of $-\operatorname{sgn}(x)$. When $\alpha \geqslant 2$ the constant $C=0$, and we obtain weighted Sobolev norms instead. The procedure is straightforward, following Section 3 so we omit the details here. The result is that

$$\|\partial_x^\alpha u(x, t)\, \xi_\alpha^{1/2}\|_{L^2} \leqslant \exp(C_\alpha t)\, \|f(x)\|_{H,\alpha}. \quad \blacksquare \tag{5.10}$$

The last result that we discuss has to do with the localizing effect of zeros of the dispersion coefficient. Dispersive problems such as (1.1) have infinite propagation speed, and we have seen that localized data for a dispersive problem have nonetheless a global effect on the solution. In fact, the global properties of the behavior of high frequency components of the solution are responsible for the dispersive smoothing effect. A zero of the dispersion coefficient modifies this global property, localizing solutions on one side or other of this point of degeneracy.

LEMMA 5.3. *Let $a(x) \in C^\infty(\mathbf{R})$ vanish at the origin, with $\partial_x a(0) \leqslant 0$ and $a(x)<0$ for $x>0$ (bounded as in the hypothesis of Lemma 5.2). If initial* *data $f(x)$ are given for problem* (5.1) *such that $f(x)=0$ for $x>0$, then the solution $u(x, t)$ also vanishes to the right of the origin.*

Since dispersive eqautions of Airy type are third order, it might be thought that the coefficient need vanish to second order at its zero in order to achieve a localizing effect. In fact, in this well-posed case it suffices to have a simple zero.

Proof. In the integrations by parts to achieve the identity (5.8), this time modify the procedure by integrating only over the positive half-line. The new identity is

$$\tfrac{1}{2}\partial_t \int_0^\infty u^2\xi\,dx$$

$$= \int_0^\infty \tfrac{3}{2}\,\partial_x(a\xi)(\partial_x u)^2 - \tfrac{1}{2}\,\partial_x^3(a\xi)\,u^2\,dx$$

$$+ [(a\xi)(\partial_x^2 uu + \tfrac{1}{2}(\partial_x u)^2) - \partial_x(a\xi)u\,\partial_x u + \tfrac{1}{2}\,\partial_x^2(a\xi)\,u^2]|_0^\infty. \qquad (5.11)$$

Let $\eta(x)\in C^\infty(\mathbf{R})^+$ be any smooth function which coincides with $kx^2/2$ near the origin, while $\partial_x\eta(x)>0$ otherwise; then define $\xi(x)= -\eta(x)/a(x)$, the appropriate weight function. Clearly $\partial_x(a\xi)= -\partial_x\eta\leqslant 0$, and we have only to check that $|\partial_x^3(a\xi)/\xi|_{L^\infty}<\infty$, and that among smooth solutions $u(x, t)$ all boundary terms vanish. Thus

$$\partial_t \int_0^\infty u^2\xi\,dx \leqslant |\partial_x^3(a\xi)/\xi|_{L^\infty} \int_0^\infty u^2\xi\,dx, \qquad (5.12)$$

and the uniqueness theorem follows. ∎

References

1. A. Cohen, Solutions of the Korteweg de Vries equation, *in* "Nonlinear Partial Differential Equations in Engineering and Applied Science" (R. Sternberg, Ed.), Dekker, New York, 1980.
2. P. Constantin and J. C. Saut, Effets régularisants locaux pour des équations dispersives générales, *Math. C. R. Acad. Sci. Paris Sér. I* **304** (1987), 407–410.
3. P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.* **1**, No. 2 (1988), 413–439.
4. A. Cordoba and C. Fefferman, Wave packets and Fourier intergral operators, *Comm. Partial Differential Equations* **3**, No. 11 (1978), 979–1005.
5. T. Kato, On the Cauchy problem for the (generalized) Korteweg deVries equation, *in* "Studies Appl. Math.," pp. 93–128, Adv. Math. Suppl. Studies, Vol. 18, Academic Press, Orlando, FL, 1983.
6. T. Kappeler, Solutions to the Korteweg deVries equation with irregular initial profile, *Comm. Partial Differential Equations* **11**, No. 9 (1986), 927–945.

7. J. B. KELLER, A geometrical theory in diffraction, *in* "Calculus of Variation and Its Applications" (L. M. Graves, Ed.), pp. 27–52, Proceedings of Symposia on Applied Math., Vol. 8, Amer. Math. Soc., Providence, RI, 1958.
8. S. N. KRUZHKOV AND A. V. FAMINSKII, Generalized solutions to the Cauchy problem for the Korteweg deVries equation, *Math. USSR-Sb.* **48** (1984), 93–138.
9. G. PONCE, Regularity of solutions to nonlinear dispersive equations, *J. Differential Equations*, in press.
10. R. SACHS, Classical solutions of the Korteweg deVries equation for nonsmooth initial data via inverse scattering, *Comm. Partial Differential Equations* **10**, No. 1 (1985), 29–98.
11. G. WHITHAM, "Linear and Nonlinear Waves," Wiley–Interscience, New York, 1974.