Supersolvable LL-lattices of binary trees

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Abstract

Some posets of binary leaf-labeled trees are shown to be supersolvable lattices and explicit EL-labelings are given. Their characteristic polynomials are computed, recovering their known factorization in a different way.

1 Introduction

The aim of this article is to study some posets on forests of binary leaf-labeled trees. These posets first appeared as an essential ingredient in the combinatorial description of the coproduct in the Hopf operad introduced by the second author in [4]. They have since been shown in [5] to have some nice properties, mainly that the characteristic polynomials of all intervals factorize completely with positive integer roots. By a theorem of Stanley [8], this factorization property is true in general for the so-called semimodular supersolvable lattices. Since these intervals are not semimodular in general, one can not use this theorem to recover the result of [5]. For a class of lattices, called LL-lattices, containing the semimodular-supersolvable ones, a theorem due to Blass and Sagan [3] generalizes Stanley's theorem.

The first main theorem of our article states that these intervals are indeed lattices, which was not known before. The proof uses a new description of the intervals using admissible partitions. Our second main result is the fact that these lattices are supersolvable. We prove it by giving explicit S_n EL-labelings and using the recent criterion of McNamara [6]. As a third result, we show that these intervals are LL-lattices and, using the theorem of Blass and Sagan mentioned above, we give a different proof of the factorization of characteristic polynomials and the explicit description of roots which were found in [5].

2 Notation, definitions and preliminaries

In this section we give some definitions, notation and results that will be used in the rest of this work. Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{Z}$ the set of integers. For every $n \in \mathbb{N}$, let $[n] := \{1, 2, \ldots, n\}$. The cardinality of a finite set A is denoted by $|A|$.

2.1 Posets

We follow Chapter 3 of [9] for any undefined notation and terminology concerning posets. In all the paper we consider only finite posets. Given a finite poset $(P, \leq)$ and $x, y \in P$ with $x \leq y$ we let $[x, y] := \{z \in P \ : \ x \leq z \leq y\}$ and call this an *interval* of P. We denote by $\mathrm{Int}(P)$ the set of all intervals of P. We say that y *covers* x, denoted $x \lhd y$, if $|[x, y]| = 2$. A poset is said to be *bounded* if it has one minimal and one maximal element, denoted by $\hat{0}$ and $\hat{1}$ respectively. The *Möbius function* of P, $\mu : \mathrm{Int}(P) \to \mathbb{Z}$, is defined recursively by

$$\mu(x, y) := \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x \neq y. \end{cases}$$

If $x, y \in P$ are such that $\{z \in P : \ z \geq x, z \geq y\}$ has a minimum element then we call it the *join* of x and y, denoted by $x \vee y$. Similarly, we define the *meet* of x and y if $\{z \in P : \ z \leq x, \ z \leq y\}$ has a maximum element, denoted by $x \wedge y$. A *lattice* is a poset L for which every pair of elements has a meet and a join. A well-known criterion is the following (see e.g. [9, Proposition 3.3.1]).

Proposition 2.1 *If P is a finite poset with $\hat{1}$ such that every pair of elements has a meet then P is a lattice.*

A lattice L that satisfies the following condition

$$\text{if } x \text{ and } y \text{ both cover } x \wedge y, \text{ then } x \vee y \text{ covers both } x \text{ and } y, \tag{1}$$

is said to be *semimodular*. The set of *atoms* of a finite lattice L, *i.e.* the elements a covering $\hat{0}$, is denoted by $\mathsf{A}(L)$.

2.2 Edge-labelings

If $x, y \in P$, with $x \leq y$, a *chain* from x to y of *length* k is a $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ such that $x = x_0 < x_1 < \ldots < x_k = y$. A chain $x_0 \lhd x_1 \lhd \ldots \lhd x_k$ is said to be *saturated*. A poset P with a $\hat{0}$ is said to be *graded* if, for any $x \in P$, all saturated chains from $\hat{0}$ to x have the same length, called the *rank* of x and denoted by $\mathrm{rk}(x)$. We denote by $\mathcal{M}(P)$ the set of all maximal chains of P.

A function $\lambda : \{(x, y) \in P^2 : x \lhd y\} \to \mathbb{N}$ is an *edge-labeling* of P. For any saturated chain $m : x = x_0 \lhd x_1 \lhd \ldots \lhd x_k = y$ of the interval $[x, y]$ we set

$$\lambda(m) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k)).$$

The chain m is said to be *increasing* if $\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k)$. Let $\leq_L$ be the lexicographic order on finite integer sequences, *i.e.* $(a_1, \ldots, a_k) <_L (b_1, \ldots, b_k)$ if and only if $a_i < b_i$ where $i = \min\{j \in [k] : a_j \neq b_j\}$.

An edge-labeling of P is said to be an EL-*labeling* if the following two conditions are satisfied:

i) Every interval $[x, y]$ has exactly one increasing saturated chain m.

ii) Any other saturated chain m' from x to y satisfies $\lambda(m) <_L \lambda(m')$.

A graded poset is said to be *edge-wise lexicographically shellable* or EL-*shellable*, if it has an EL-labeling. EL-shellable posets were first introduced by Björner [1]. Several connections with shellable, Cohen-Macaulay complexes and Cohen-Macaulay posets can be found in the survey paper [2]. In particular EL-shellable posets are Cohen-Macaulay [1].

A particular class of EL-labelings has an interesting property.

An EL-labeling λ is said an S_n EL-*labeling* if, for any maximal chain $m : \hat{0} = x_0 \lhd x_1 \lhd \cdots \lhd x_n = \hat{1}$ of P, the label $\lambda(m)$ is a permutation of $[n]$. If a poset P has an S_n EL-labeling, then it is said to be S_n EL-*shellable*.

Following [8], we introduce the following definition. A finite lattice L is said to be *supersolvable* if it contains a maximal chain, called an M-*chain* of L, which together with any other chain in L generates a distributive sublattice. Examples of supersolvable lattices include modular lattices, the partition lattice Π_n and the lattice of subgroups of a finite supersolvable group.

McNamara [6, Theorem 1] has recently shown that supersolvable lattices are completely characterized by S_n EL-shellability.

Theorem 2.2 *A finite graded lattice of rank n is supersolvable if and only if it is S_n* EL-*shellable.*

2.3 Poset of forests

A *tree* is a leaf-labeled rooted binary tree and a *forest* is a set of such trees. Vertices are either inner vertices (valence 3) or leaves and roots (valence 1). By convention, edges are oriented towards the root. Leaves are bijectively labeled by a finite set. Trees and forests are pictured with their roots down and their leaves up, by choosing an arbitrary plane embedding. A leaf is an *ancestor leaf* of a vertex if there is a path from the leaf to the root going through the vertex. For a forest F, we denote by $\mathcal{V}(F)$ the set of its *inner vertices* and by $\mathcal{L}(F)$ the set of *leaves*. By a forest F on I, we mean a forest with leaf set $\mathcal{L}(F) = I$. If $F_1, F_2, \ldots, F_k$ are forests on $I_1, I_2, \ldots, I_k$, let $F_1 \sqcup F_2 \sqcup \cdots \sqcup F_k$ be their disjoint union. The number of trees in a forest F on I is the difference between the cardinal of I and the cardinal of $\mathcal{V}(F)$. By a *subtree* T_v we mean the union of all paths starting from any vertex v and going up to the leaves. Note that any subtree T_v can be further divided in two parts denoted by T_v^L and T_v^R as shown in Figure 1. If the symbols L and R are taken to mean left and right, then this notation of course depends on the choice of a plane embedding. In fact, the words "left" and "right" and symbols L and R will always be used just as a convenient set of cardinality 2.

Two ancestor leaves of an inner vertex v are said to be on the same side of v if the paths from these leaves to the root enter v by the same edge.

Following [5] and using a simple reformulation, we introduce a partial order on the *set of forests on I* denoted by $\mathrm{For}(I)$.

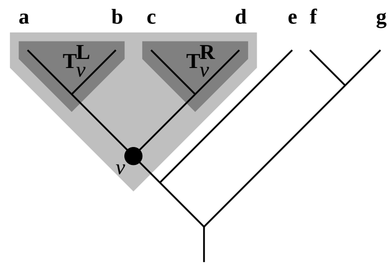

Figure 1: The subtree T_v, and its parts T_v^R and T_v^L.

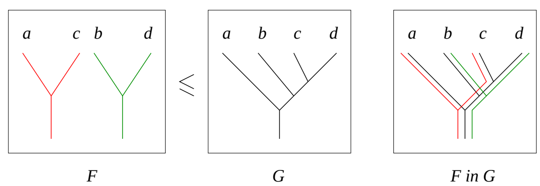

Figure 2: $F \leq G$.

Definition Let F and G be forests on the label set I. Then $F \leq G$ if there is an injective map φ from the set of inner vertices $\mathcal{V}(F)$ to the set of inner vertices $\mathcal{V}(G)$ such that :

(D_1) For each inner vertex v of F, the set of ancestor leaves of v in F is contained, as a subset of I, in the set of ancestor leaves of $\varphi(v)$ in G.

(D_2) For each inner vertex v of F, two ancestors leaves of v in F are on the same side of v in F if and only if they are on the same side of $\varphi(v)$.

Let us remark that such a map φ is unique when it exists. Indeed the image of an inner vertex v is determined by its set of ancestor leaves S as the highest possible inner vertex of G whose set of ancestor leaves contains S.

One can depict such a map φ by a drawing of F inside G where the image $\varphi(v)$ of an inner vertex v of F is joined in G with the leaves of G which were the ancestor leaves of v in F.

The following proposition can be found in [5, Proposition 3.1].

Proposition 2.3 *The poset* $\mathrm{For}(I)$ *is graded by the number of inner vertices.*

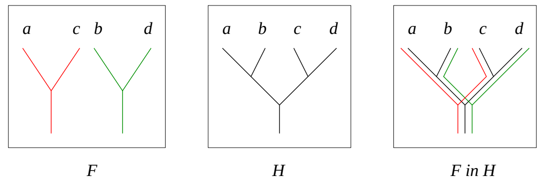

Figure 3: F and H are not comparable.

It was proved in [5] that the maximal elements of the poset $\mathrm{For}(I)$ are the trees. The forest without inner vertices is the unique minimal element and is denoted by $\hat{0}$. For any $J \subseteq I$, we denote by $|_J$ the forest such that $\mathcal{V}(|_J) = \emptyset$ and $\mathcal{L}(|_J) = J$. Note that $\hat{0} = |_I$.

3 Intervals are lattices

In this section we fix a finite set of leaves I of cardinality $n+1$ and consider a tree T on I. We study the interval $[\hat{0}, T]$ that is a graded bounded subposet of $\mathrm{For}(I)$. Our main goal is to show that $[\hat{0}, T]$ is a lattice.

Any two distinct leaves $i, j \in I$ determine an inner vertex $v_{(i,j)} \in \mathcal{V}(T)$, as the intersection of the two paths starting from these leaves and going down to the root. Sometimes we will write $i \overset{v}{\longleftrightarrow} j$ instead of $v = v_{(i,j)}$. For any $J \subseteq I$, let

$$\mathcal{S}(J) := \{v \in \mathcal{V}(T) \,:\, v = v_{(i,j)} \text{ for some distinct } i, j \in J\}.$$

Remark 1 For any subset $J \subseteq I$, it is easy to see that $|\mathcal{S}(J)| = |J| - 1$.

Lemma 3.1 *For any $J \subseteq I$, there exists a unique tree T_J on J such that*

$$T_J \sqcup |_{I \setminus J} \leq T.$$

Proof. We define T_J to be the union of all the paths starting from the leaves in J and going down to the root. It is easy to check that all conditions in the definition of the partial order of forests are satisfied. ■

Remark 2 Let $J_1 \subseteq J_2$ be two subsets of I. Then $T_{J_1} \sqcup |_{I \setminus J_1} \leq T_{J_2} \sqcup |_{I \setminus J_2}$

The following definition is crucial in the rest of this paper.

Let $\pi = (\pi_1, \ldots, \pi_k)$ be a partition of I. We say that π is T-*admissible* if and only if $\mathcal{S}(\pi_i) \cap \mathcal{S}(\pi_j) = \emptyset$ for all $i \neq j \in [k]$. We denote the set of all T-admissible partitions of I by $\mathrm{Ad}(T)$.

For example, let $T = F''$ be the tree in Figure 3 on the set $I = \{a, b, c, d\}$. Then $\{\{a, b\}, \{c, d\}\} \in \mathrm{Ad}(T)$, but $\{\{a, c\}, \{b, d\}\}$ is not a T-admissible partition of I, as in fact $\mathcal{S}(\{a, c\}) = \mathcal{S}(\{b, d\}) = v_{(a,c)}$.

It is easy to see that $\mathrm{Ad}(T)$ is a poset by refinement order $\leq_r$, *i.e.* $(\pi_1, \ldots, \pi_n) \leq_r (\tau_1, \ldots, \tau_m)$ if and only if each block π_i is contained in some block τ_j.

For example $\{\{a\}, \{b, c\}, \{d\}\} \leq_r \{\{a\}, \{b, c, d\}\}$.

Let $F \in [\hat{0}, T]$, $F = T_1 \sqcup \ldots \sqcup T_k$, we define

$$\Pi(F) := (\pi_1, \ldots, \pi_k),$$

where $\pi_i := \mathcal{L}(T_i)$ for all $i \in [k]$. It follows from the definition of the partial order on forests that $\Pi(F)$ is a T-admissible partition.

Proposition 3.2 *The map* $\Pi : ([\hat{0}, T], \leq) \longrightarrow (\mathrm{Ad}(T), \leq_r)$ *is an isomorphism of posets.*

Proof. First we prove that Π is a bijection. For every $\pi = (\pi_1, \ldots, \pi_k) \in \mathrm{Ad}(T)$, let

$$\Gamma(\pi) := T_{\pi_1} \sqcup \ldots \sqcup T_{\pi_k}, \tag{2}$$

where each tree T_{π_i} is defined by Lemma 3.1.

It is clear that $\Pi \circ \Gamma = \mathrm{Id}$. By the uniqueness in Lemma 3.1, it follows that $\Gamma \circ \Pi = \mathrm{Id}$, and so Γ is the inverse of Π.

Now let $F, G \in [\hat{0}, T]$ with $F \leq G$. Then, by definition of $\leq$, for all $T_F \in F$ there exists a $T_G \in G$ such that $\mathcal{L}(T_F) \subseteq \mathcal{L}(T_G)$. It follows that $\Pi(F) \leq_r \Pi(G)$. Conversely, if $\pi \leq_r \pi'$, then, by Remark 2, we have $\Gamma(\pi) \leq \Gamma(\pi')$. This concludes the proof. ∎

From now on, forests in $[\hat{0}, T]$ and T-admissible partitions are identified via the bijection Π.

We are ready to state and prove the main theorem of this section.

Theorem 3.3 *For each tree T on the set I, the interval $[\hat{0}, T]$ is a lattice.*

Proof. As the interval has a $\hat{1}$, by Proposition 2.1 it suffices to prove that each $F, G \in [\hat{0}, T]$ have a meet. Let $\Pi(F) = \pi = (\pi_1, \ldots, \pi_n)$ and $\Pi(G) = \tau = (\tau_1, \ldots, \tau_m)$. We show that the meet of π and τ as partitions, defined by

$$\pi \wedge \tau := (\pi_1 \cap \tau_1) \cup (\pi_1 \cap \tau_2) \cup \ldots \cup (\pi_n \cap \tau_1) \cup \ldots \cup (\pi_n \cap \tau_m),$$

is also in $\mathrm{Ad}(T)$. For every $(i, j) \neq (i', j') \in [n] \times [m]$ we have that

$$\mathcal{S}(\pi_i \cap \tau_j) \cap \mathcal{S}(\pi_{i'} \cap \tau_{j'}) \subseteq \mathcal{S}(\pi_i) \cap \mathcal{S}(\tau_j) \cap \mathcal{S}(\pi_{i'}) \cap \mathcal{S}(\tau_{j'}) = \emptyset.$$

In fact, since π and τ are in $\mathrm{Ad}(T)$, either $\mathcal{S}(\pi_i) \cap \mathcal{S}(\pi_{i'})$ or $\mathcal{S}(\tau_j) \cap \mathcal{S}(\tau_{j'})$ is empty. It is immediate to see that $\pi \wedge \tau$ is the meet also in $\mathrm{Ad}(T)$; hence $\mathrm{Ad}(T)$ is a lattice and we are done. ■

4 S_n EL-labelings on $[\hat{0}, T]$

In this section we introduce an edge-labeling on the poset $[\hat{0}, T]$ and prove that it is an S_n EL-labeling. By Theorem 2.2 it follows that the lattice $[\hat{0}, T]$ is supersolvable.

A partial order $\preceq$ is defined on the vertex set $\mathcal{V}(T)$ in the following way.

Definition A vertex v is smaller than a vertex v', denoted by $v \preceq v'$, if v' is on the path between the root and v. Any total order extending this partial order on $\mathcal{V}(T)$ is called a *nice* total order, still denoted by $\preceq$.

Using a nice total order, one can label the inner vertices by integer numbers from 1 to n. From now on, inner vertices and labels are identified in this way using a fixed nice total order. Note that the bottom vertex is the maximum element for the order $\preceq$. An example is drawn in Figure 4.

Now we introduce an edge-labeling as follows. First remark that for all $F \leq G \in [\hat{0}, T]$, one has $\mathcal{V}(F) \subseteq \mathcal{V}(G) \subseteq \mathcal{V}(T)$ by definition of the ordering. Moreover if $F \lessdot G$, by Proposition 2.3, there exists a unique $v \in \mathcal{V}(G)$ such that $\mathcal{V}(G) = \mathcal{V}(F) \cup \{v\}$.

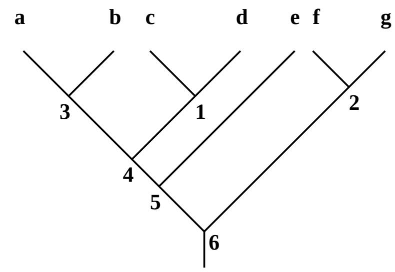

Figure 4: Example of nice total order on $\mathcal{V}(T)$.

Definition Let $F \lhd G \in [\hat{0}, T]$. If

$$\mathcal{V}(G) = \mathcal{V}(F) \cup \{v\},$$

then we define $\lambda(F, G)$ to be the label of v.

An example of this edge-labeling is shown in Figure 5. The proof of the following Lemma is immediate.

Lemma 4.1 *The label of a maximal chain of $[F, G]$ is a permutation of the set $\mathcal{V}(G) \setminus \mathcal{V}(F)$.*

Lemma 4.2 *For each $F \in [\hat{0}, T] \setminus \{T\}$, there exists a unique $G \in [\hat{0}, T]$ covering F such that*

$$\lambda(F, G) = \min(\mathcal{V}(T) \setminus \mathcal{V}(F)). \tag{3}$$

Proof. Let $\Pi(F) = \pi$ and let $v_0 := \min(\mathcal{V}(T) \setminus \mathcal{V}(F))$. Consider the two subtrees starting from v_0, as explained in §2.3, denoted $T^L_{v_0}$ and $T^R_{v_0}$. We show that $\mathcal{L}(T^R_{v_0})$ is contained in one part of π.
Each $w \in \mathcal{V}(T^R_{v_0})$ is such that $w \prec v_0$. It follows that $w \in \mathcal{V}(F)$ by minimality of v_0. Let $i \neq j \in \mathcal{L}(T^R_{v_0})$. Then there is $v \in \mathcal{V}(T^R_{v_0}) \subseteq \mathcal{V}(F)$ such that $i \overset{v}{\longleftrightarrow} j$. Hence i, j are in the same part of π. Therefore $\mathcal{L}(T^R_{v_0})$ is contained in only one part of π denoted by π_R. The same result is true for $T^L_{v_0}$, and we denote the corresponding part by π_L. As $v_0 \notin \mathcal{V}(F)$, the parts π_L and π_R are distinct. We define a new partition

$$\pi' := (\pi_L \sqcup \pi_R, \pi_1, \dots, \pi_k),$$

where π_j are the remaining parts of π. From now on, we denote $\pi_L \sqcup \pi_R$ by π_{LR}.
To show that $\pi' \in \mathrm{Ad}(T)$, it suffices to prove that

$$\mathcal{S}(\pi_{LR}) \cap \mathcal{S}(\pi_j) = \emptyset, \quad \text{for all } j \in [k]. \tag{4}$$

We have that $\mathcal{S}(\pi_{LR}) \supseteq \mathcal{S}(\pi_L) \cup \mathcal{S}(\pi_R) \cup \{v_0\}$. On the other hand, by Remark 1, we have that $|\mathcal{S}(\pi_L)| + |\mathcal{S}(\pi_R)| + 1 = |\mathcal{S}(\pi_{LR})|$, and so we have an equality.
Now, for any $j \in [k]$, the vertex v_0 is not in $\mathcal{S}(\pi_j)$, because all the ancestor leaves of v_0 are in π_L or in π_R; hence condition (4) is verified. Now, it is clear that $G := \Gamma(\pi_{LR}, \pi_1, \ldots, \pi_k)$, where Γ is defined in (2), is the unique forest covering F satisfying (3). ■

The preceding Lemma can be extended as follows.

Proposition 4.3 *For all $F, H \in [\hat{0}, T]$ with $F < H$, there exists a unique $G \in [\hat{0}, T]$ covering F such that*

$$\lambda(F, G) = \min(\mathcal{V}(H) \setminus \mathcal{V}(F)).$$

Proof. If $H = T$ then the result is given by Lemma 4.2. Otherwise let $H = H_1 \sqcup H_2 \sqcup \ldots \sqcup H_k$, where H_j is a tree for all $j \in [k]$. Since $F \leq H$, we have $F = F_1 \sqcup F_2 \sqcup \ldots \sqcup F_k$ where F_j is a forest and $F_j \leq H_j$ for all $j \in [k]$. It was observed in [5, Proposition 2.1] that the interval $[F, H]$ is isomorphic to $\prod_{j=1}^{k} [F_j, H_j]$.
Let $v_1 := \min(\mathcal{V}(H) \setminus \mathcal{V}(F))$. We have $\mathcal{V}(H) = \mathcal{V}(H_1) \cup \mathcal{V}(H_2) \cup \ldots \cup \mathcal{V}(H_k)$ and, after re-ordering, we can assume that $v_1 \in \mathcal{V}(H_1)$. Then, by Lemma 4.2 applied to $[F_1, H_1]$, there exists a unique $G_1 \in [F_1, H_1]$ covering F_1 such that $\lambda(F_1, G_1) = v_1$. Define $G = G_1 \sqcup F_2 \sqcup \ldots \sqcup F_k$ in $[F, H]$. Then G is the unique forest of $[F, H]$, covering F, such that $\lambda(F, G) = v_1$. This concludes the proof. ■

Theorem 4.4 *The lattice $[\hat{0}, T]$ is EL-shellable.*

Proof. By Lemma 4.1, for any interval $[F, G]$ of $[\hat{0}, T]$, the unique possible increasing label for a saturated chain from F to G is given by the unique increasing permutation of the elements of $\mathcal{V}(G) \setminus \mathcal{V}(F)$.
Then Proposition 4.3 implies that there exists an unique chain m from F to G with

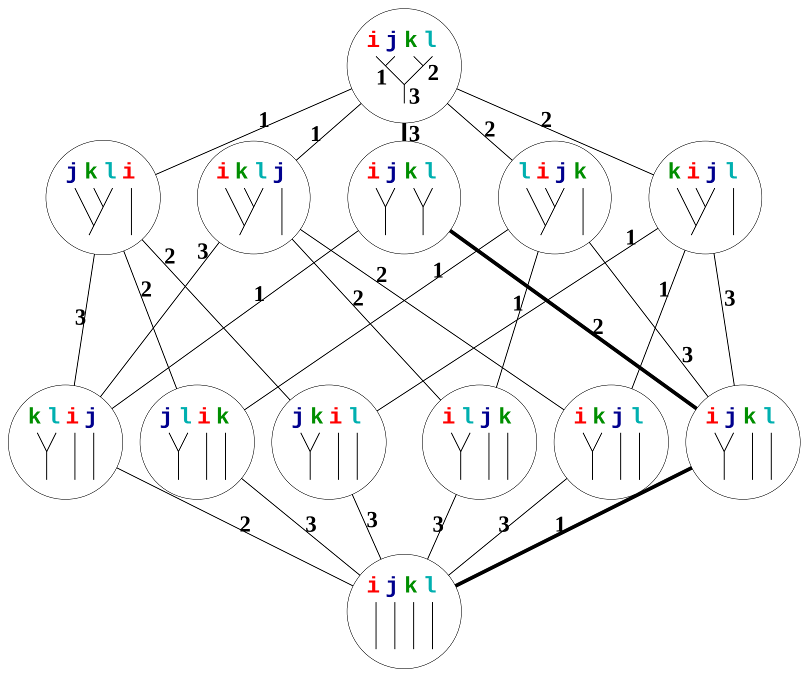

Figure 5: S_3 EL-labeling of the interval $[\hat{0}, T]$.

this label. The other maximal chains of $[F, G]$ are labeled by different permutations, which are lexicographically greater than the increasing one.
Hence the edge-labeling λ is an EL-labeling. ■

Corollary 4.5 *The lattice $[\hat{0}, T]$ is supersolvable.*

Proof. By Theorem 4.4, λ is an EL-labeling and by Lemma 4.1, $\lambda(m)$ is a permutation of $[n]$ for each maximal chain m. Hence λ is an S_n EL-labeling and the result follows from Theorem 2.2. ■

Remark 3 Note that $[\hat{0}, T]$ is not semimodular in general. For example, the atoms $\{\{j, k\}, \{i\}, \{l\}\}$ and $\{\{i, l\}, \{j\}, \{k\}\}$ in Figure 5 do not satisfy the condition (1).

5 Characteristic polynomials

In this section, we recover the results of [5] concerning the characteristic polynomials of the intervals $[\hat{0}, T]$. Note that, by Remark 3, the well-known theorem of Stanley

[8, Theorem 4.1] (see also [7, Theorem 6.2]) on the factorization of the characteristic polynomials of semimodular supersolvable lattices, does not apply. We use instead a stronger theorem due to Blass and Sagan [3].

5.1 LL-lattices

Recall that the characteristic polynomial of a graded finite lattice L of rank n is

$$\chi_L(t) = \sum_{y \in L} \mu(\hat{0}, y) t^{n-\mathrm{rk}(y)},$$

where μ is the Möbius function of L and $\mathrm{rk}(y)$ is the rank of y.

Following [3], we define an element x of a lattice L to be *left-modular* if, for all $y \leq z$,

$$y \vee (x \wedge z) = (y \vee x) \wedge z.$$

A maximal chain $m \in \mathcal{M}(L)$ is said to be *left-modular* if all its elements are left-modular.

Remark 4 From [8, Proposition 2.2], it follows that if L is a supersolvable lattice then its M-chain is left-modular.

Any maximal chain $m : \hat{0} = x_0 \lhd x_1 \lhd \cdots \lhd x_n = \hat{1}$ defines a partition of the set of atoms A into subsets called *levels* indexed by $i \in [n]$:

$$\mathsf{A}_i = \{a \in \mathsf{A} : \, a \leq x_i \text{ and } a \not\leq x_{i-1}\}.$$

The partial order $\sqsubset_m$ on A *induced* by the maximal chain m is defined by

$$a \sqsubset_m b \text{ if and only if } a \in \mathsf{A}_i \text{ and } b \in \mathsf{A}_j \text{ with } i < j.$$

Then the *level condition* with respect to m is:

$$\text{if } a_0 \sqsubset_m a_1 \sqsubset_m a_2 \sqsubset_m \cdots \sqsubset_m a_k, \text{ then } a_0 \not\leq \bigvee_{i=1}^{k} a_i.$$

A lattice L having a maximal chain m that is left-modular and satisfies the level condition is called an LL-*lattice*.

The following theorem is due to Blass and Sagan [3, Theorem 6.5].

Theorem 5.1 *Let P be an* LL*-lattice of rank n. Let A_i be the levels with respect to the left-modular chain of P. Then*

$$\chi_P(t) = \prod_{i=1}^{n}(t - |\mathsf{A}_i|).$$

5.2 Factorization of characteristic polynomials

A tree T with n inner vertices and leaf set I is fixed. A nice total order on $\mathcal{V}(T)$ is chosen, defining an edge-labeling as in §4.

The set A of atoms of $[\hat{0}, T]$ is the set of pairs (i, j) of distinct elements of I. To each atom (i, j) is associated an inner vertex $v_{(i,j)}$ of T as defined in §3. The covering edge $\hat{0} \lhd (i, j)$ is labeled by the integer in $[n]$ corresponding to $v_{(i,j)}$ in the chosen total order on $\mathcal{V}(T)$.

Proposition 5.2 *Let $a_1, a_2, \dots, a_k \in \mathsf{A}$ associated with pairwise distinct vertices $v_1, v_2, \dots, v_k$ in $\mathcal{V}(T)$. Then $\mathcal{V}(a_1 \vee a_2 \vee \dots \vee a_k) = \{v_1, v_2, \dots, v_k\}$.*

Proof. Let $V = \{v_1, v_2, \dots, v_k\}$. Let $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(k)}$ be the partitions of I associated to $a_1, a_2, \dots, a_k$. Let π be the join $\pi^{(1)} \vee \pi^{(2)} \vee \dots \vee \pi^{(k)}$ in the lattice of partitions. We want to show that $\pi \in \mathrm{Ad}(T)$ and that $\mathcal{V}(\pi) = V$.
Let p be a part of π. Let V_p be the set of vertices in V whose corresponding atoms in $\{a_1, \dots, a_k\}$ have their leaves in p. Observe that the sets V_p form a partition of V because atoms in $\{a_1, \dots, a_k\}$ have pairwise distinct vertices. Let v be a vertex in $\mathcal{S}(p)$. This means that there exists i, j in p such that $i \xleftrightarrow{v} j$. As p is a part of a join, there exists a chain

$$i = i_0 \xleftrightarrow{t_0} i_1 \xleftrightarrow{t_1} i_2 \dots i_{\ell-1} \xleftrightarrow{t_{\ell-1}} i_\ell \xleftrightarrow{t_\ell} i_{\ell+1} = j,$$

where each $i_r \xleftrightarrow{t_r} i_{r+1}$ is an atom in $\{a_1, \dots, a_k\}$ with vertex in V_p.
In the rest of the proof, the symbol $\preceq$ stands for the partial order introduced in §4.
Let us prove by induction on the length ℓ of the chain that there exists θ_ℓ in V_p such that $\theta_\ell \succeq t_0$ and $\theta_\ell \succeq t_\ell$.
If $\ell = 0$, then one can take $\theta_0 = t_0$. Assume that there exists $\theta_{\ell-1}$ in V_p such that $\theta_{\ell-1} \succeq t_0$ and $\theta_{\ell-1} \succeq t_{\ell-1}$. The path joining the leaf i_ℓ to the root contains the vertices

$t_{\ell-1}$,t_ℓ and hence also by induction hypothesis the vertex $\theta_{\ell-1}$. Either $t_\ell \preceq \theta_{\ell-1}$, and one can take $\theta_\ell = \theta_{\ell-1}$ or $t_\ell \succeq \theta_{\ell-1}$ and one can take $\theta_\ell = t_\ell$. This concludes the induction.
Therefore $\theta_\ell \in V_p$ is such that $i \overset{\theta_\ell}{\longleftrightarrow} j$. Hence $\theta_\ell = v \in V_p$ and so $\mathcal{S}(p) \subseteq V_p$. The converse inclusion is clear.
Now let p and p' be two different parts of π. Then $\mathcal{S}(p) \cap \mathcal{S}(p') = V_p \cap V_{p'}$ is empty. Hence π is T-admissible.
We have proved that π is T-admissible and that the vertices of π are exactly V. It follows that π defines the join $a_1 \vee \ldots \vee a_k$ in $[\hat{0}, T]$ and the proposition is proved. ■

Define another partition of A indexed by $i \in [n]$:

$$\mathsf{B}_i = \{a \in \mathsf{A} : \lambda(\hat{0}, a) = i\}.$$

Let $m : \hat{0} = x_0 \lhd x_1 \lhd \cdots \lhd x_n = T$ be the fixed left-modular chain of $[\hat{0}, T]$, *i.e.* the unique increasing maximal chain for the fixed labeling.

Lemma 5.3 *Let $i \in [n]$. For each $j \in [i]$, let a_j be an atom in B_j. Then*

$$x_i = a_1 \vee a_2 \vee \ldots \vee a_i.$$

Proof. The proof is by induction on i. By Proposition 4.3, $x_1 = a_1$ is the unique atom in B_1. Assume that $x_{i-1} = a_1 \vee \ldots \vee a_{i-1}$. Then $a_1 \vee \ldots \vee a_{i-1} \vee a_i$ is $x_{i-1} \vee a_i$ and has rank i by Proposition 5.2. Moreover we have that $\lambda(x_{i-1}, x_{i-1} \vee a_i) = i$. By uniqueness in Proposition 4.3, it follows that $x_i = x_{i-1} \vee a_i$. ■

Lemma 5.4 *Let A_i be the levels with respect to m. Then for each $i \in [n]$,*

$$\mathsf{A}_i = \mathsf{B}_i.$$

Proof. It suffices to prove that

$$\{a \in \mathsf{A} : a \leq x_i\} = \{a \in \mathsf{A} : \lambda(\hat{0}, a) \in [i]\}.$$

If $a \leq x_i$, then $\lambda(\hat{0}, a)$ is one of the vertices of x_i, *i.e.* belongs to $[i]$. Conversely, take any atom a with $\lambda(\hat{0}, a)$ in $[i]$. Choose other atoms to have one atom in each B_j for $j \in [i]$. Then, by Lemma 5.3, x_i is the join of a and the other chosen atoms, so $a \leq x_i$. ■

Proposition 5.5 *The lattice $[\hat{0}, T]$ is an* LL*-lattice.*

Proof. This lattice is supersolvable, so by Remark 4 the M-chain is a left-modular chain. It remains to check the level condition. Take atoms $a_0, a_1, \ldots, a_k$ which belong to pairwise different A_i. By Lemma 5.4, these atoms belong to pairwise different B_i. Then by Proposition 5.2 the set of vertices of the join $a_1 \vee \ldots \vee a_k$ does not contain the vertex of the atom a_0. This ensures the level condition. ■

Now we are ready to state and prove the main result of this section, which was already proved in [5, Theorem 4.6].

Theorem 5.6 *The characteristic polynomial of $[\hat{0}, T]$ is*

$$\chi_{[\hat{0},T]}(t) = \prod_{v \in \mathcal{V}(T)} (t - e(v)),$$

where $e(v)$ is the product of the number of left ancestor leaves of v by the number of right ancestor leaves of v.

Proof. By Proposition 5.5, one can apply Theorem 5.1 to $[\hat{0}, T]$. Let us count the number of elements of A_i for each i. By Lemma 5.4, this is equal to the cardinality of B_i. Let v be the vertex of T with index i. It is easy to see that the number of atoms in B_i is the number of left ancestor leaves of v times the number of right ancestor leaves of v. ■

For example, the characteristic polynomial of the interval $[\hat{0}, T]$ where T is the tree in Figure 6 is $\chi_{[\hat{0},T]}(t) = (t-1)^3(t-4)^2(t-10)$.

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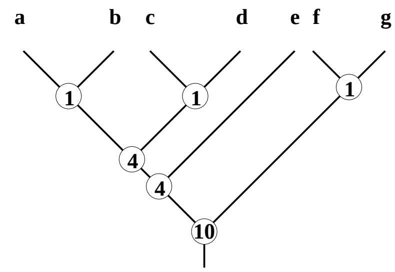

Figure 6: Example of roots of the characteristic polynomial.

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