Higher-Order Finite Elements for Hybrid Meshes Using New Pyramidal Elements

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Abstract

An arbitrarily high-order finite element space is given for pyramidal elements, such that these elements can be used in hybrid meshes which include a high percentage of hexahedra, and some tetrahedra, wedges and pyramids. Numerical results are given to demonstrate the efficiency of these new elements.

Introduction

Highly efficient finite element methods using hexahedral meshes have been developed by Cohen and others, but currently the only way to automatically generate unstructured hexahedral meshes for a complex geometry is to generate a tetrahedral mesh, and split each tetrahedron into four hexahedra, which introduce needlessly substantial increase in the cost. However, some mesh generators are able to produce hexahedral-dominant meshes that include a minor number of tetrahedra, wedges and pyramids. The aim here is to study finite element methods on hybrid meshes in order to preserve the efficiency of the method developed for hexahedra.

Finite elements for tetrahedra, hexahedra and wedges are detailed in Hesthaven [2], Cohen [1] and Šolín [3]. In this work, the main effort is devoted to the construction of pyramidal finite elements, preserving conformity with the other types of elements. Only few papers are dealing with pyramidal elements (Bedrosian [4], Graglia [5], Chatzi [6], Nigam and Phillips [7]) since obtaining a proper base for these elements is a tricky point.

1 Arbitrary High-Order Pyramidal Element

1.1 A Pyramidal Finite Element Space of Order r

We consider the transformation F given by Bedrosian [4] using rational fractions, transforming the reference pyramid $\hat{K}$ (taken as the unit symmetrical pyramid, centered at the origin, whose apex is on the z-axis) into any pyramid K of the mesh.

The finite element space V_h on an open set Ω of $\mathbb{R}^3$ is given by

$$V_h = \{u \in H^1(\Omega) \mid u_{|K} \in V_r^F\},$$

where V_r^F is the real space of order r for an element K of the mesh defined by

$$V_r^F = \{u \mid u \circ F \in \hat{V}_r\},$$

and the finite element space $\hat{V}_r$ of order r on $\hat{K}$ is $\mathbb{P}_r$ when K is a tetrahedron, $\mathbb{Q}_r$ when K is a hexahedron, $\mathbb{P}_r(\hat{x},\hat{y}) \otimes \mathbb{P}_r(\hat{z})$ when K is a wedge, and is as defined below when K is a pyramid.

To obtain a method of order r, the real space V_r^F for a pyramidal element K of the mesh must be such that

$$\mathbb{P}_r(x,y,z) \subset V_r^F.$$

We prove that this inclusion implies that $\hat{V}_r$ is defined by

$$\hat{V}_r = \mathbb{P}_r(\hat{x},\hat{y},\hat{z}) + \sum_{0\le k\le r-1} \left(\frac{\hat{x}\hat{y}}{1-\hat{z}}\right)^{r-k} \mathbb{P}_k(\hat{x},\hat{y}),$$

and is of dimension

$$\dim \hat{V}_r = \frac{1}{6}(r+1)(r+2)(2r+3).$$

With this choice of $\hat{V}_r$ for pyramidal elements, we check that a function u in V_h is continuous across the interface between elements, whatever the type of the elements adjacent to the face, and therefore belongs to $H^1(\Omega)$. The same local space $\hat{V}_r$ will be used for pyramidal elements for discontinuous Galerkin methods.

1.2 Location of the Degrees of Freedom

To link pyramids with other elements of the mesh, interpolatory basis functions are used with Gauss-Lobatto points on each quadrangle and Hesthaven points on each triangle [2]. The number of degrees of freedom on the faces n_f is

$$n_f = 3r^2 + 2.$$

We add n_i degrees of freedom inside the pyramid, placed on $(r-2)$ parallel planes of k^2 degrees of freedom

$$n_i = \frac{1}{6}(r-1)(r-2)(2r-3) = \sum_{1\le k\le r-2} k^2.$$

The total number of degrees of freedom $n_e = n_i + n_f$ for the pyramidal element is then equal to the dimension of $\hat{V}_r$.

2 Numerical Results

2.1 Dispersion

In order to study the pyramidal elements, a dispersion analysis is performed on the wave equation, relying on the computation of the phase error on infinite periodic meshes. The periodic cell is a cube that can be made of a single hexahedron; of two wedges; of two pyramids and two tetrahedra (hybrid); or of-six tetrahedra. The analysis has been carried out on periodic cells made up of distorted cubes in order to check the consistency of our method when the base of the pyramid is not a parallelogram : we obtain an $O(h^{2r})$ phase error for both regular and distorted meshes.

2.2 Stability

The stability condition (CFL) is also computed on a periodic infinite mesh. The CFL for each type of element is given in Table 1 for discontinuous Galerkin methods, up to order 3.

Table 1: CFL for regular meshes with discontinuous elements

Element	Order 1	Order 2	Order 3
Hexahedron	0.1583625	0.0745356	0.0442645
Wedge	0.1168082	0.0632206	0.0399083
Hybrid	0.0718421	0.0446415	0.0286480
Tetrahedron	0.0621769	0.0353414	0.0227479

2.3 Numerical Experiments

Numerical experiments are performed on the scalar wave equation. The scattering of a cone-sphere with an incident plane wave striking its tip is shown in the Fig. 1.

Whereas a pure hexahedral mesh (made of split tetrahedra) of order 3 with 1,077,000 degrees of freedom provides an error of 9.0 %, the hybrid mesh with 247,000 degrees of freedom gives an error of 7.7 %.

The elements have been extensively studied for the time-domain wave equation by using discontinuous Galerkin methods. In this case, not only does the hybrid mesh contains fewer degrees of freedom, but also the CFL of the hybrid mesh is larger than the CFL of the hexahedral mesh. A local time stepping scheme will be tested on these new elements.

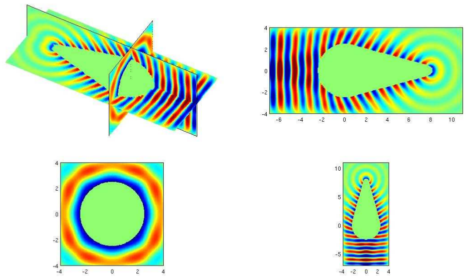

Figure 1: Real part of the field diffracted by a cone-sphere

Conclusion

Highly efficient pyramidal elements of any order are constructed using the finite element space $\hat{V}_r$. Tested up to order 6, these new elements are characterized by a low phase error, a quite good CFL, and a very good behaviour in a hybrid mesh.

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