High-Order Optimal Edge Elements for Pyramids, Prisms and Hexahedra

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Talk Abstract

Edge elements are a popular method to solve Maxwell's equations especially in time-harmonic domain. When non-affine elements are considered however, elements of the Nedelec's first family are not providing an optimal rate of the convergence of the numerical solution toward the solution of the exact problem in $H(curl)$-norm. We propose new finite element spaces for pyramids, prisms, and hexahedra in order to recover the optimal convergence. In the particular case of pyramids, a comparison with other existing elements found in the literature is performed. Numerical results show the good behaviour of these new finite elements.

Introduction

We are interested in the resolution of time-harmonic Maxwell's equations

$$-\omega^2 \varepsilon E + \text{curl}(\frac{1}{\mu}\text{curl} E) \;=\; f \tag{1}$$

where ε, μ may be heterogeneous. Many numerical methods have been developed to solve this equation. Using nodal finite elements is possible by considering either a weighted regularization [4] or a Discontinous Galerkin formulation [12]. A more natural choice consists of using edge elements, and considering the following variational formulation:

$$-\omega^2 \int_\Omega \varepsilon E \cdot \varphi \, dx + \int_\Omega \frac{1}{\mu} \text{curl} E \cdot \text{curl}\varphi \, dx \;=\; \int_\Omega f \cdot \varphi \, dx \tag{2}$$

We restrict ourselves to the study of elements of Nedelec's first family since optimal convergence of the convergence of the numerical solution toward the solution of the exact problem is sought in $H(curl)$-norm. These elements are well-known in the case of tetrahedra, prisms, and hexahedra (see for example [8]).

However, these elements are not providing an optimal convergence when non-affine hexahedra and prisms are considered, that is to say when the transformation from the reference element to the real element of the mesh is in $\mathbb{Q}_1$ instead of being affine. An improved finite element space has been proposed in [1] in 2-D for quadrilaterals, and in 3-D in [5] for first-order hexahedra. We propose here optimal finite element spaces in 3-D for hexahedra, prisms and pyramids at any order of approximation.

For pyramidal elements, we compare the spaces found with edge elements proposed in the literature, e.g. [13], [10], [7].

High-Order Optimal Finite Element Spaces

Let us introduce the following classical spaces:

$$\mathbb{Q}_{m,n,p} \;=\; Span\left\{ x^i y^j z^k, \quad \begin{array}{c} 0 \leq i \leq m \\ 0 \leq j \leq n \\ 0 \leq k \leq p \end{array} \right\}$$

$$\mathbb{P}_r \;=\; Span\left\{ x^i y^j z^k, \quad \begin{array}{c} i,j,k \geq 0 \\ i+j+k \leq r \end{array} \right\}$$

We also define the polynomial spaces for the tetrahedal edge element introduced by Nedelec [9]

$$\widetilde{\mathbb{P}}_r \;=\; Span\left\{ x^i y^j z^k, \quad \begin{array}{c} i,j,k \geq 0 \\ i+j+k = r \end{array} \right\}$$

$$\mathcal{S}_r \;=\; \left\{ u \in \widetilde{\mathbb{P}_r} \text{ so that } u_1 x + u_2 y + u_3 z = 0 \right\}$$

$$\mathcal{R}_r \;=\; \mathbb{P}_{r-1}^3 \oplus \mathcal{S}_r$$

and the approximation space for continuous pyramidal elements introduced in [3]

$$\mathbb{B}_r \;=\; \mathbb{P}_r(x,y,z) \;\oplus\; \sum_{k=0}^{r-1} \mathbb{P}_k(x,y) \left(\frac{xy}{1-z}\right)^{r-k}$$

The finite element approximation is constructed by considering a transformation F from a reference element $\hat{K}$ (unit tetrahedron, unit cube, etc) to a real element K on the mesh. This transformation writes

$$F = \sum_{1 \leq i \leq n_i} S_i \; \hat{\varphi}_i^1, \tag{3}$$

where n_i is the number of vertices of the element, $S_i = (x_i, y_i, z_i)$ are the vertices and $\hat{\varphi}_i^1$ are mapping functions depending on the considered type of the element.

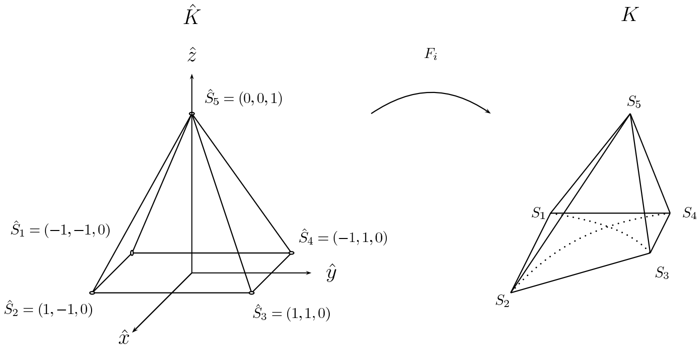

Figure 1: transformation F for a pyramid

For pyramidal elements, the mapping functions given in [2] using rational fractions, are the following:

$$\begin{cases} \hat{\varphi}_1^1 = \frac{1}{4}\left(1 - \hat{x} - \hat{y} - \hat{z} + \frac{\hat{x}\hat{y}}{1-\hat{z}}\right) \\ \hat{\varphi}_2^1 = \frac{1}{4}\left(1 + \hat{x} - \hat{y} - \hat{z} - \frac{\hat{x}\hat{y}}{1-\hat{z}}\right) \\ \hat{\varphi}_3^1 = \frac{1}{4}\left(1 + \hat{x} + \hat{y} - \hat{z} + \frac{\hat{x}\hat{y}}{1-\hat{z}}\right) \\ \hat{\varphi}_4^1 = \frac{1}{4}\left(1 - \hat{x} + \hat{y} - \hat{z} - \frac{\hat{x}\hat{y}}{1-\hat{z}}\right) \\ \hat{\varphi}_5^1 = \hat{z} \end{cases}$$

The obtained transformation is shown in Fig. 1

The electric field E and test function φ of the variational formulation (2) are belonging to the following finite element space:

$$V_h = \{u \in H(\text{curl}, \Omega) \text{ so that } u|_K \in P_r^F\}$$

where $P_r^F(K)$ denotes the finite element space on the real element K. This space is induced by the choice of the space $\hat{P}_r(\hat{K})$ thanks to Piola transform

$$P_r^F(K) = \{u \text{ so that } DF^*\, u \circ F \in \hat{P}_r(\hat{K})\}$$

This space $\hat{P}_r(\hat{K})$ is independent from the element K, since it depends only on the reference element $\hat{K}$ and on the order of approximation r. We denote by DF the jacobian matrix of transformation F.

In order to obtain an optimal convergence in $O(h^r)$ for $H(curl)$-norm, where h denotes the mesh size, a sufficient condition is that the space P_r^F includes all polynomials of space $\mathcal{R}_r$.

Definition 1 *$\hat{P}_r(\hat{K})$ is said to be optimal if for any element K, $P_r^F \supset \mathcal{R}r$*

After a careful examination of $DF^*p \circ F$ for all the polynomials p in $\mathcal{R}_r$, the following optimal finite element spaces have been obtained.

Theorem 1 *The optimal space $\hat{P}_r(\hat{K})$ is equal to:*
Tetrahedron:

$$\hat{P}_r(\hat{K}) = \mathcal{R}_r$$

Hexahedron:

$$\hat{P}_r(\hat{K}) = \mathbb{Q}_{r-1,r+1,r+1} \times \mathbb{Q}_{r+1,r-1,r+1} \times \mathbb{Q}_{r+1,r+1,r-1}$$

Prism:

$$\hat{P}_r(\hat{K}) = (\mathcal{R}_r(\hat{x},\hat{y}) \otimes \mathbb{P}_{r+1}(\hat{z})) \times (\mathbb{P}_{r+1}(\hat{x},\hat{y}) \otimes \mathbb{P}_{r-1}(\hat{z}))$$

Pyramid:

$$\hat{P}_r(\hat{K}) = \mathbb{B}_{r-1}^3$$

$$\oplus \left\{ \frac{\hat{x}^p \hat{y}^p}{(1-\hat{z})^{p+2}} \left| \begin{array}{l} \hat{y}(1-\hat{z}) \\ \hat{x}(1-\hat{z}) \\ \hat{x}\hat{y} \end{array} \right. ,\ 0 \le p \le r-1 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^m \hat{y}^{n+2}}{(1-\hat{z})^{m+2}} \left| \begin{array}{l} (1-\hat{z}) \\ 0 \\ \hat{x} \end{array} \right. ,\ 0 \le m \le n \le r-2 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^{n+2} \hat{y}^m}{(1-\hat{z})^{m+2}} \left| \begin{array}{l} 0 \\ (1-\hat{z}) \\ \hat{y} \end{array} \right. ,0 \le m \le n \le r-2 \right\}$$

$$\oplus \left\{ \frac{\hat{x}^p \hat{y}^q}{(1-\hat{z})^{p+q+1-r}} \left| \begin{array}{l} (1-\hat{z}) \\ 0 \\ \hat{x} \end{array} \right. ,\ \left| \begin{array}{l} 0 \le p \le r-1 \\ 0 \le q \le r+1 \end{array} \right. \right\}$$

$$\oplus \left\{ \frac{\hat{x}^q \hat{y}^p}{(1-\hat{z})^{p+q+1-r}} \left| \begin{array}{l} 0 \\ (1-\hat{z}) \\ \hat{y} \end{array} \right. ,\ \begin{array}{l} 0 \le p \le r-1 \\ 0 \le q \le r+1 \end{array} \right\}$$

The proposed optimal pyramidal space is completely new.

"Nodal" basis functions and hierarchical basis functions have been constructed for these spaces. For hexahedral elements, the same space as in [5] has been found for $r = 1$. The prismatic space can also be found as a combination of optimal quadrilateral and triangular elements.

Comparison With Other Pyramidal Elements

A pyramidal finite element space compatible with classical hexahedral and prismatic elements of the Nedelec's first family can also be constructed by replacing in theorem 1

$$0 \le p \le r-1, \quad 0 \le q \le r+1$$

by

$$0 \leq p \leq r-1, \quad 0 \leq q \leq r.$$

This modified space will be considered as the Nedelec's first family for pyramids, and denoted as $\hat{P}_r^1$. We notice that this modification removes all the components of degree $r+1$ so that $(r+1)^3$ quadrature points are sufficient to evaluate the integrals.

For $r = 1$, this space is exactly the same as proposed in [13], [7], [6], [10].

For $r = 2$, the basis functions proposed in [13] generate a finite element space that does not contain $\hat{P}_1$, and is included in $\hat{P}_3$. Numerical experiments show that the additional basis functions of second order are not improving the accuracy. Spurious modes are also observed and the basis functions proposed for the faces vanish completely on the other faces, whereas only tangential component should vanish. The basis functions proposed by [7] for $r = 2$ provide spurious modes as well, and the finite element space they generate do not contain $\hat{P}_1$. The use of the two basis functions proposed in [13] and [7] should then be avoided.

The finite element space proposed in [10] contains $\hat{P}_{r-1}$, but provides a sub-optimal convergence for non-affine pyramids, even if optimal convergence is recovered for affine pyramids. It is free from spurious modes. This finite element space is a good alternative for affine pyramids, despite the increase of the number of degrees of freedom when r is high. A second finite element space with reduced dimension has been proposed in [11], but this space does not contain $\hat{P}_1$, thus being non-consistent for non-affine pyramids. This second space also provides an optimal convergence for affine pyramids.

A dispersion analysis has been performed for the different finite element spaces for a mesh made of a repeated cell composed of non-affine pyramids and affine pyramids, as shown in Fig. 2. The dispersion error obtained for these kinds of meshes is displayed in Fig. 3.

We have also checked the accuracy of the source problem (1) for the same family of meshes, the error obtained between the numerical solution and a reference solution is displayed in Fig. 3. We can see that the optimal finite element space gives a better accuracy than the spaces described in [13], [10], [7].

Numerical Results

Our finite elements have been tested on general hybrid meshes, and they give accurate results as expected. We have for example used Nedelec's first family $\hat{P}_r^1$ for pyra-

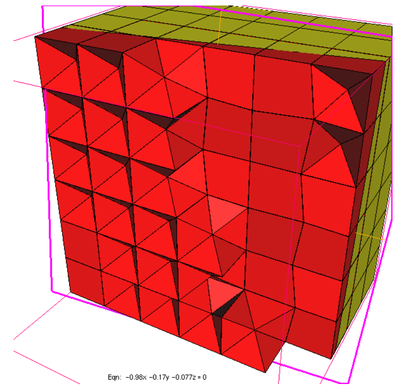

Figure 2: Pyramidal mesh used for the dispersion analysis

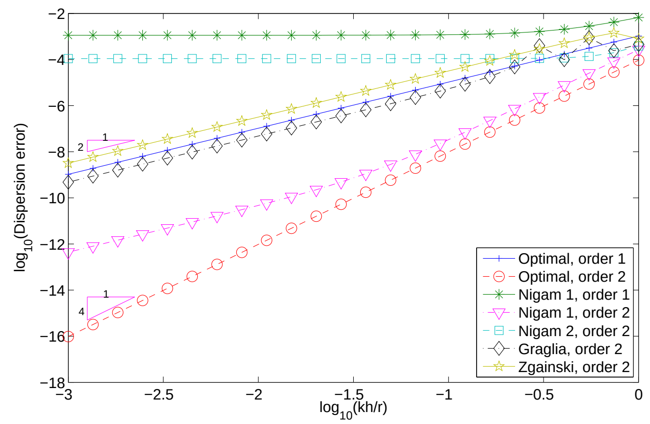

Figure 3: Dispersion error in log-log scale for a mesh comprising affines and non-affine pyramids.

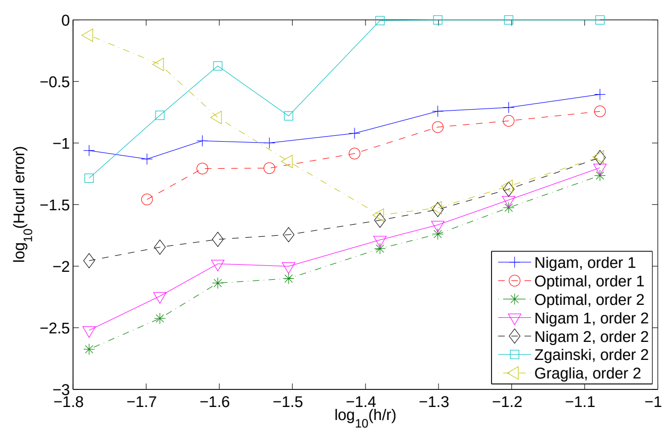

Figure 4: $H(curl)$-error in log-log scale for a mesh comprising affines and non-affine pyramids (Gaussian source inside a cubic cavity).

mids, tetrahedra and hexahedra on the geometry of an aircraft for which an hybrid mesh containing only affine elements can be generated (see Fig. 5).
The numerical solution obtained for $r = 2$ with 2.43 millions of degrees of freedom is displayed in Fig 6,

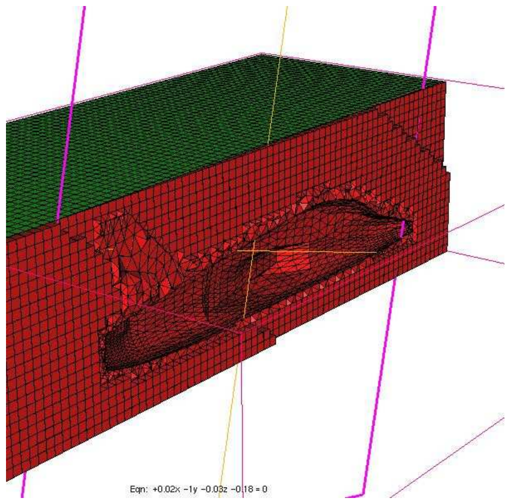

Figure 5: Mesh used for the scattering by an aircraft

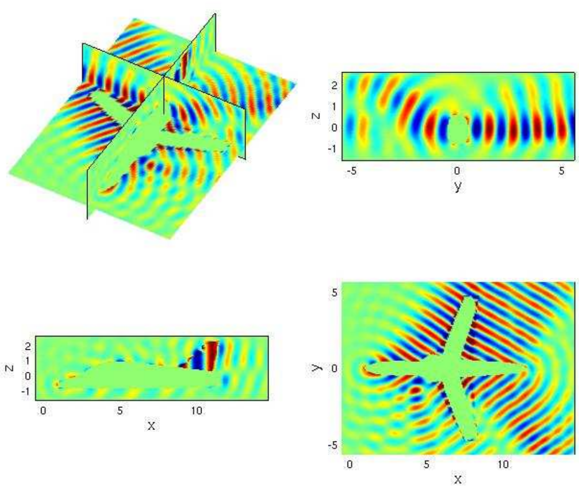

Figure 6: Scattering by an aircraft, real part of diffracted field E_3

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