Pyramidal Finite Elements for Hybrid Meshes

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Abstract

We propose new families of arbitrarily high-order finite elements defined on pyramids that can be used in hybrid meshes which include hexahedra, tetrahedra, wedges and pyramids. These finite elements will be used for equations requiring H^1 or $H(curl)$ spaces in continuous cases, and can also be used with discontinuous Galerkin methods. Numerical results in time domain demonstrate the efficiency of hybrid meshes compared to pure tetrahedral meshes or hexahedral meshes obtained by splitting tetrahedra into hexahedra.

1. Introduction

Higher-order finite elements have exhibited a very good efficiency for hexahedral elements [1], but the automatic generation of unstructured hexahedral meshes is still challenging. A solution consists of generating hexahedral dominant meshes including a small number of tetrahedra, triangular prisms and pyramids. We explain here how to construct high order pyramidal finite elements that are optimal in a sense that will be defined, and we compute numerical experiments in time domain on hybrid meshes.

2. General Scheme of the Equations

2.1 Space Discretisation

We consider time domains hyperbolic equations. After a space discretisation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} M_h U + K_h U = 0,$$

where

- $K_h = R_h + S_h$ in the discontinuous case;
- $K_h = R_h$ in the continuous case

and

- $(M_h)_{i,j} = \displaystyle\int_K \varphi_i \cdot \varphi_j \, dx$ is the mass matrix;
- $(R_h)_{i,j} = \displaystyle\int_K \sum_{1 \le k \le d} \left(A_k \frac{\partial \varphi_j}{\partial x_k} \cdot \varphi_i - B_k \varphi_j \cdot \frac{\partial \varphi_i}{\partial x_k} \right) dx$ is the stiffness matrix;
- $(S_h)_{i,j} = \displaystyle\int_{\partial K} \sum_{1 \le k \le d} \left(A_k n_k \left[\varphi_j\right] + B_k n_k \left\{\varphi_j\right\} \right) \cdot \varphi_i \, ds$ is the flux matrix,

with

- $\{\varphi_i\} = \dfrac{1}{2}(\varphi_1 + \varphi_2)$;
- $[\varphi_i] = \dfrac{1}{2}(\varphi_2 - \varphi_1) + \alpha \displaystyle\int_{\partial K} C \frac{1}{2}(\varphi_2 - \varphi_1) \, ds$

where $\alpha \le 0$ and C a symmetric positive matrix.

2.2 Time Discretisation

Using for example the classical leap-frog scheme, we get

$$U^{n+1} = U^{n-1} - 2\Delta t \; M_h^{-1} K_h U^n$$

when K_h is antisymmetric ($\alpha = 0$, no absorbing condition)

$$U^{n+1} = U^{n-1} - 2\Delta t \; M_h^{-1}(K_h U^n + L_h U^{n-1})$$

where L_h is a symmetric positive matrix containing the flux part associated with α and absorbing conditions.

3. Optimal Finite Elements

3.1 Definition

Definition 3.1. *Let $\Omega = \bigcup K$ an open of $\mathbb{R}^3$,*

- *$u \in V$ solution of the continuous problem on Ω*
- *$u_h \in V_h \subset V$ solution of the corresponding discrete problem.*

For a given order r, denoting by h the caracteristic size of the mesh, we want

$$\|u - u_h\|_{V,\Omega} = O(h^r)$$

that is

$$\|u - u_h\|_{V,K} = O(h^r)$$

3.2 Characterization

Theorem 3.2. *Let (K, P_r^F, Σ) be any finite element of the mesh*

- *For $V = H^1$, $\mathbb{P}_r \subset P_r^F \iff \|u - u_h\|_{H^1,K} = O(h^r)$*
- *For $V = H(rot)$, $\mathrm{R}^r \subset P_r^F \iff \|u - u_h\|_{H(rot),K} = O(h^r)$*

where $\mathbb{P}_r$ is the space of polynomials of order $\le r$, and R^r is Nédélec space of order r (see [2])

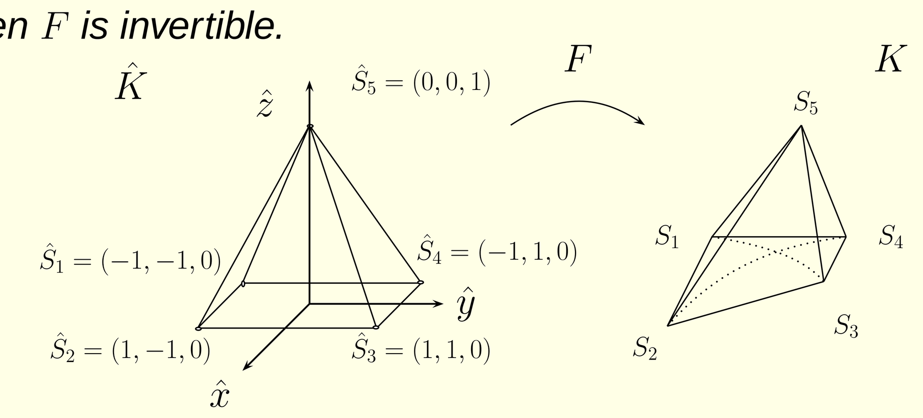
4. Application to Pyramidal Elements

4.1 Definition

Definition 4.1. *A pyramid K is the image of the reference pyramid $\hat{K}$ by the transformation F*

$$\begin{aligned} 4F = & \,(S_1 + S_2 + S_3 + S_4) \\ & + \hat{x}\,(-S_1 + S_2 + S_3 - S_4) \\ & + \hat{y}\,(-S_1 - S_2 + S_3 + S_4) \\ & + \hat{z}\,(4\,S_5 - S_1 - S_2 - S_3 - S_4) \\ & + \frac{\hat{x}\hat{y}}{1 - \hat{z}}\,(S_1 + S_3 - S_2 - S_4) \end{aligned}$$

when F is invertible.



We transform the element (K, P_r^F, Σ) into the reference element $(\hat{K}, \hat{P}_r, \hat{\Sigma})$ via F^{-1}.

4.2 H^1 Nodal Element

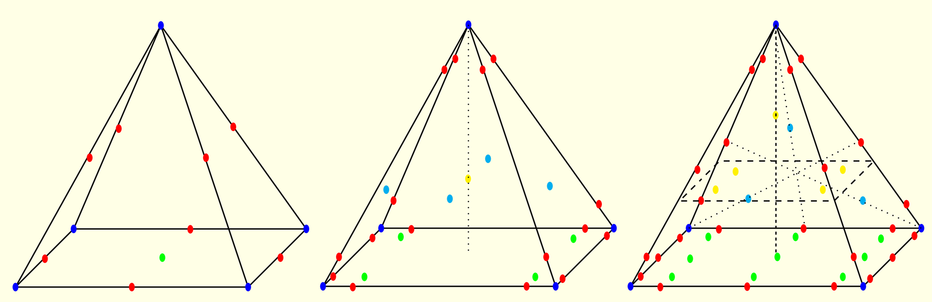
Definition 4.2. *The reference finite element $(\hat{K}, \hat{P}_r, \hat{\Sigma})$ is:*

$\hat{K}$ **:** *Unit symmetrical pyramid, centered at the origin;*

$\hat{P}_r$ **:** $\mathbb{P}_r(\hat{x}, \hat{y}, \hat{z}) \oplus \displaystyle\sum_{0 \le k \le r-1} \left(\frac{\hat{x}\hat{y}}{1 - \hat{z}} \right)^{r-k} \mathbb{P}_k(\hat{x}, \hat{y})$;

$\hat{\Sigma}$ **:** *degrees of freedom placed to ensure the continuity with the other types of elements:*

- *Hesthaven points (see [3]) on the triangular faces;*
- *Gauss-Lobatto points on the quadrangular base;*
- *interior points.*



4.3 $H(curl)$ Element - First Family

Definition 4.3. *The reference finite element $(\hat{K}, \hat{P}_r, \hat{\Sigma})$ is:*

$\hat{K}$ **:** *Unit symmetrical pyramid, centered at the origin;*

$\hat{P}_r$ **:**

$$\nabla \left(\mathbb{P}_r(\hat{x}, \hat{y}, \hat{z}) \oplus \sum_{0 \le k \le r-1} \left(\frac{\hat{x}\hat{y}}{1 - \hat{z}} \right)^{r-k} \mathbb{P}_k(\hat{x}, \hat{y}) \right)$$

$$\oplus \begin{bmatrix} \dfrac{\hat{x}^j \, \hat{y}^{k+2}}{(1-\hat{z})^{j+2}} \\ 0 \\ \dfrac{\hat{x}^{j+1} \hat{y}^{k+2}}{(1-\hat{z})^{j+2}} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \dfrac{\hat{x}^{k+2} \, \hat{y}^j}{(1-\hat{z})^{j+2}} \\ \dfrac{\hat{x}^{k+2} \hat{y}^{j+1}}{(1-\hat{z})^{j+2}} \end{bmatrix}, \quad 0 \le j \le k \le r-2$$

$$\oplus \begin{bmatrix} \dfrac{\hat{x}^j \, \hat{y}^i}{(1-\hat{z})^{i+j-r}} \\ 0 \\ \dfrac{\hat{x}^{j+1} \, \hat{y}^i}{(1-\hat{z})^{i+j-r}} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \dfrac{\hat{x}^i \, \hat{y}^j}{(1-\hat{z})^{i+j-r}} \\ 0 \\ \dfrac{\hat{x}^i \, \hat{y}^{j+1}}{(1-\hat{z})^{i+j-r}} \end{bmatrix}, \quad 0 \le i \le r+1, \, 0 \le j \le r-1$$

$$\oplus \begin{bmatrix} \dfrac{\hat{x}^k \, \hat{y}^{k+1}}{(1-\hat{z})^{k+1}} \\ \dfrac{\hat{x}^{k+1} \, \hat{y}^k}{(1-\hat{z})^{k+1}} \\ \dfrac{\hat{x}^{k+1} \, \hat{y}^{k+1}}{(1-\hat{z})^{k+2}} \end{bmatrix}, \quad 0 \le k \le r-1$$

$\hat{\Sigma}$ **:** *classical degrees of freedom of Nédélec (cf [2]) to ensure the continuity with the other types of elements.*

4.4 Extension to Discontinuous Elements

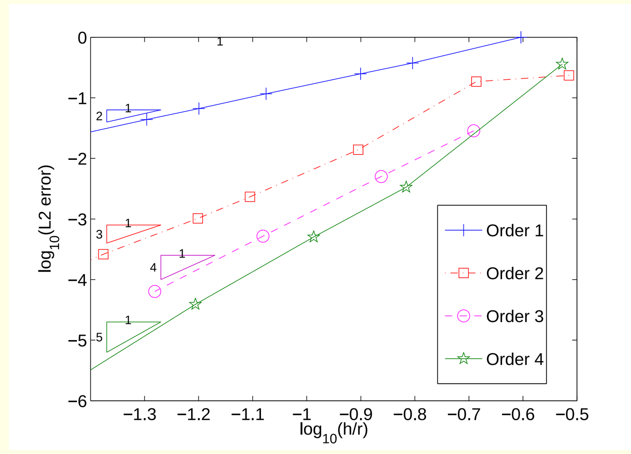
In the discontinuous case, we take the same finite element space as for H^1 but we use orthogonal basis functions to make the mass matrix sparser, and then easier to invert.

5. Numerical Results

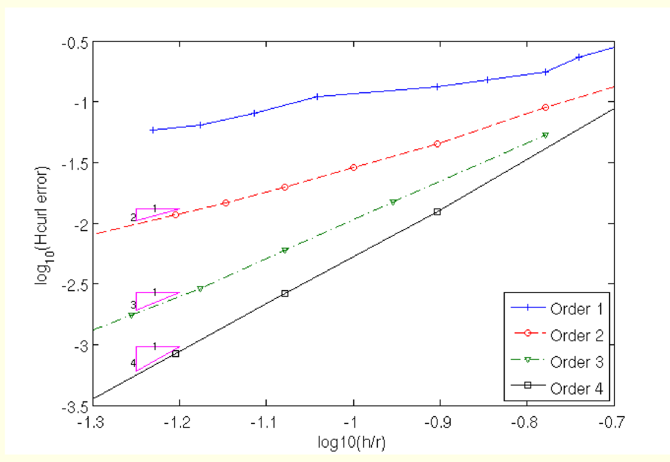
5.1 Convergence on Continuous Elements

To demonstrate the optimality of pyramidal elements in a hybrid mesh, we consider a cubic cavity $[-1, 1]^3$ and a gaussian source centered at the origin, with

- Helmholtz equation and homogeneous Dirichlet boundary conditions



- Time-harmonic Maxwell equations and perfectly conducting boundary conditions



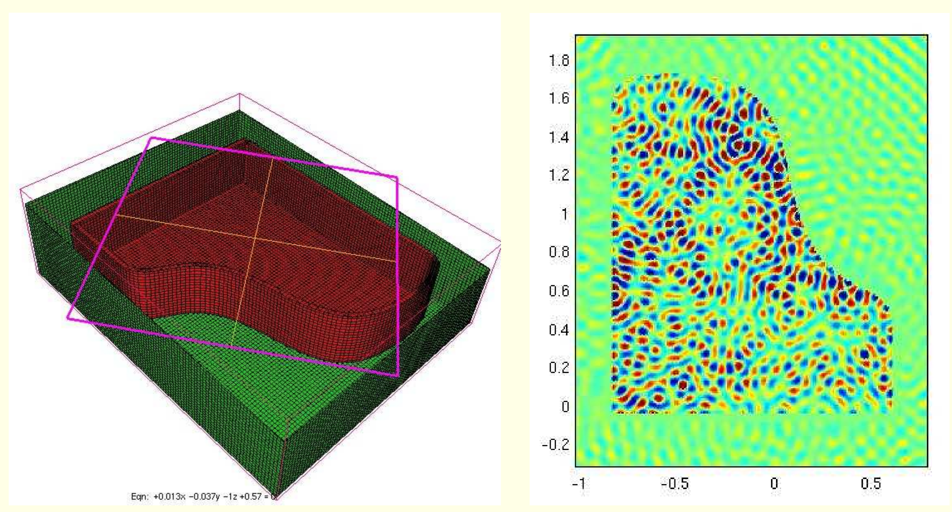
5.2 Acoustic Equation - Discontinuous Finite Elements

Diffraction by a piano with a source chosen as

$$f(x, t) = \frac{1}{r_0^2} e^{-13\left(\frac{r}{r_0}\right)^2} e^{-4(t - t_0)^2} sin(2\pi f_0 t),$$

where r is the distance to the center of the source, r_0 the distribution radius of the Gaussian, f_0 the frequency, and t_0 a constant. We compute the solution from $t = 0$ until $t = 6$, and for a same accuracy, we obtain the following results

Type of mesh	Hexahedral	Tetrahedral	Hybrid
Error	9.4 %	5.7 %	6.3 %
Dof	49.3 millions	16.9 millions	14.88 millions
Δt	0.0002	0.0004	0.0005
Comp. Time	12j 6h	4j 7h	1j 4h



Remark. A fast matrix-vector product is used to speed-up the computation (see [4] for more details).

References

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