On the stability of large amplitude semi-discrete shock profiles by means of an Evans function in infinite dimensions

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Introduction

Sophisticated numerical schemes have been developed for hyperbolic systems of conservation laws, which often provide excellent results. However, their analysis (stability, convergence) is mostly limited to scalar conservation laws (apart from Glimm-type schemes). Regarding their qualitative properties, we often argue from analogy with viscous conservation laws. But very few results are really demonstrated (in the case of systems). In particular, the issue of discrete shocks is widely open, despite great efforts in the last decades. On one hand, discrete shocks of rational speed have been well understood since the work of Majda and Ralston [17]. On the other hand, the existence and stability of discrete shocks of irrational Diophantian speed have been shown by Liu and Yu [14, 15], for the (modified) Lax-Friedrichs scheme. But such restrictions on the discrete shock speed seem irrelevant to practical applications. We propose an alternative approach to tackle the problem. It is based on the observation that properties of a conservative scheme are mostly encoded in its flux, which corresponds to the discretization in space of the exact flux. The discretization in time is usually of Euler type, and should not lead to discriminate one scheme among others. This motivates the analysis of semi-discrete schemes, which seems more accessible than that of fully-discrete schemes. This approach is encouraged by the recent work of Chow, Mallet-Paret and Shen [6] on **Lattice Dynamical Systems**. They have shown that sufficiently stable semi-discrete traveling waves give rise to fully-discrete traveling waves, no matter the nature of the discrete speed (Theorem B in [6]). Translating to our context, the existence of spectrally stable semi-discrete shocks would show the existence of discrete shocks of arbitrary speed, for small enough Courant numbers.

Here we shall focus on the very simple "upwind" scheme, which applies to systems with characteristic speeds of one sign only. The reason for this (restrictive) choice will appear below. Semi-discrete upwind profiles can be viewed as heteroclinic orbits of a system of **Retarded Differential Equations** [11, 7]. We proved their existence in an earlier work [2], for Lax shocks of small strength. For the existence of semi-discrete profiles associated with more general schemes, which involve differential equations with both delay and advance, we refer to the recent work of Huot [10]. Assuming the existence of a profile, we are concerned here by its stability properties. Let us note that, by the translation invariance, one cannot expect asymptotic stability but only orbital stability (that is, with an asymptotic shift). This also holds true for the more classical viscous profiles, associated with a parabolic regularization of the conservation laws.

We concentrate on the **linearized stability** analysis, for shocks of arbitrary amplitude. The idea is to construct an **Evans function** and to proceed as Gardner and Zumbrun did for viscous profiles [9]. This function, introduced by Evans in the 70's [8], was developed and used by many authors in several contexts (see [1], [18], [12] and the references therein). It is an analytic function, usually denoted by D, that plays the role of the characteristic polynomial for infinite dimensional operators. It cannot be computed explicitly in general. However, in favorable cases, its low/high-frequency behaviors can be related to each other, in a way that determines the parity of the number of unstable modes. In case of translational invariance we always have $D(0) = 0$, hence the low-frequency behavior depends on $D'(0)$. We point out that the Evans function, especially its low-frequency behavior, is also a building block for the nonlinear stability analysis. We refer to the important work by Zumbrun and Howard concerning viscous profiles [19]. Furthermore, the requirement $D'(0) \neq 0$ is undoubtedly related to one of the basic assumptions made by Chow, Mallet-Paret and Shen [6], even though the relationship between their approach and ours still needs some clarification.

The first section of this paper is devoted to the construction of an Evans function for semi-discrete profiles. It relies on a dynamical system formulation of the eigenvalue equations for the linearized operator about the wave. This looks like the classical approach, but there is an additional difficulty. In the work by Gardner and Zumbrun, the operator about the wave is a standard second order differential operator, and the dynamical system is just the first order version of a second order ODE. In our case, the operator about the wave is a **retarded differential operator** and yields an **infinite dimensional** dynamical system. However, we shall take advantage of the fact that the **unstable manifold** of this dynamical system is **finite dimensional**. This is where the special choice of the scheme is involved. We hope that a finite dimension reduction can also be performed for more general schemes but we do not see how at the moment. We construct a "mixed type" Evans function by using the adjoint dynamical system (see [5] for a detailed presentation of a similar Evans function in the context of viscous profiles).

In Section 2, we shall derive stability conditions from this Evans function. This will be performed by relating the adjoint dynamical system, which is rather unusual and difficult to handle, to the adjoint eigenvalue equations, which are nothing but **advanced differential equations**. For an extreme Lax shock, the stability condition that we obtain is the same as for viscous profiles associated with the "artificial" (or numerical) viscosity. This is in accordance with the common analogy. We also obtain a stability condition for undercompressive shocks. It is less explicit than for viscous profiles, due to the infinite dimensions. In this case, the analogy between semi-discrete and viscous profiles is not so clear.

This work was announced in [3]. An extended version can be found in [4].

1 The Evans function

1.1 Description of profiles

We consider a hyperbolic system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \tag{1}$$

where $u(x,t) \in \Omega \subset \mathbb{R}^N$, $f \in \mathcal{C}^2(\Omega; \mathbb{R}^N)$, and the semi-discrete system

$$\partial_t u + \frac{f(u(x,t)) - f(u(x-\Delta x, t))}{\Delta x} = 0, \tag{2}$$

where Δx is a fixed space step. A semi-discrete profile is, by definition, a traveling wave $u(x,t) = U\left(\frac{x-\sigma t}{\Delta x}\right)$, solution of Equation (2), attaining some limits $u_\pm$ at $\pm\infty$. Thus the function U must belong to $\mathcal{C}^1(\mathbb{R};\Omega)$ and satisfy the retarded differential equation

$$\sigma\, U'(y) = f(U(y)) - f(U(y-1)). \tag{3}$$

We see by integrating once Equation (3), that the existence of a semi-discrete profile implies the Rankine-Hugoniot condition

$$f(u_+) - f(u_-) = \sigma(u_+ - u_-). \tag{4}$$

From now on, we fix $(u_-, u_+, \sigma) \in \Omega \times \Omega \times \mathbb{R}$ such that (4) holds. Furthermore, we assume that the **constant states** $u_\pm$ are linearly **stable solutions** of both (1) and (2). This requires the hyperbolicity of the flux f in (1) and positive characteristic speeds. For simplicity we shall assume strict hyperbolicity.

(H) The jacobian matrices $A^\pm := df(u_\pm)$ have distinct positive eigenvalues,

$$0 < a_1^\pm < \cdots < a_N^\pm\,. \tag{5}$$

Equation (3) is a rather simple retarded differential equation, the occurrence of the delay only being local (or pointwise) (see [11, 7] for more general equations and related theory). As we shall see below, we may easily regard it as a dynamical system, which is nevertheless infinite dimensional. A semi-discrete profile can then be viewed as a heteroclinic connection. Because of the translational invariance, the corresponding fixed points are not hyperbolic. But, using the Rankine-Hugoniot condition (4) and the integrated form,

$$\sigma\, U(x) \;-\; \int_{-1}^{0} f(U(x+\theta))\, d\theta \;=\; \sigma u_\pm \;-\; f(u_\pm)\,, \tag{6}$$

of Equation (3) we can prove the exponential decay of a profile towards the endstates. This will be important for the low frequency analysis of the Evans function, in particular to show the analogous of the gap lemma of Gardner and Zumbrun [9].

Lemma 1 *If U is a semi-discrete profile, that is a global solution of* (3) *such that*

$$\lim_{x \to \pm\infty} U(x) = u_\pm ,$$

with $u_\pm$ satisfying (4) *and* (H), *and if there exist p, q, such that*

$$a^-_{p-1} < \sigma < a^-_p \,, \quad a^+_q < \sigma < a^+_{q+1} \,, \tag{7}$$

then there exist $C > 0$, $\alpha > 0$, such that

$$|U(x) - u_-| \leq Ce^{\alpha x} \quad \forall x \leq 0 \,, \qquad |U(x) - u_+| \leq Ce^{-\alpha x} \quad \forall x \geq 0 \,. \tag{8}$$

If $u_\pm$ are close enough, the case $p = q$ in (7) is known as the Lax shock case. A center manifold argument shows the existence of semi-discrete profiles in this case (and the exponential decays (8) are consequences of the proof), see [2]. In the following, we shall assume the existence of a profile and the inequalities (7), disregarding the amplitude and the nature of the discontinuity. Thus we shall treat undercompressive shocks (which are typically not of small strength) as well as Lax shocks.

1.2 The linearized operator about a profile

As usual in the analysis of traveling waves, we look at the stability in a moving frame. We also perform a rescaling. Through the change of variables $(x, t) \mapsto (\tilde{x} := (x - \sigma t)/\Delta x, \tilde{t} := t/\Delta x)$, the semi-discretized system (2) becomes

$$\partial_t u \, - \, \sigma \, \partial_x u \, + f(u(x,t)) - f(u(x-1,t)) = 0 \,, \tag{9}$$

where we have dropped the tildas for convenience. The profile U is then a stationary solution of (9). Linearizing Equation (9) about U yields

$$\partial_t w = L \cdot w := \sigma \, \partial_x w \, - A(x) w(x,t) + A(x-1) w(x-1,t) \,, \tag{10}$$

where $A(x) := df(U(x))$. We emphasize that, provided that (5) and (7) hold true, we have $\sigma > 0$. Without loss of generality we may even assume that $\sigma = 1$. The retarded differential operator L is then a closed unbounded operator on $L^2(\mathbb{R})$, with dense domain $H^1(\mathbb{R})$. We are concerned with the spectrum of L, which should lie in the left half-plane in order to ensure stability of the profile U.

1.3 Essential spectrum of L

We recall the definition $\sigma_{\text{ess}}(L) := \mathbb{C}\backslash\{\lambda\ ;\ L-\lambda \text{ is Fredholm}, \text{ind}(L-\lambda)=0\}$. Like for classical differential operators, $\sigma_{\text{ess}}(L)$ is linked to the spectra of the asymptotic operators, defined by $(L_{\pm}\cdot w)(x) := w'(x) - A^{\pm}(w(x)-w(x-1))$. We easily show by means of a Fourier transform that

$$\sigma(L_{\pm}) = \{\, \lambda \in \mathbb{C}\ ;\ \exists \xi \in \mathbb{R}\,,\ \det(i\xi - A^{\pm}(1-e^{-i\xi}) - \lambda) = 0\,\}.$$

Furthermore, elementary arguments yield the following.

Lemma 2 *Assuming* (H), *for all* $\lambda \in \mathbb{C}$ *such that Re* $\lambda > 0$, *the "characteristic equations"*

$$\det(\mu - A^{\pm}(1-e^{-\mu}) - \lambda) = 0 \tag{11}$$

do not have imaginary solutions $\mu \in i\mathbb{R}$, *and have exactly* N *solutions of positive real part, counting with multiplicity.*

As a consequence of Lemma 2 and the results in [13, 16], we infer

Theorem 1 *If* (H) *holds and* $\sigma > 0$, *then for all* $\lambda \in \mathbb{C}$ *such that Re* $\lambda > 0$, $L-\lambda$ *is Fredholm of index 0, that is,* $\lambda \notin \sigma_{ess}(L)$.

1.4 Eigenvalues

We are now concerned with possible unstable eigenvalues of L. For $\lambda \in \mathbb{C}$, λ is an eigenvalue of L if and only if there exists $w \in H^1(\mathbb{R};\mathbb{C}^N)$, $w \not\equiv 0$, such that

$$w'(x) \,=\, (A(x)+\lambda)\,w(x) \,-\, A(x-1)\,w(x-1) \tag{12}$$

for all $x \in \mathbb{R}$. Equation (12) is a (simple) variational retarded differential equation, which can be formulated as a dynamical system. This is the basic idea to construct an Evans function. We consider the Hilbert space

$$H_0 := L^2((-1,0);\mathbb{C}^N) \times \mathbb{C}^N\,,$$

and the dense subspace

$$H_1 := \left\{ \begin{pmatrix} \phi \\ c \end{pmatrix} \in H_0\,;\ \phi \in H^1((-1,0);\mathbb{C}^N) \text{ and } \phi(0) = c \right\}.$$

Associating to a (smooth enough) function w and $x \in \mathbb{R}$ the element of H_1,

$$W(x) := \begin{pmatrix} w_x : \theta \mapsto w(x+\theta) \\ w(x) \end{pmatrix}\,,$$

we obtain the following equivalence.

Proposition 1 *Let $\lambda \in \mathbb{C}$ be of positive real part. There exists a solution $w \in H^1(\mathbb{R};\mathbb{C}^N)$, $w \not\equiv 0$, of (12) if and only if there exists a solution $W \in \mathcal{C}(\mathbb{R};H_1) \cap \mathcal{C}^1(\mathbb{R};H_0)$, $W \not\equiv 0$, of the dynamical system*

$$\frac{dW}{dx} = \mathbb{A}(x;\lambda)\cdot W\,, \tag{13}$$

where

$$\begin{array}{llcc} \mathbb{A}(x;\lambda): & H_1 & \to & H_0 \\ & \begin{pmatrix} \phi \\ c \end{pmatrix} & \mapsto & \begin{pmatrix} d\phi/d\theta \\ (A(x)+\lambda)\,c \,-\, A(x-1)\,\phi(-1) \end{pmatrix}, \end{array} \tag{14}$$

with W tending to zero at $\pm\infty$.

Lemma 2 actually shows that both asymptotic operators

$$\begin{array}{llcc} \mathbb{A}^{\pm}(\lambda): & H_1 & \to & H_0 \\ & \begin{pmatrix} \phi \\ c \end{pmatrix} & \mapsto & \begin{pmatrix} d\phi/d\theta \\ (A^{\pm}+\lambda)\,c \,-\, A^{\pm}\,\phi(-1) \end{pmatrix} \end{array} \tag{15}$$

have unstable manifolds of dimension N for Re $\lambda > 0$. This suggests that the "unstable manifold" of (13) is of dimension N and that the "stable manifold" is of co-dimension N. This is the purpose of Theorem 2 below, where the "stable manifold" is characterized by means of the adjoint dynamical system.

1.5 Definition of the Evans function

Our basic result concerning the variational system (13) is the following.

Theorem 2 *We assume that* (H) *and (7) hold true with $\sigma = 1$. If $\lambda \in \mathbb{C}$ is of positive real part, there exist solutions $W_1(\cdot;\lambda), \cdots, W_N(\cdot;\lambda)$ of (13) and solutions $Z_1(\cdot;\lambda), \cdots, Z_N(\cdot;\lambda)$ of the adjoint dynamical system*

$$\frac{dZ}{dx} = -\mathbb{A}(x;\lambda)^*\cdot Z, \tag{16}$$

such that $W \in \mathcal{C}(\mathbb{R};H_1)\cap\mathcal{C}^1(\mathbb{R};H_0)$ is a solution of (13) decaying to 0 at $\pm\infty$ if and only if

$$W(x) \in \ \textit{Span}\ \{\,W_1(x;\lambda), \cdots, W_N(x;\lambda)\,\} \quad \cap \tag{17}$$

$$\{\,\Phi \in H_0\,;\ \langle Z_i(x;\lambda), \Phi\rangle = 0 \quad \forall i = 1, \cdots, N\,\}$$

for some $x \in \mathbb{R}$. Equivalently, $D(\lambda) = 0$, where D is defined by

$$D(\lambda) := \det(\langle Z_i(x;\lambda), W_j(x;\lambda)\rangle)_{1\le i,j\le N}\,. \tag{18}$$

The functions W_j, Z_i, and thus also D, depend analytically on λ on any compact subset of $\{\lambda;\ Re\ \lambda > 0\}$. Furthermore, they have analytic extensions to a neighborhood of $\lambda = 0$.

Note that the inner products $\langle Z_i(x;\lambda), W_j(x;\lambda)\rangle$ do not depend on x. The proof of this theorem is based on technical constructions, inspired from [9, 19], as well as on the notion of **exponential dichotomy** for solution operators (in infinite dimensions, see [13]). More specifically, the fundamental solutions W_j are constructed in such a way that, for Re $\lambda > 0$, we have

$$\lim_{x\to-\infty} e^{-\mu_j^-(\lambda)x} W_j(x;\lambda) \,=\, \Phi_j^-(\lambda) := \begin{pmatrix} \phi_j(\lambda) : \theta \mapsto e^{\mu_j^-(\lambda)\theta} r_j^- \\ r_j^- \end{pmatrix},$$

where the vector r_j^- denotes as usual the eigenvector of A^- associated with a_j^-. The element $\Phi_j^-(\lambda)$ of H_1 is an eigenvector of $\mathbb{A}^-(\lambda)$ associated with $\mu_j^-(\lambda)$, which is defined as the unique root of $\mu - a_j^- - \lambda + a_j^- e^{-\mu}$ of positive real part. Similarly, the fundamental solutions Z_i are characterized by

$$\lim_{x\to+\infty} e^{\overline{\mu_i^+(\lambda)x}} Z_i(x;\lambda) \,=\, \Psi_i(\lambda)\,,$$

where $\Psi_i(\lambda)$ are respectively eigenvectors of $\mathbb{A}^+(\lambda)^*$ associated with $\overline{\mu_i^+(\lambda)}$, $\mu_i^+(\lambda)$ being the root of positive real part of $\mu - a_i^+ - \lambda + a_i^+ e^{-\mu}$. The important point to note for the extension to $\lambda = 0$ is that, in view of (7), we have in the limit

$$\mu_j^-(0) \,=\, 0 \quad \forall j \le p-1\,, \quad \mu_i^+(0) \,=\, 0 \quad \forall i \le q\,.$$

This extension is shown similarly as the gap lemma in [9] (also see [19] for a more "practical" treatment).

2 Stability conditions

Once the Evans function is well defined, the derivation of stability conditions follows the same lines as for viscous profiles [5].

We require that sgn $D'(0)$ sgn $D(+\infty) \,\ge\, 0$, otherwise the (real valued) Evans function necessarily vanishes on the real axis. The aim is thus to compute $D'(0)$ and sgn $D(\lambda)$ for $\lambda \gg 1$. For this we need to properly handle the adjoint dynamical system (16), which is somewhat weird. In particular, we easily check that the domain of $\mathbb{A}(x;\lambda)^*$ depends on x. However, things are restored through the following remark. There is a one-to-one relation between solutions of (16) and solutions of the **adjoint eigenvalue equation** $L^* \cdot z = \bar{\lambda} z$, which equivalently reads

$$z'(x) \,=\, z(x+1)\,A(x) \,-\, z(x)\,(A(x) + \bar{\lambda})\,. \tag{19}$$

The proof of this fact is based on the observation that, if w is a solution of (12) and z is a solution of (19) then the quantity

$$\{z,w\}(x) \,:=\, z(x)\,\overline{w(x)} \,-\, \int_{-1}^{0} z(x+1+\theta)\,A(x+\theta)\,\overline{w(x+\theta)}\,d\theta$$

actually does not depend on x.

2.1 Lax shocks

Theorem 3 *Assuming (H), (4), and (7) with* $p = q = N$*, a necessary condition for a semi-discrete profile* U *associated with* (u_-, u_+, σ) *to be stable reads*

$$\ell_N^- \cdot (u_+ - u_-) \geq 0\,, \tag{20}$$

where ℓ_N^- *is a left eigenvector of* A^- *associated with* a_N^- *such that* $\ell_N^- \cdot U'(x) > 0$ *for* $-x \gg 1$.

The condition (20) is the same as for a viscous profile associated with the numerical viscosity matrix (see [9, 5]). It is always satisfied by shocks of small strength. On the other hand, if the Majda-Liu determinant, that is $\ell_N^- \cdot (u_+ - u_-)$ vanishes at some N-shock, then generically there are neighboring shocks that are strongly unstable (see [5], Section 5).

2.2 Undercompressive shocks

Theorem 4 *We assume (H), (4), and (7) with* $p = q + 1 = N$ *and* $\sigma = 1$*, supplemented with the transversality condition*

$$Span\,\{r_1^-, \cdots, r_{N-1}^-, r_N^+\} = \mathbb{R}^N. \tag{21}$$

A necessary condition for a semi-discrete profile U *associated with* (u_-, u_+, σ) *to be stable reads*

$$\gamma\, \ell_N^- \cdot r_N^+ \geq 0\,, \tag{22}$$

where the orientation of ℓ_N^- *is determined by the condition* $\ell_N^- \cdot U'(x) > 0$, $-x \gg 1$*, and* γ *is a "transversality coefficient". It is defined by*

$$\gamma := \lim_{x \to +\infty} \{z_N, v\}(x), \tag{23}$$

where z_N *is the unique solution (up to a positive constant factor) of*

$$z_N'(x) = z_N(x+1)A(x) - z_N(x)A(x)$$

such that $\lim_{x\to+\infty} z_N(x) = 0$ *and* $z_N(x) \cdot r_N^+ > 0$ *for* $x \gg 1$*, and* v *satisfies*

$$\begin{aligned} v(x) - \int_{-1}^{0} A(x+\theta)v(x+\theta)\,d\theta \ &= U(x) - u_- - \sum_{j=1}^{N-1} \gamma_j (1 - a_j^-) r_j^- \ , \\ &= U(x) - u_+ + \gamma_N r_N^+ \end{aligned}$$

the coefficients γ_j *being uniquely defined because of (21) (and (7)).*

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