Fully discrete traveling waves from semi-discrete traveling waves

S. Benzoni-Gavage

May 20th, 2003

The purpose of this note is to state and sketch the proof of Theorem B in [4]. For the reader's convenience we adopt the same notations as Chow, Mallet-Paret and Shen. Their result concerns a general Lattice Dynamical System (LDS)

$$\dot{x} \,=\, F(x)\,, \tag{1}$$

where F is a smooth function in $\mathcal{X} \,=\, \ell^\infty(\mathbb{Z}, \mathbb{R}^d)$ that commutes with the shift operator,

$$S \,:\, x \,\mapsto\, Sx\,;\, (Sx)_j \,=\, x_{j-1}\,,$$

and the fully discrete counterpart of (1) obtained by Euler discretization

$$x^{n+1} \,=\, x^n \,+\, h\,F(x^n)\,. \tag{2}$$

This is called a Coupled Map Lattice, associated with the map

$$G_h \,:\, x \,\mapsto\, G_h(x) \,:=\, x \,+\, h\,F(x)\,.$$

The result of Chow, Mallet-Paret and Shen reported here shows that spectrally stable traveling wave solutions to (1) give rise to traveling wave solutions to (2) for small enough h. Their spectral stability requirement needs some explanation. Assume that $x = p(t)$ is a *traveling wave* solution of (1), of positive speed c, i. e. $p_j(t) \,=\, \varphi(j \,-\, c\,t)$ for every $j \in \mathbb{Z}$ and $t \in \mathbb{R}$. Introducing the "return time" $T \,=\, 1/c$, a traveling wave of speed c is characterized by

$$p(t+T) \,=\, S\,p(t)\,,$$

and the corresponding function φ is uniquely determined by

$$\varphi(y) \,=\, p_0(-y\,T)\,.$$

Given a traveling wave solution of (1), its derivative, $\dot{p}$, is a traveling wave solution of the variational linear system

$$\dot{x} \,=\, \mathrm{D}F(p)\cdot x\,. \tag{3}$$

Denoting by $A(t, t_0)$ the solution operator of (3), we thus infer that

$$A(T, 0)\cdot \dot{p} \,=\, S\,\dot{p}\,.$$

In other words, $\dot{p}$ is an eigenvector of the operator

$$R \,:=\, S^{-1}\, A(T,0)\,,$$

associated with the eigenvalue 1. It is the operator R that encodes the spectral stability of the wave p. This is rather natural, since we easily prove by induction that

$$R^m \,=\, S^{-m}\, A(mT,0)\,,$$

and S is isometric in $\mathcal{X}$.

Definition 1 *The wave $x = p(t)$ is said spectrally stable if and only if*

- *the spectrum of the operator R lies in $\{\,\zeta \in \mathbb{C}\,;\, |\zeta| \,<\, 1\,\} \,\cup\, \{1\}\,,$*
- *the eigenvalue 1 is simple and isolated in the spectrum of R.*

Theorem 1 (Chow, Mallet-Paret, Shen) *Suppose that $x = p(t)$ is a spectrally stable traveling wave solution of* (1) *such that*

$$\lim\,\inf_{t\to\pm\infty}\, \|\, p(t) \,-\, p(0)\,\| \,>\, 0\,.$$

Then there is a positive h_0 so that for $0 \,<\, h \,\leq\, h_0$, there exists a smooth one-dimensional manifold M_h, close to $M \,:=\, \{\,p(\theta)\,;\, \theta \in \mathbb{R}\}$ in $\mathcal{X} \,=\, \ell^\infty(\mathbb{Z},\mathbb{R}^d)$, which is invariant under the CML (2)*. Moreover, this manifold contains traveling wave solutions of* (2)*, of speed ρ_h close to $c\,h$.*

It is remarkable that the speeds of the fully discrete waves obtained this way are either rational or irrational. In the latter case, the result crucially relies on the smoothness assumption on F. To be precise, it is required that F be $\mathcal{C}^3$, in order to apply Denjoy's theorem on $\mathcal{C}^2$ circle diffeomorphisms, as it should be clear from the sketch of the proof below.

Sketch of proof.

Step 1 : change of coordinates. The proof is based on a change of coordinates inspired from the study of periodic solutions in (finite dimensional) ODEs. Using the connectedss of GL($\mathcal{X}$), the authors first prove the following.

Lemma 0.0.1 *If F is of class $\mathcal{C}^r$, there exists $Z \in \mathcal{C}^r(\,\mathbb{R}, GL(\mathcal{X})\,)$ such that*

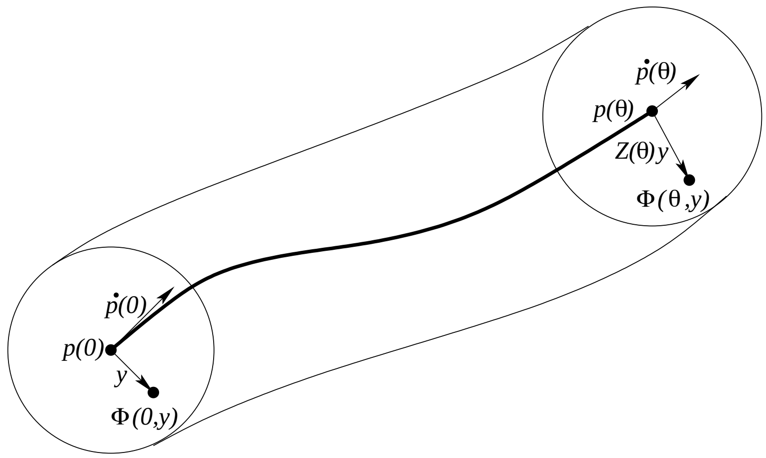
$$Z(0) \,=\, I\,, \quad Z(\theta + T) \,=\, S\,Z(\theta)\,, \quad Z(\theta)\,\dot{p}(0) \,=\, \dot{p}(\theta)$$

for every $\theta \in \mathbb{R}$.

We shall not reproduce here the proof of this technical result. It is more interesting to see how it can be used to compare the flow of (1) around p with the flow of (2).

A useful change of coordinates is obtained by choosing $\nu \,\in\, \mathcal{X}'$ – the dual space of $\mathcal{X}$ – normalized in such a way that

$$\langle\, \nu\,,\, \dot{p}(0)\,\rangle \,=\, 1\,.$$



Then, considering $\mathcal{Y} = \nu^\perp$ and the $\mathcal{C}^r$ map

$$\begin{array}{rccl} \Phi: & \mathbb{R}\times\mathcal{Y} & \to & \mathcal{X} \\ & (\theta, y) & \mapsto & \Phi(\theta, y) := p(\theta) + Z(\theta)\cdot y\,, \end{array}$$

we have local coordinates around the manifold $M = \{\, p(\theta)\,;\ \theta \in \mathbb{R}\}$.

In these coordinates the LDS (1) reads

$$\left\{ \begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \Theta(\theta, y)\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= Y(\theta, y)\,, \end{aligned} \right. \tag{4}$$

where the functions Θ and Y are implicitly defined by

$$\mathrm{D}\Phi(\theta, y)\cdot(\Theta, Y) = F(\,\Phi(\theta, y)\,)\,,$$

i. e.

$$\Theta\,\dot{p}(\theta) + (\,\mathrm{D}Z(\theta)\cdot\Theta\,)\cdot y + Z(\theta)\cdot Y = F(\,\Phi(\theta, y)\,)\,. \tag{5}$$

We clearly have

$$\Theta(\theta, 0) = 1 \quad \text{and} \quad Y(\theta, 0) = 0\,,$$

which expresses that $(\theta(t), y(t)) = (t, 0)$, the coordinates of the wave $x = p(t)$, are solution of (4). In the (θ, y) coordinates, the invariant manifold M is just the straight line $\mathbb{R}\times\{0\}$.

We now call E the solution operator at time T of the (autonomous) system (4). Since the map F commutes with the shift operator S, and

$$Z(\theta + T) = S\,Z(\theta)\,, \quad p(\theta + T) = S\,p(\theta)\,,$$

we readily see from (5) that

$$\Theta(\theta + T, y) = \Theta(\theta, y) \quad \text{and} \quad Y(\theta + T, y) = Y(\theta, y)$$

for every (θ, y) with y close to 0. This periodicity property means that E actually operates on the manifold

$$\mathbb{R}/T\mathbb{Z}\times\mathcal{Y}\,,$$

and leaves invariant

$$V \;:=\; \mathbb{R}/T\mathbb{Z} \times \{0\}\,,$$

(each point of V being a fixed point of E). Additionally, V is a *normally hyperbolic* invariant manifold. Indeed, it is not difficult to see that $\mathrm{D}E(\theta_0, 0)$ is conjugated to R since

$$\mathrm{D}E(\theta_0, 0) \;=\; \mathrm{D}\Phi(\theta_0, 0)^{-1}\, S^{-1}\, A(\theta_0 + T, \theta_0)\, \mathrm{D}\Phi(\theta_0, 0)$$

and

$$S^{-1}\, A(\theta_0 + T, \theta_0)\, A(\theta_0, 0) \;=\; A(\theta_0, 0)\, S^{-1}\, A(T, 0) \;=\; A(\theta_0, 0)\, R$$

by definition of R. Because of the spectral assumption on R, this shows that V is normally hyperbolic for the map E.

Step 2 : Persistence of the invariant manifold. A recent work of Bates, Lu and Zeng [2, 3] has extended to infinite dimensional settings the persistence of normally hyperbolic invariant manifolds. Their result applies in particular to the map E and its invariant manifold V. Every map close to E thus admits a unique invariant manifold close to V.

Now the CML (2) can also be rewritten in the (θ, y) coordinates. Denoting

$$\Gamma_h \;=\; \Phi^{-1}\, G_h\, \Phi\,,$$

(2) is equivalent to

$$(\theta^{n+1}, y^{n+1}) \;=\; \Gamma_h(\theta^n, y^n)\,, \tag{6}$$

where $x^n \;=\; \Phi(\theta^n, y^n)$. Choosing N so that

$$N - 1 \;<\; \frac{T}{h} \;<\; N + 1\,,$$

standard estimates of the Euler method show that

$$\Gamma_h^N \;-\; E \;=\; \mathcal{O}(h)$$

in the $\mathcal{C}^{r-1}$ topology. Therefore, for h small enough, there exists a manifold V_h that is invariant for the map Γ_h^N. Furthermore, by a classical uniqueness argument, V_h is invariant under Γ_h itself. As a matter of fact, $\Gamma_h(V_h)$ is also an invariant manifold, and it is close to V because Γ_h is close to identity for small h, hence $\Gamma_h(V_h)$ coincides with V_h.

Step 3 : Dynamics on the perturbed manifold. Being close to V, the manifold V_h is the graph of some function ζ_h, i. e.

$$V_h \;=\; \{\, (\theta, \zeta_h(\theta)\,;\; \theta \in \mathbb{R}\,\}\,.$$

The invariance of V_h under Γ_h means there exists β_h so that

$$\Gamma_h(\,\theta, \zeta_h(\theta)\,) \;=\; (\,\beta_h(\theta), \zeta_h(\beta_h(\theta))\,)\,.$$

The map β_h is of class $\mathcal{C}^{r-1}$ and is close to identity for small h. Therefore it is a circle diffeomorphism

$$\beta_h \;:\; \mathbb{R}/T\mathbb{Z} \xrightarrow{\sim} \mathbb{R}/T\mathbb{Z}\,.$$

Of course, coming back to the original coordinates, we find that

$$M_h := \{ p(\theta) + Z(\theta) \cdot \zeta_h(\theta) ;\ \theta \in \mathbb{R} \}$$

is invariant under G_h, and we have by definition of β_h :

$$G_h(q_h(\theta)) = q_h(\beta_h(\theta)) , \quad q_h(\theta) := p(\theta) + Z(\theta) \cdot \zeta_h(\theta) .$$

Therefore,

$$x^n := q_h(\beta_h^n(\theta_0)) \tag{7}$$

is a solution of the CML (2) for every θ_0.

Step 4 : (7) **defines the searched traveling wave.** The proof consists in showing that (7) defines a traveling of speed equal to

$$\rho_h = \lim_{n \to \infty} \frac{\beta_h^n(\theta)}{n\,T} ,$$

which is independent of θ and called the *rotation number* of the circle $(\mathbb{R}/T\mathbb{Z})$ diffeomorphism β_h. The fact that ρ_h is close to $c\,h = h/T$ merely follows from the first order Taylor expansion of β_h :

$$\beta_h(\theta) = \theta + h\,\Theta(\theta, \zeta_h(\theta)) + \mathcal{O}(h^2) = \theta + h + \mathcal{O}(h^2) .$$

We shall repeatedly use the traveling wave identity satisfied by the map q_h,

$$q_h(\theta + T) = S \cdot q_h(\theta) .$$

Rational case. If $\rho_h = p/q$, $p \wedge q = 1$, then there exists θ_0 so that

$$\beta_h^q(\theta_0) = \theta_0 + p\,T .$$

Choosing this point θ_0 in (7), we see that

$$x^{n+q} = q_h(\beta_h^n(\beta_h^q(\theta_0))) = q_h(\beta_h^n(\theta_0 + pT)) =$$

$$= q_h(\beta_h^n(\theta_0) + pT) = S^p \cdot q_h(\beta_h^n(\theta_0)) = S^p \cdot x^n .$$

Irrational case. This is the trickiest one. The smoothness of β_h is here crucial. If $r \geq 3$, β_h is at least $\mathcal{C}^2$ and thus, by a well-known theorem of Denjoy (see for instance [1, 5]), β_h is topologically conjugated to a rotation. More precisely, there exists a homeomorphism η_h of $\mathbb{R}$, with

$$\eta(\theta + T) = \eta_h(\theta) + T$$

for every θ, such that

$$\beta_h = \eta_h^{-1}\, R_h\, \eta_h , \quad R_h(\theta) = \theta + \rho_h\, T .$$

Defining

$$\psi_h : \xi \mapsto \psi_h(\xi) := q_h(\eta_h^{-1}(\,\eta_h(\theta_0) - \xi\,T\,)) ,$$

and

$$P \,:\, x \,=\, (x_j)_{j\in\mathbb{Z}} \,\mapsto\, x_0\,,$$

some elementary computations show that the sequence defined in (7) satisfies the identity

$$x_j^n \,=\, P \cdot \psi_h(j \,-\, \rho_h\, n)\,.$$

As a matter of fact,

$$\eta_h^{-1}(\,\eta_h(\theta_0) \,-\, (j \,-\, \rho_h\, n)\,T) \,=\, \eta_h^{-1}(\,\eta_h(\theta_0) \,+\, \rho_h\, n\,T) \,-\, j\,T$$
$$=\, \eta_h^{-1}(\,R_h^n(\eta_h(\theta_0))\,) \,-\, j\,T \,=\, \beta_h(\theta_0) \,-\, j\,T\,,$$

hence

$$\psi_h(j \,-\, \rho_h\, n) \,=\, q_h(\,\eta_h^{-1}(\,\eta_h(\theta_0) \,-\, (j \,-\, \rho_h\, n)\,T)\,) \,=\, q_h(\,\beta_h(\theta_0) \,-\, j\,T\,) \,=$$
$$S^{-j}\cdot q_h(\,\beta_h(\theta_0)\,) \,=\, S^{-j}\cdot x^n\,.$$

The result thus follows from the obvious fact that

$$P\cdot S^{-j}\cdot x^n \,=\, x_j^n\,.$$

References

[1] V. I. Arnol′d. *Geometrical methods in the theory of ordinary differential equations*, volume 250 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York, 1983. Translated from the Russian by Joseph Szücs, Translation edited by Mark Levi.

[2] Peter W. Bates, Kening Lu, and Chongchun Zeng. Existence and persistence of invariant manifolds for semiflows in Banach space. *Mem. Amer. Math. Soc.*, 135(645):viii+129, 1998.

[3] Peter W. Bates, Kening Lu, and Chongchun Zeng. Invariant foliations near normally hyperbolic invariant manifolds for semiflows. *Trans. Amer. Math. Soc.*, 352(10):4641–4676, 2000.

[4] S.-N. Chow, J. Mallet-Paret, and W. Shen. Traveling waves in lattice dynamical systems. *J. Differential Equations*, 149(2):248–291, 1998.

[5] Jack K. Hale. *Ordinary differential equations*. Robert E. Krieger Publishing Co. Inc., Huntington, N.Y., second edition, 1980.