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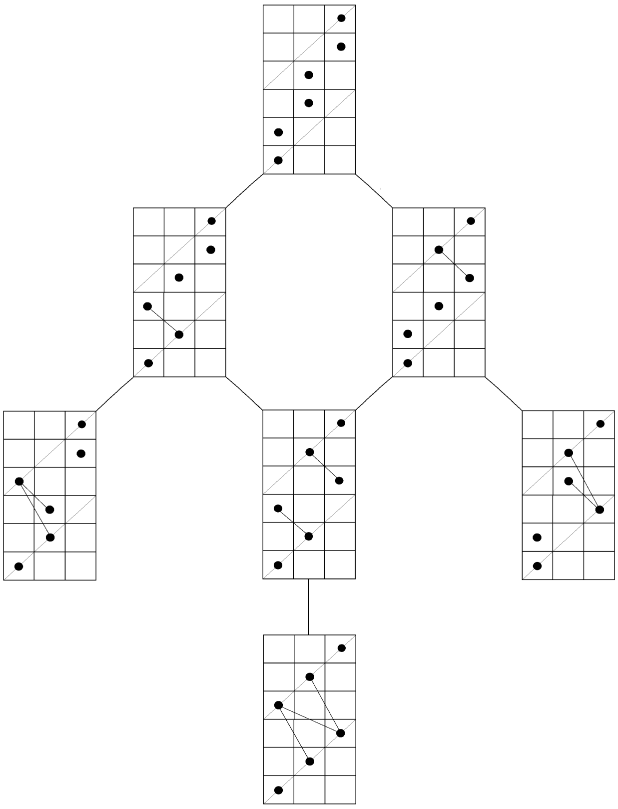
UMR 5208 du CNRS

Combinatoire bijective des permutations et nombres de Genocchi



Ange Bigeni

Thèse de doctorat

Université Claude Bernard Lyon 1

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Ange Bigeni

devant le Jury composé de :

M. Jean-Christophe Aval	Université de Bordeaux	Rapporteur
M. Guoniu Han	Université de Strasbourg	Examinateur
M. Frédéric Jouhet	Université Lyon 1	Examinateur
M. Jean-Christophe Novelli	Université Paris-Est Marne-la-Vallée	Examinateur
M. Marc van Leeuwen	Université de Bordeaux	Rapporteur
M. Jiang Zeng	Université Lyon 1	Directeur de thèse

Résumé-Abstract

Résumé

Cette thèse a pour contexte la combinatoire énumérative et décrit la construction de plusieurs bijections entre modèles combinatoires connus ou nouveaux de suites d'entiers et polynômes, plus particulièrement celle des nombres de Genocchi (et de leurs extensions, les polynômes de Gandhi) qui interviennent dans diverses branches des mathématiques et dont les propriétés combinatoires sont de ce fait activement étudiées, et celles de polynômes q-eulériens associés aux quatre statistiques fondamentales de MacMahon sur les permutations ainsi qu'à des statistiques analogues.

On commence par définir les permutations de Dumont normalisées, un modèle combinatoire des nombres de Genocchi médians normalisés q-étendus, notés $\bar{c}_n(q)$ et définis par Han et Zeng, puis l'on construit une première bijection entre ce modèle et l'ensemble des configurations de Dellac, autre interprétation combinatoire de $\bar{c}_n(q)$ mise en évidence par Feigin dans le contexte de la géométrie des grassmanniennes de carquois. En s'appuyant sur la théorie des fractions continues de Flajolet, on en construit finalement un troisième modèle combinatoire à travers les histoires de Dellac, que l'on relie aux premiers modèles sus-cités au moyen d'une seconde bijection.

On s'intéresse ensuite à la classe combinatoire des k-formes irréductibles définies par Hivert et Mallet dans l'étude des k-fonctions de Schur, et qui faisaient l'objet d'une conjecture supposant que les polynômes de Gandhi sont générés par les k-formes irréductibles selon la statistique des k-sites libres. On construit une bijection entre les k-formes irréductibles et les pistolets surjectifs de hauteur $k-1$ (connus pour générer les polynômes de Gandhi selon la statistique des points fixes) envoyant les k-sites libres des premières sur les points fixes des seconds, démontrant de ce fait la conjecture.

Enfin, on établit une nouvelle identité combinatoire entre deux polynômes

q-eulériens définis par des statistiques eulériennes et mahoniennes sur l'ensemble des permutations d'un ensemble fini, au moyen d'une dernière bijection sur les permutations, qui envoie une suite finie de statistiques sur une autre.

Abstract

This work is set in the context of enumerative combinatorics and constructs several statistic-preserving bijections between known or new combinatorial models of sequences of integers or polynomials, espacially the sequence of Genocchi numbers (and their extensions, the Gandhi polynomials) which appear in numerous mathematical theories and whose combinatorial properties are consequently intensively studied, and two sequences of q-Eulerian polynomials associated with the four fundamental statistics on permutations studied by MacMahon, and with analog statistics.

First of all, we define normalized Dumont permutations, a combinatorial model of the q-extended normalized median Genocchi numbers $\bar{c}_n(q)$ introduced by Han and Zeng, and we build a bijection between the latter model and the set of Dellac configurations, which have been proved by Feigin to generate $\bar{c}_n(q)$ by using the geometry of quiver Grassmannians. Then, in order to answer a question raised by the theory of continued fractions of Flajolet, we define a new combinatorial model of $\bar{c}_n(q)$, the set of Dellac histories, and we relate them with the previous combinatorial models through a second statistic-preserving bijection.

Afterwards, we study the set of irreducible k-shapes defined by Hivert and Mallet in the topic of k-Schur functions, which have been conjectured to generate the Gandhi polynomials with respect to the statistic of free k-sites. We construct a statistic-preserving bijection between the irreducible k-shapes and the surjective pistols of height $k-1$ (well-known combinatorial interpretation of the Gandhi polynomials with respect to the fixed points statistic) mapping the free k-sites to the fixed points, thence proving the conjecture.

Finally, we prove a new combinatorial identity between two eulerian polynomials defined on the set of permutations thanks to Eulerian and Mahonian statistics, by constructing a bijection on the permutations, which maps a finite sequence of statistics on another.

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Notations

Soit (n, m) un couple d'entiers naturels tels que $n < m$, et soit σ une permutation de l'ensemble $\{1, 2, \ldots, n\}$. On se permet d'identifier σ avec le mot $\sigma(1)\sigma(2)\ldots\sigma(n)$.

Si un ensemble d'entiers $S = \{n_1, n_2, \ldots, n_k\}$ vérifie $n_1 < n_2 < \ldots < n_k$, on utilise parfois la notation $S = \{n_1, n_2, \ldots, n_k\}_<$.

$[\|n, m\|]$	Ensemble d'entiers $\{n, n+1, \ldots, m\}$.
$[n]$	Ensemble d'entiers $\{1, 2, \ldots, n\}$.
$\binom{n}{2}$	Nombre triangulaire $n(n-1)/2$.
$\|S\|$	Cardinal d'un ensemble fini S.
$\mathbb{N}$	Ensemble des entiers naturels $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{N}_{>0}$	Ensemble $\mathbb{N} \setminus \{0\}$.
$\mathbb{Z}$	Ensemble des entiers relatifs.
$\mathfrak{S}_n$	Ensemble des permutations de $[n]$.
Sym	Espace des fonctions symétriques $f(x_1, x_2, \ldots)$ sur l'anneau $\mathbb{Z}$.
Id_S	Fonction identité d'un ensemble S.
$Im(f)$	Ensemble image d'une fonction f.

Introduction

La combinatoire énumérative est le domaine des mathématiques qui s'intéresse au dénombrement des éléments d'ensembles discrets. Elle se subdivise en plusieurs théories, comme par exemple celles des séries génératrices, des posets ou encore des P-partitions (voir [Sta11]). Pour mettre ces théories en pratique, on se ramène souvent à construire des bijections entre deux ensembles et qui démontrent incidemment certaines propriétés des nombres ou polynômes dont les ensembles considérés constituent une interprétation combinatoire (c'est-à-dire, qui sont comptés par les nombres en question, ou qui génèrent les polynômes en question par le biais de statistiques, autrement dit des fonctions qui envoient les éléments des ensembles considérés sur des entiers naturels), telles que la parité d'un nombre ou la symétrie d'un polynôme.

Le premier thème de cette thèse est la suite des nombres de Genocchi, intervenant dans diverses branches des mathématiques et dont les relations combinatoires sont nombreuses, notamment avec la classe combinatoire des permutations. On s'intéresse dans un premier temps à des q-analogues (polynômes en q généralisant un nombre) d'une suite de nombres reliés aux nombres de Genocchi, introduits par Han et Zeng : les nombres de Genocchi médians normalisés q-étendus, dont Feigin a déterminé une nouvelle interprétation combinatoire à travers les configurations de Dellac dans le contexte de la géométrie des grassmanniennes de carquois, et à l'aide de la théorie des fractions continues de Flajolet. On fait ici le lien entre ce résultat et l'étude combinatoire de ces polynômes en construisant deux bijections entre les configurations de Dellac et des modèles combinatoires préexistants. L'autre travail consacré aux nombres de Genocchi dans cette thèse a pour sujet la résolution d'une conjecture formulée par Hivert et Mallet, selon laquelle une famille d'objets intervenant dans la théorie des k-fonctions de Schur, les k-formes irréductibles, constitue un nouveau modèle combinatoire des nombres de Genocchi, et plus précisément des polynômes de Gandhi, extensions bien connues des nombres de Genocchi. On démontre la conjecture en construisant une bijection entre les k-formes irréductibles et un modèle combinatoire connu des polynômes de Gandhi, les pistolets surjectifs de hauteur $k-1$.

La dernière partie de ce mémoire se place dans le contexte des statistiques de permutations, sujet initié par MacMahon au début du $\text{XX}^{\text{ème}}$ siècle et devenu par la suite une branche importante de la combinatoire énumérative. On y établit une nouvelle égalité entre polynômes q-eulériens, des q-analogues des polynômes eulériens classiques apparus dans les travaux d'Euler, riches en interprétations combinatoires (notamment sur les permutations et selon les statistiques introduites par MacMahon) et activement étudiés en mathématiques et informatique. L'égalité proprement dite est la conséquence d'une bijection que l'on construit sur les permutations et qui envoie un triplet de statistiques définissant le premier polynôme eulérien sur un triplet définissant le second. Cette égalité généralise au passage un résultat obtenu par Shareshian et Wachs au moyen de techniques faisant intervenir les fonctions quasi-symétriques.

Dans les lignes suivantes, on commence par donner des éléments de vocabulaire de la combinatoire énumérative et bijective, puis l'on introduit la classe combinatoire des permutations où l'on définit les quatre statistiques fondamentales de MacMahon, et l'on finit par présenter la suite des nombres de Genocchi, leurs extensions et interprétations combinatoires, avant de résumer les trois chapitres qui constituent le corps de ce mémoire.

La combinatoire énumérative

On nomme classe combinatoire un ensemble $\mathcal{A}$ muni d'une fonction *taille*, c'est-à-dire une fonction $\tau : \mathcal{A} \to \mathbb{N}$ telle que le sous-ensemble $\mathcal{A}_n = \{a \in \mathcal{A} : \tau(a) = n\}$, dit des *objets de* $\mathcal{A}$ *de taille* n, soit fini pour tout $n \in \mathbb{N}$. Quand il n'y a pas d'ambiguïté sur la nature de la fonction taille d'une classe combinatoire $(\mathcal{A}, \tau)$, on se permet de désigner cette dernière par l'ensemble sous-jacent $\mathcal{A}$. Une statistique d'une classe combinatoire $\mathcal{A}$ est une fonction $\text{st} : \mathcal{A} \to \mathbb{N}$. Si $\text{st}_1, \text{st}_2, \ldots, \text{st}_m$ sont m statistiques de $\mathcal{A}$, le polynôme générateur de $\mathcal{A}_n$ selon le m-uplet de statistiques $(\text{st}_1, \text{st}_2, \ldots, \text{st}_m)$ est défini comme le polynôme en m variables

$$P_n(x_1, x_2, \ldots, x_m) = \sum_{a \in \mathcal{A}_n} x_1^{\text{st}_1(a)} x_2^{\text{st}_2(a)} \ldots x_m^{\text{st}_m(a)}.$$

On dit alors que $P_n(x_1, x_2, \ldots, x_m)$ est généré par $\mathcal{A}_n$ selon le m-uplet de statistiques $(\mathrm{st}_1, \mathrm{st}_2, \ldots, \mathrm{st}_m)$, ou encore que la donnée de $\mathcal{A}_n$ et de $(\mathrm{st}_1, \mathrm{st}_2, \ldots, \mathrm{st}_m)$ constitue une interprétation combinatoire de $P_n(x_1, x_2, \ldots, x_m)$. Si deux m-uplets de statistiques $(\mathrm{st}_1^1, \mathrm{st}_2^1, \ldots, \mathrm{st}_m^1)$ et $(\mathrm{st}_1^2, \mathrm{st}_2^2, \ldots, \mathrm{st}_m^2)$ sur une même classe combinatoire $\mathcal{A}$ vérifient

$$\sum_{a \in \mathcal{A}_n} x_1^{\mathrm{st}_1^1(a)} x_2^{\mathrm{st}_2^1(a)} \ldots x_m^{\mathrm{st}_m^1(a)} = \sum_{a \in \mathcal{A}_n} x_1^{\mathrm{st}_1^2(a)} x_2^{\mathrm{st}_2^2(a)} \ldots x_m^{\mathrm{st}_m^2(a)},$$

on dit qu'ils sont *équidistribués*.

Déterminer une formule close d'un polynôme générateur, et, inversement, interpréter combinatoirement un polynôme à coefficients entiers (c'est-à-dire déterminer une classe combinatoire et un ensemble de statistiques dont il apparaît comme le polynôme générateur) constituent deux champs d'activités de la combinatoire énumérative, dont les implications dépassent le seul contexte de la combinatoire et sont à même de fournir des informations générales sur des familles de polynômes, comme par exemple la positivité des coefficients d'un polynôme donné (voir par exemple [HRW15]) ou l'explicitation de polynômes de Poincaré (voir [Fei12, Fei11, IFR12]).

Il existe diverses façons d'énumérer les éléments d'un ensemble fini (ou plus généralement de calculer le polynôme générateur d'un ensemble fini selon un m-uplet de statistiques). Il peut par exemple s'agir d'établir une formule de récurrence. Une autre méthode repose sur une notion importante en combinatoire énumérative, celle des séries génératrices [Sta11, FS09]. Etant donnée une suite de polynômes $(P_n(x_1, x_2, \ldots, x_m))_{n \geq 0}$, les *série génératrice ordinaire* et *série génératrice exponentielle* de $(P_n(x_1, x_2, \ldots, x_m))_{n \geq 0}$ sont respectivement définies comme les séries formelles

$$\sum_{n \geq 0} P_n(x_1, x_2, \ldots, x_m) t^n \text{ et } \sum_{n \geq 0} P_n(x_1, x_2, \ldots, x_m) \frac{t^n}{n!}.$$

L'étude de ces séries formelles peut révéler des propriétés combinatoires ou analytiques des suites polynômiales considérées, et l'on s'en sert souvent pour en déterminer des formules exactes ou encore des égalités entre deux suites polynômiales, obtenues en montrant l'égalité des séries génératrices correspondantes.

Une dernière méthode, plus fondamentale, consiste à construire une bijection entre l'ensemble fini dont on veut déterminer le cardinal ou le polynôme générateur, et un ensemble pour lequel les caractéristiques en question sont connues. La combinatoire bijective est la partie de la combinatoire énumérative s'intéressant à cette approche, et vise ainsi à établir des bijections entre classes combinatoires, c'est-à-dire, pour deux classes combinatoires $\mathcal{A}$ et $\mathcal{B}$ données, des suites de fonctions $(f_n)_{n\geq 0}$ où $f_n : \mathcal{A}_n \to \mathcal{B}_n$ est une bijection envoyant un m-uplet de statistiques $(\mathrm{st}_1^{\mathcal{A}}, \mathrm{st}_2^{\mathcal{A}}, \ldots, \mathrm{st}_m^{\mathcal{A}})$ sur un m-uplet de statistiques $(\mathrm{st}_1^{\mathcal{B}}, \mathrm{st}_2^{\mathcal{B}}, \ldots, \mathrm{st}_m^{\mathcal{B}})$, c'est-à-dire

$$\big(\mathrm{st}_1^{\mathcal{B}}(f_n(a)), \mathrm{st}_2^{\mathcal{B}}(f_n(a)), \ldots, \mathrm{st}_m^{\mathcal{B}}(f_n(a))\big) = \big(\mathrm{st}_1^{\mathcal{A}}(a), \mathrm{st}_2^{\mathcal{A}}(a), \ldots, \mathrm{st}_m^{\mathcal{A}}(a)\big)$$

pour tout $n \geq 0$ et $a \in \mathcal{A}_n$. Les classes $\mathcal{A}$ et $\mathcal{B}$, munies des statistiques sus-nommées, apparaissent alors comme deux interprétations combinatoires d'une même suite polynômiale. L'intérêt de la démarche peut être de déterminer de nouvelles interprétations combinatoires d'une suite polynômiale donnée à partir d'interprétations connues, ou de déterminer de nouvelles identités entre suites polynômiales. Il est par ailleurs naturel de chercher à faire correspondre deux modèles combinatoires d'une même expression (par exemple obtenue en montrant l'égalité des séries génératrices des deux classes combinatoirées considérées) par une bijection reliant les statistiques associées.

La classe combinatoire des permutations

On note $\mathfrak{S}_n$ l'ensemble des permutations de $[n] = \{1, 2, \ldots, n\}$, c'est-à-dire les bijections de $[n]$ dans lui-même. L'ensemble $\bigsqcup_{n\geq 1} \mathfrak{S}_n$, muni de la fonction $\tau : \sigma \mapsto |Im(\sigma)|$, constitue une classe combinatoire classique.

Au début du XX$^{\text{ème}}$ siècle, MacMahon [Mac15] a étudié quatre statistiques fondamentales sur les permutations[1] : les statistiques respectives des descentes, exédents, inversions, et l'indice majeur.

Une *descente* (respectivement un *excédent*) d'une permutation $\sigma \in \mathfrak{S}_n$ est un entier $i \in [n-1]$ vérifiant $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$). On note $\mathrm{DES}(\sigma)$ (resp. $\mathrm{EXC}(\sigma)$) l'ensemble des descentes (resp. excédents) de σ, et l'on note $\mathrm{des}(\sigma)$ (resp. $\mathrm{exc}(\sigma)$) son cardinal. MacMahon [Mac15] a montré que les statistiques des et exc sont équidistribuées, c'est-à-dire que

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

pour tout $n \geq 1$, en prouvant plus précisément que les séries génératrices de ces deux polynômes en t sont égales. Riordan [RS73] a plus tard démontré

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[1] Plus généralement sur les permutations de multiensembles.

que ce polynôme est en réalité le $n^{\text{ème}}$ polynôme eulérien $A_n(t)$ apparaissant (voir [Eul55] et [Sta11, page 34]) dans la formule

$$\sum_{n\geq 0} A_n(t)\frac{x^n}{n!} = \frac{t-1}{t-exp((t-1)x)}. \tag{1}$$

Les polynômes eulériens apparaissent dans différents contextes des mathématiques et possèdent une combinatoire riche (voir par exemple [FS70] et [Knu98, section 5.1]). De manière générale, une statistique st sur les permutations, dont le polynôme générateur associé $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{st}(\sigma)}$ vaut $A_n(t)$, est dite *eulérienne*. Outre les statistiques des et exc, la statistique asc, qui envoie $\sigma \in \mathfrak{S}_n$ sur le nombre d'*ascensions* $\mathrm{asc}(\sigma) = |\,\mathrm{ASC}(\sigma)|$ de σ, c'est-à-dire les entiers $i \in [i-1]$ vérifiant $\sigma(i) < \sigma(i+1)$, et la statistique ides définie par $\mathrm{ides}(\sigma) = \mathrm{des}(\sigma^{-1})$, sont deux autres exemples de statistiques eulériennes.

Une *inversion* de $\sigma \in \mathfrak{S}_n$ est un couple $(i,j) \in [n]^2$ tel que $i < j$ et $\sigma(i) > \sigma(j)$. On note $\mathrm{INV}(\sigma)$ l'ensemble des inversions de σ et $\mathrm{inv}(\sigma)$ son cardinal. Il est bien connu (voir par exemple [Mac15] ou [Sta11, Sta99]) que

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!$$

où le *q-factoriel* $[n]_q!$, q-analogue du factoriel $n!$, est défini comme le polynôme en q

$$[n]_q! = \prod_{i=1}^{n} \frac{1-q^i}{1-q}.$$

Toute statistique st sur les permutations est alors dite *mahonienne* si le polynôme générateur associé $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{st}(\sigma)}$ vaut le q-factoriel $[n]_q!$. MacMahon [Mac15] a défini l'*indice majeur*, une statistique sur les permutations envoyant $\sigma \in \mathfrak{S}_n$ sur l'entier naturel

$$\mathrm{maj}(\sigma) = \sum_{i\in\mathrm{DES}(\sigma)} i,$$

et a démontré qu'elle est mahonienne en montrant l'égalité des séries génératrices des suites $\left(\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}\right)_{n\geq 1}$ et $\left(\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}\right)_{n\geq 1}$.

Nombres de Genocchi et polynômes de Gandhi

Les nombres de Bernoulli et de Genocchi apparaissent dans des domaines aussi divers que la théorie des nombres, l'analyse asymptotique, la topologie différentielle ou encore la théorie des formes modulaires. Leurs propriétés combinatoires font ainsi l'objet d'études intensives (voir par exemple [Ara14, AABS13, AAS13, Kim10, Gen52, SKS12, Sri11]).

Les nombres de Bernoulli $(B_n)_{n\geq 0} = (1, -1/2, 1/6, 0, -1/30, 0, 1/42, \ldots)$ [OEIa, OEIb] sont des nombres rationnels pouvant être définis par la formule

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

Les nombres de Genocchi $(G_{2n})_{n\geq 1} = (1, 1, 3, 17, \ldots)$ [OEIc] (de signes constants, il s'agit d'une définition alternative des nombres de Genocchi de signes alternés présentés dans [Sta99, page 74]) ont pour série génératrice

$$\frac{2t}{1 + e^t} = t + \sum_{n\geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}$$

et sont reliés aux nombres de Bernoulli par la formule

$$G_{2n} = 2(-1)^n (1 - 2^{2n}) B_{2n}.$$

Il s'agit d'entiers naturels, propriété plus visible dans la définition suivante, fournie par le triangle de Seidel [DV80] $(g_{i,j})_{1\leq j\leq \lceil i/2\rceil}$ défini par $g_{1,1} = 1$ et, pour tout $p \geq 1$,

$$g_{2p,j} = g_{2p-1,j} + g_{2p,j+1} \text{ pour tout } 1 \leq j \leq p,$$
$$g_{2p+1,j} = g_{2p+1,j-1} + g_{2p,j} \text{ pour tout } 1 \leq j \leq p+1,$$

en posant également $g_{2p,p+1} = g_{2p+1,0} = 0$ pour tout $p \geq 1$: les nombres de Genocchi $(G_{2n})_{n\geq 1}$ sont les entiers $(g_{2n-1,n})_{n\geq 1}$, qui apparaissent encadrés dans la Figure 1 représentant le triangle de Seidel, chaque entier $g_{i,j}$ y étant la somme des entiers desquels partent une flèche vers $g_{i,j}$. Toujours dans la Figure 1, les entiers $(g_{2n,1})_{n\geq 1}$, apparaissant encerclés, sont nommés *nombres de Genocchi médians* et notés $(H_{2n+1})_{n\geq 0} = (1, 2, 8, 56, \ldots)$ [OEId]. Il est bien connu [BD81] que l'entier H_{2n+1} est divisible par 2^n pour tout n. Les nombres de Genocchi médians normalisés $(h_n)_{n\geq 0} = (1, 1, 2, 7, \ldots)$ [OEIe] sont alors définis par

$$h_n = H_{2n+1}/2^n. \tag{2}$$

Les nombres de Genocchi $(G_{2n+2})_{n\geq 1}$ apparaissent également comme des évaluations des polynômes de Gandhi définis ci-après.

$$\begin{tikzcd}
i\backslash j & 0 & 1 & 2 & 3 & 4 & \cdots \\
1 & & \boxed{1} \arrow[d] & & & & \\
2 & & \enclose{circle}{1} \arrow[d] & 0 \arrow[l] \arrow[d] & & & \\
3 & 0 \arrow[r] & 1 \arrow[r] \arrow[d] & \boxed{1} \arrow[d] & & & \\
4 & & \enclose{circle}{2} \arrow[d] & 1 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & & \\
5 & 0 \arrow[r] & 2 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & \boxed{3} \arrow[d] & & \\
6 & & \enclose{circle}{8} \arrow[d] & 6 \arrow[l] \arrow[d] & 3 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \\
7 & 0 \arrow[r] & 8 \arrow[r] \arrow[d] & 14 \arrow[r] \arrow[d] & 17 \arrow[r] \arrow[d] & \boxed{17} \arrow[d] & \\
8 & & \enclose{circle}{56} \arrow[d] & 48 \arrow[l] \arrow[d] & 34 \arrow[l] \arrow[d] & 17 \arrow[l] \arrow[d] & 0 \arrow[l] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

FIGURE 1 – Triangle de Seidel $(g_{i,j})$ des nombres de Genocchi.

Définition 0.1 (polynômes de Gandhi [Gan70])**.** Les polynômes de Gandhi $(Q_n(x))_{n\geq 1}$ sont définis par la récurrence

$$\begin{cases} Q_1(x) & = 1, \\ Q_n(x) & = (x+1)^2 Q_{n-1}(x+1) - x^2 Q_{n-1}(x), \; n \geq 2. \end{cases}$$

Il s'agit de polynômes à coefficients entiers positifs. Par exemple, les trois premiers polynômes de Gandhi sont :

$$Q_1(x) = 1,$$
$$Q_2(x) = 2x + 1,$$
$$Q_3(x) = 6x^2 + 8x + 3.$$

Théorème 0.2 ([Car71, RS73])**.** *Pour tout $n \geq 1$, l'entier naturel $Q_n(1)$ est le nombre de Genocchi G_{2n+2}.*

Interprétations combinatoires

Dumont [Dum74, DR94] a défini plusieurs modèles combinatoires des nombres de Genocchi et nombres de Genocchi médians.

Le nombre de Genocchi médian H_{2n+1} compte entre autres les permutations de Dumont d'ordre $2n+2$ [DR94], c'est-à-dire les éléments de l'ensemble

$$\mathcal{D}_{n+1} = \{\sigma \in \mathfrak{S}_{2n+2} : \sigma(2i) < 2i \text{ et } \sigma(2i-1) > 2i-1 \text{ pour tout } i\}. \qquad (3)$$

Par exemple, pour $n = 2$, les $H_5 = 8$ éléments de $\mathcal{D}_3 \subset \mathfrak{S}_6$ sont les permutations

$$214365, 215364, 314265, 315264, 415263, 415362, 514263, 514362.$$

Le nombre de Genocchi G_{2n} (où $n \geq 2$) peut compter quant à lui les pistolets surjectifs de hauteur $n-1$. Plus précisément, le polynôme de Gandhi $Q_{n-1}(x)$ est généré par les pistolets surjectifs de hauteur $n-1$ selon la statistique des points fixes, définis ci-dessous.

Définition 0.3. Un pistolet surjectif de hauteur $k \geq 1$ est une fonction surjective de $[2k]$ dans $\{2, 4, \ldots, 2k\}$ vérifiant $f(j) \geq j$ pour tout j. On note SP_k l'ensemble de ces fonctions. On se permet d'identifier chaque pistolet surjectif $f \in \mathrm{SP}_k$ avec la suite $(f(j))_{j\in[2k]}$. On définit alors une statistique fix sur les pistolets, comptant les points fixes d'un pistolet $f \in \mathrm{SP}_k$, c'est-à-dire les entiers $j \in [2k]$ tels que $f(j) = j$.

Une manière de représenter un pistolet surjectif est de tracer un tableau constitué de k rangées de $2, 4, 6, \ldots, 2k$ cases (du bas vers le haut) justifiées à gauche (présentant ainsi la forme d'un pistolet orienté vers la droite), de sorte que chaque rangée contienne au moins un point (traduisant la surjectivité) et chaque colonne exactement un point (traduisant le statut de fonction). La fonction f correspondant à un tel tableau est alors définie par $f(j) = 2(\lceil j/2 \rceil + z_j)$ où la $j^{\text{ème}}$ colonne du tableau (de gauche à droite) contient son point dans sa $(1+z_j)^{\text{ème}}$ case (de bas en haut) pour tout $j \in [2k]$. Par exemple, si $f = (2,4,4,8,8,6,8,8) \in \mathrm{SP}_4$, le tableau correspondant à f est représenté dans la Figure 2. En particulier, un entier j est un point fixe de f si et seulement si j est pair et si le point de la $j^{\text{ème}}$ colonne du tableau associé à f se trouve dans sa case la plus basse. Par exemple, le pistolet de la Figure 2 a pour points fixes les entiers 6 et 8 (il est clair par ailleurs que $2k$ est nécessairement le point fixe de tout pistolet $f \in \mathrm{SP}_k$, d'où $\mathrm{fix}(f) \geq 1$).

On doit à Dumont le résultat suivant.

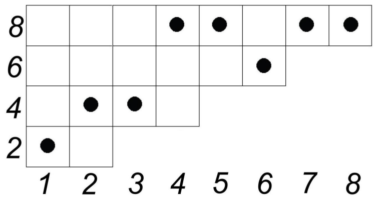


FIGURE 2 – Pistolet surjectif $f = (2, 4, 4, 8, 8, 6, 8, 8) \in \mathrm{SP}_4$.

Théorème 0.4 ([Dum74]). *Pour tout $k \geq 1$, le polynôme de Gandhi $Q_k(x)$ est généré par les pistolets surjectifs de hauteur k selon l'égalité*

$$Q_k(x) = \sum_{f \in SP_k} x^{\mathit{fix}(f)-1}.$$

En particulier, en évaluant $Q_k(x)$ en $x = 1$, on obtient effectivement $G_{2k+2} = |SP_k|$. Par exemple, les $G_6 = 3$ elements de SP_2 sont

$$\begin{aligned} f_1 &= (2, 2, 4, 4) \\ f_2 &= (2, 4, 4, 4) \\ f_3 &= (4, 2, 4, 4) \end{aligned}$$

et l'on retrouve $x^{\mathrm{fix}(f_1)-1} + x^{\mathrm{fix}(f_2)-1} + x^{\mathrm{fix}(f_3)-1} = x + 1 + x = Q_2(x)$.

Résumé du mémoire

Chapitre 1 : étude combinatoire des configurations de Dellac et des nombres de Genocchi médians q-étendus normalisés

Une configuration de Dellac de taille n[Del00] est un tableau rectangulaire C constitué de $2n$ rangées de n cases et contenant $2n$ points répartis selon les conditions suivantes :

- chaque rangée de C contient exactement un point ;
- chaque colonne de C contient exactement deux points ;
- si le coin inférieur gauche de la case la plus basse et la plus à gauche de C a pour coordonnées cartésiennes $(0, 0)$, chaque point de C est situé entre les deux droites $y = x$ et $y = n + x$.

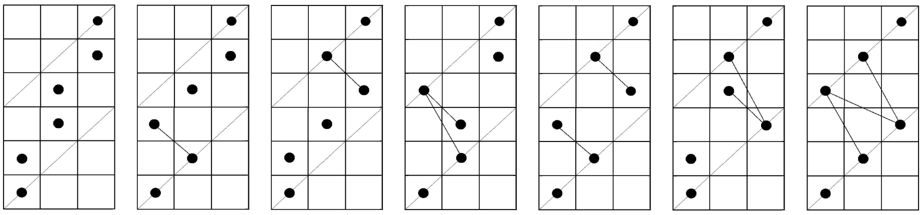


FIGURE 3 – Les $h_3 = 7$ éléments de DC(3).

On note $DC(n)$ l'ensemble des configurations de Dellac de taille n. Une statistique, nommée fal, compte le nombre de *chutes* (falls) d'une configuration de Dellac, c'est-à-dire les couples de points de coordonnées cartésiennes (x_1, y_1) et (x_2, y_2) telles que $x_1 < x_2$ et $y_1 > y_2$. Il est connu (voir par exemple [Fei11]) que $DC(n)$ a pour cardinal le nombre de Genocchi médian normalisé h_n pour tout $n \geq 1$. Par exemple, la Figure 3 illustre les $h_3 = 7$ configurations de Dellac de taille 3, dont les chutes sont représentées par des segments.

A travers la géométrie des grassmaniennes de carquois [IFR12] et la théorie des fractions continues de Flajolet [Fla80], Feigin [Fei12] a montré que la fonction génératrice $\tilde{h}_n(q)$ (voir Definition 1.5) des configurations de Dellac $C \in DC(n)$ selon une statistique reliée à fal, égale le nombre de Genocchi médian q-étendu normalisé $\bar{c}_{n+1}(q)$ introduit par Han et Zeng [HZ99b]. Plus précisément, en définissant la suite $(\lambda_n(q))_{n\geq 1}$ par

$$\lambda_{2p-1}(q) = (1 - q^{p+1})(1 - q^p)/((1 - q^2)(1 - q))$$

et $\lambda_{2p}(q) = q\lambda_{2p-1}(q)$ pour tout $p \geq 1$, Feigin a démontré que la série génératrice $\sum_{n\geq 0} \tilde{h}_n(q)t^n$ égale la fraction continue

$$\cfrac{1}{1 - \cfrac{\lambda_1(q)t}{1 - \cfrac{\lambda_2(q)t}{1 - \cfrac{\lambda_3(q)t}{\ddots}}}} \tag{4}$$

que Han et Zeng ont déjà prouvé être la série génératrice $\sum_{n\geq 0} \bar{c}_{n+1}(q)t^n$.

Dans ce chapitre, on fait la jonction entre ces résultats par la combinatoire bijective en définissant tout d'abord deux nouveaux modèles combinatoires $\mathcal{D}'_{n+1}$ et $DH(n)$ du polynôme $\bar{c}_{n+1}(q)$, puis en construisant deux bijections $\phi : DC(n) \to \mathcal{D}'_{n+1}$ et $\phi : DC(n) \to \mathrm{DH}(n)$ entre les différents modèles.

Le premier modèle $\mathcal{D}'_{n+1} \subset \mathfrak{S}_{2n+2}$ est l'ensemble des *permutations de Dumont normalisées* d'ordre $2n+2$ (voir Définition 1.8). La construction de $\phi : DC(n) \to \mathcal{D}'_{n+1}$ (voir Définition 1.24) consiste à étiqueter les points des configurations de Dellac avec des entiers de 1 à $2n$ de sorte que la permutation $\phi(C)$ apparaisse naturellement sur la configuration étiquetée $C \in DC(n)$.

Le second modèle $\mathrm{DH}(n)$ est l'ensemble des *histoires de Dellac* de longueur $2n$ (voir Définition 1.45). Il s'agit de chemins de Dyck à $2n$ pas et dont les pas descendants sont étiquetés par des couples d'entiers vérifiant certaines conditions. La construction de l'application $\psi : \mathrm{DC}(n) \to \mathrm{DH}(n)$ (voir Définition 1.53) consiste à reprendre l'étiquetage des points d'une configuration de Dellac $C \in DC(n)$ défini dans la première partie du chapitre, de considérer les points de C dans un certain ordre et de tracer successivement des pas ascendants ou descendants selon la parité des étiquettes des points, tout en étiquetant les pas descendants par des couples d'entiers dépendant du nombre de chutes faisant intervenir les points considérés. Par rapport à la première bijection ϕ, qui se contente d'impliquer l'égalité $\tilde{h}_n(q) = \bar{c}_{n+1}(q)$, la bijection ψ implique également que la série génératrice $\sum_{n\geq 1} \tilde{h}_n(q)t^n$ égale la fraction continue de la Formule (4) via la théorie de Flajolet [Fla80].

Chapitre 2 : une bijection entre k-formes irréductibles et pistolets surjectifs de hauteur $k-1$

Soit k un entier naturel. L'étude des k-formes (k-shapes) intervient dans celle des k-fonctions de Schur (voir [LLMS13]). Les k-fonctions de Schur [LM05] sont des fonctions symétriques formant une base de l'ensemble $\mathrm{Sym}^{(k)}$, sous-ensemble de Sym généré par les fonctions symétriques complètement homogènes h_λ, où λ parcourt les partitions dont aucune part n'excède k. Les k-formes sont des partitions (c'est-à-dire des suites finies et décroissantes d'entiers naturels) particulières apparaissant dans la décomposition des k-fonctions de Schur dans la base des fonctions de Schur classiques, les coefficients de ladite décomposition faisant intervenir l'énumération de chemins dans le poset des k-formes [LLMS13].

Hivert et Mallet [HM11] ont montré que la fonction génératrice des k-formes (selon une statistique comptant le nombre de cases dont la longueur d'équerre, *hook length* en anglais, vaut au plus k dans le diagramme de Ferrers des k-formes) est une fraction rationnelle dont le numérateur $P_k(t)$ est le polynôme générateur de k-formes particulières nommées k-formes irréductibles (voir Définition 2.6). L'appellation *irréductible* vient d'une opération sur les k-formes, définie par Hivert et Mallet, permettant de définir de nouvelles k-formes à partir d'une k-forme donnée, les k-formes irréductibles apparaissant comme les k-formes ne pouvant être obtenues par cette opération. La suite des nombres de k-formes irréductibles $(P_k(1))_{k\geq 1}$ semblait alors être celle des nombres de Genocchi $(G_{2k})_{k\geq 1} = (1, 1, 3, 17, 155, 2073, \ldots)$. Plus précisément, en définissant une statistique sur les k-formes comptant le nombre de «k-sites libres» d'une k-forme donnée, Hivert et Mallet ont conjecturé que le polynôme générateur des k-formes irréductibles relativement à cette dernière statistique était le $(k-1)^{\text{ème}}$ polynôme de Gandhi $Q_{k-1}(x)$, extension bien connue du nombre de Genocchi G_{2k} [Car71, RS73].

Le but de ce chapitre est de construire une bijection $\varphi : \mathrm{SP}_{k-1} \to \mathrm{IS}_k$ (voir Définition 2.16) entre les pistolets surjectifs de hauteur $k-1$, modèle combinatoire des polynômes de Gandhi, et les k-formes irréductibles, envoyant les points fixes des premiers sur les k-sites libres des secondes et prouvant ainsi la conjecture de Hivert et Mallet tout en fournissant un algorithme permettant de générer les k-formes irréductibles à partir des pistolets surjectifs.

On étudie ensuite les nouvelles interprétations combinatoires des polynômes de Dumont-Foata (extensions des polynômes de Gandhi) produites par cette bijection.

Chapitre 3 : une nouvelle bijection connectant des polynômes q-eulériens

Dans [SW14], Shareshian et Wachs manipulent des statistiques raffinant les notions de descentes, ascensions et inversions des permutations. On s'intéresse ici aux statistiques des_2, asc_2 et inv_2 comptant respectivement les *2-descentes* (2-descents), *2-ascensions* (2-ascents) et *2-inversions* d'une permutation $\sigma \in \mathfrak{S}_n$, c'est-à-dire les éléments des ensembles respectifs

$$\begin{aligned} \mathrm{DES}_2(\sigma) &= \{i \in [n-1] : \sigma(i) > \sigma(i+1)+1\}, \\ \mathrm{ASC}_2(\sigma) &= \{i \in [n-1] : \sigma(i) < \sigma(i+1)+1\}, \\ \mathrm{INV}_2(\sigma) &= \{(i,j) \in [n]^2 : \sigma(i) = \sigma(j)+1\}. \end{aligned}$$

A noter que la statistique inv_2 est eulérienne par l'égalité facilement vérifiable $\mathrm{inv}_2(\sigma) = \mathrm{des}(\sigma^{-1})$ pour toute permutation σ. On considère également l'*indice 2-majeur maj*$_2$ défini par $\mathrm{maj}_2(\sigma) = \sum_{i\in \mathrm{DES}_2(\sigma)} i$, et son analogue amaj_2 pour les 2-ascensions, défini par $\mathrm{amaj}_2(\sigma) = \sum_{i\in \mathrm{ASC}_2(\sigma)} i$.

Un polynôme q-eulérien à m variables est un polynôme $P_n(x_1, x_2, \dots, x_m)$ égalant le polynôme eulérien classique $A_n(t)$ en spécifiant $x_{i_0} = t$ pour un certain $i_0 \in [m]$ et $x_i = 1$ pour tout $i \neq i_0$. Dans le cadre de la théorie des fonctions quasisymétriques, Shareshian et Wachs [SW14] ont démontré l'égalité entre polynômes q-eulériens à 2 variables

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)}. \tag{5}$$

De façon analogue, Hance et Li [HL12] ont prouvé l'égalité entre polynômes q-eulériens à 3 variables

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{amaj}_2(\sigma)} y^{\widetilde{\mathrm{asc}_2}(\sigma)} z^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{exc}(\sigma)} \tag{6}$$

où la statistique $\widetilde{\mathrm{asc}_2}$ envoie une permutation σ sur $\mathrm{asc}_2(\sigma)$ si $\sigma(1) = 1$, et sur $\mathrm{asc}_2(\sigma) + 1$ autrement.

Le but de ce chapitre est de définir une statistique $\widetilde{\mathrm{des}_2}$ (analogue à $\widetilde{\mathrm{asc}_2}$ en cela qu'elle compte le nombre $\mathrm{des}_2(\sigma)$ d'une permutation σ en ajoutant 1 à la somme si σ vérifie une condition géométrique au niveau d'un certain graphe), puis de construire une bijection $\Psi : \mathfrak{S}_n \to \mathfrak{S}_n$ (voir §3.4) qui envoie le triplet de statistiques $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$ sur le triplet $(\mathrm{maj} - \mathrm{exc}, \mathrm{des}, \mathrm{exc})$, fournissant en particulier l'égalité entre polynômes q-eulériens à 3 variables

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\widetilde{\mathrm{des}_2}(\sigma)} z^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{exc}(\sigma)}, \tag{7}$$

qui généralise au passage la Formule (5).

Nomenclature

Let (n, m) be a pair of positive integers such that $n < m$, and σ be a permutation of the set $\{1, 2, \ldots, n\}$. By abuse of notation, we identify σ with the word $\sigma(1)\sigma(2)\ldots\sigma(n)$.

If a set $S = \{n_1, n_2, \ldots, n_k\}$ of integers is such that $n_1 < n_2 < \ldots < n_k$, we sometimes use the notation $S = \{n_1, n_2, \ldots, n_k\}_<$.

$[\|n, m\|]$	Set of integers $\{n, n+1, \ldots, m\}$.
$[n]$	Set of integers $\{1, 2, \ldots, n\}$.
$\binom{n}{2}$	Triangular number $n(n-1)/2$.
$\|S\|$	Cardinality of a finite set S.
$\mathbb{N}$	Set of the nonnegative integers $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{N}_{>0}$	Set of the positive integers $\mathbb{N} \setminus \{0\}$.
$\mathbb{Z}$	Set of all integers.
$\mathfrak{S}_n$	Set of the permutations of $[n]$.
Sym	Set of the symmetric functions $f(x_1, x_2, \ldots)$ over the ring $\mathbb{Z}$.
Id_S	Identity map of a set S.
$Im(f)$	Image of a function f.

Chapter 1

Combinatorial study of Dellac configurations and q-extended normalized median Genocchi numbers

1.1 Abstract

In two recent chapters [Fei12, Fei11], by using mainly geometric considerations, Feigin proved that the Poincar polynomials of the degenerate flag varieties have a combinatorial interpretation through Dellac configurations, and related them to the q-extended normalized median Genocchi numbers $\bar{c}_n(q)$ introduced by Han and Zeng [HZ99a].

In this chapter, we give combinatorial proofs of these results by constructing statistic-preserving bijections between Dellac configurations and two other combinatorial models of $\bar{c}_n(q)$.

1.2 Introduction

This chapter largely follows [Big14].

The following lines recall the definition of Genocchi numbers. The new content starts at §1.2.1.

The Genocchi numbers $(G_{2n})_{n\geq 1} = (1, 1, 3, 17, \ldots)$ [OEIc] and the median Genocchi numbers $(H_{2n+1})_{n\geq 0} = (1, 2, 8, 56, \ldots)$ [OEId] can be defined (see

$$\begin{tikzcd}
i\backslash j & 0 & 1 & 2 & 3 & 4 & \dots \\
1 & & \boxed{1} \arrow[d] & & & & \\
2 & & \textcircled{1} \arrow[d] & 0 \arrow[l] \arrow[d] & & & \\
3 & 0 \arrow[r] & 1 \arrow[r] \arrow[d] & \boxed{1} \arrow[d] & & & \\
4 & & \textcircled{2} \arrow[d] & 1 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & & \\
5 & 0 \arrow[r] & 2 \arrow[r] \arrow[d] & 3 \arrow[r] \arrow[d] & \boxed{3} \arrow[d] & & \\
6 & & \textcircled{8} \arrow[d] & 6 \arrow[l] \arrow[d] & 3 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \\
7 & 0 \arrow[r] & 8 \arrow[r] \arrow[d] & 14 \arrow[r] \arrow[d] & 17 \arrow[r] \arrow[d] & \boxed{17} \arrow[d] & \\
8 & & \textcircled{56} \arrow[d] & 48 \arrow[l] \arrow[d] & 34 \arrow[l] \arrow[d] & 17 \arrow[l] \arrow[d] & 0 \arrow[l] \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

Figure 1.1: Seidel triangle $(g_{i,j})$ of the Genocchi numbers.

[DV80]) as the positive integers $(g_{2n-1,n})_{n\geq 1}$ and $(g_{2n+2,1})_{n\geq 0}$ respectively in the Seidel triangle $(g_{i,j})_{1\leq j\leq \lceil i/2\rceil}$ (see Figure 1.1 where the Genocchi numbers are boxed and the median Genocchi numbers are circled) defined by $g_{1,1}=1$ and, for all $p\geq 1$,

$$g_{2p,j} = g_{2p-1,j} + g_{2p,j+1} \text{ for all } 1\leq j\leq p,$$
$$g_{2p+1,j} = g_{2p+1,j-1} + g_{2p,j} \text{ for all } 1\leq j\leq p+1$$

(where $g_{2p,p+1} = g_{2p+1,0} = 0$ for all $p\geq 1$). It is well known that H_{2n+1} is divisible by 2^n (see [BD81]) for all $n\geq 0$. The *normalized median Genocchi numbers* $(h_n)_{n\geq 0} = (1,1,2,7,\dots)$ [OEIe] are the positive integers defined by

$$h_n = H_{2n+1}/2^n.$$

1.2.1 Combinatorial interpretations of the (normalized) median Genocchi numbers

Dumont [DR94] gave several combinatorial models of the Genocchi numbers and the median Genocchi numbers, among which are the *Dumont permutations.* We denote by $\mathfrak{S}_n$ the set of permutations of the set $[n] := \{1, 2, \ldots, n\}$, and by $\mathrm{inv}(\sigma)$ the number of inversions of a permutation $\sigma \in \mathfrak{S}_n$, *i.e.*, the number of pairs $(i, j) \in [n]^2$ with $i < j$ and $\sigma(i) > \sigma(j)$. Broadly speaking, the number of inversions $\mathrm{inv}(w)$ of a word $w = l_1 l_2 \ldots l_n$ with n letters in the alphabet $\mathbb{N}$ is the number of pairs $(i, j) \in [n]^2$ such that $i < j$ and $l_i > l_j$. In particular, the number $\mathrm{inv}(\sigma)$ associated with a permutation $\sigma \in \mathfrak{S}_n$ is the number $\mathrm{inv}(w)$ associated with the word $w = \sigma(1)\sigma(2) \ldots \sigma(n)$.

Definition 1.1. A Dumont permutation of order $2n$ is a permutation $\sigma \in \mathfrak{S}_{2n}$ such that $\sigma(2i) < 2i$ and $\sigma(2i - 1) > 2i - 1$ for all i. We denote by $\mathcal{D}_n$ the set of these permutations.

It is well-known (see [DR94]) that $H_{2n+1} = |\mathcal{D}_{n+1}|$ for all $n \geq 0$. In [HZ99b], Han and Zeng introduced the set $\mathcal{G}''_n$ of *normalized Genocchi permutations*, which consists of permutations $\sigma \in \mathcal{D}_n$ such that for all $j \in [n-1]$, the two integers $\sigma^{-1}(2j)$ and $\sigma^{-1}(2j + 1)$ have the same parity if and only if $\sigma^{-1}(2j) < \sigma^{-1}(2j + 1)$, and they proved that $h_n = |\mathcal{G}''_{n+1}|$ for all $n \geq 0$. The number h_n also counts the Dellac configurations of size n (see [Fei11]).

Definition 1.2. A Dellac configuration of size n is a tableau of width n and height $2n$ which contains $2n$ dots between the lines $y = x$ and $y = n + x$, such that each row contains exactly one dot and each column contains exactly two dots. Let $\mathrm{DC}(n)$ be the set of Dellac configurations of size n. A *fall* of $C \in \mathrm{DC}(n)$ is a pair (d_1, d_2) of dots whose Cartesian coordinates in C are respectively (j_1, i_1) and (j_2, i_2) such that $j_1 < j_2$ and $i_1 > i_2$. We denote by $\mathrm{fal}(C)$ the number of falls of C.

For example, the tableau depicted in Figure 1.2 is a Dellac configuration $C \in \mathrm{DC}(3)$ with $\mathrm{fal}(C) = 2$ falls (represented by two segments).

1.2.2 q-extended normalized median Genocchi numbers

In [HZ99a, HZ99b], Han and Zeng defined the *q-Gandhi polynomials of the second kind* $(C_n(x, q))_{n\geq 1}$ by $C_1(x, q) = 1$ and $C_{n+1}(x, q) = (1 +$

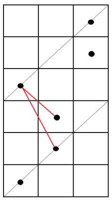


Figure 1.2: Dellac configuration $C \in \mathrm{DC}(3)$ with $\mathrm{fal}(C) = 2$ falls.

$qx)\Delta_q(xC_n(x,q))$, where

$$\Delta_q P(x) = (P(1+qx) - P(x))/(1+qx-x)$$

for all polynomial $P(x)$. They proved that the polynomials $C_n(1,q)$ are q-analogs of the median Genocchi numbers $(H_{2n+1})_{n\geq 0} = (1,2,8,56,\ldots)$ via $C_n(1,1) = H_{2n-1}$ for all $n \geq 1$. Furthermore, they gave a combinatorial interpretation of $C_n(1,q)$ through the set $\mathcal{D}_n$ of Dumont permutations.

Theorem 1.3 (Han and Zeng, 1997). *Let $n \geq 1$. For all $\sigma \in \mathcal{D}_n$, we define $st(\sigma)$ as the number*

$$st(\sigma) = n^2 - \sum_{i=1}^{n} \sigma(2i) - inv(\sigma^o) - inv(\sigma^e) \tag{1.1}$$

where σ^o and σ^e are the words $\sigma(1)\sigma(3)\ldots\sigma(2n-1)$ and $\sigma(2)\sigma(4)\ldots\sigma(2n)$ respectively (and where we define the number of inversions $inv(w)$ of a word w whose letters are integers as the number of pairs $(i,j) \in \mathbb{N}^2_{>0}$ such that w_i and w_j are defined and such that $i < j$ and $w_i > w_j$).

Then, the polynomial $C_n(1,q)$has the following combinatorial interpretation:

$$C_n(1,q) = \sum_{\sigma \in \mathcal{D}_n} q^{st(\sigma)}. \tag{1.2}$$

By introducing the subset $\mathcal{G}''_n \subset \mathcal{D}_n$ of normalized Genocchi permutations and using the combinatorial interpretation provided by Theorem 1.3, Han and Zeng proved combinatorially that the polynomial $(1+q)^{n-1}$ divides $C_n(1,q)$, which gives birth to polynomials $(\bar{c}_n(q))_{n\geq 1}$ defined by

$$\bar{c}_n(q) = C_n(1,q)/(1+q)^{n-1}. \tag{1.3}$$

This divisibility had previously been proved in the same chapter with a continued fraction approach, as a corollary of the following theorem and a well-known result on continued fractions (see [Fla80]).

Theorem 1.4 (Han and Zeng, 1997). *The generating function of the sequence* $(\bar{c}_{n+1}(q))_{n\geq 0}$ *is*

$$\sum_{n\geq 0} \bar{c}_{n+1}(q)t^n = \cfrac{1}{1-\cfrac{\lambda_1(q)t}{1-\cfrac{\lambda_2(q)t}{1-\cfrac{\lambda_3(q)t}{\ddots}}}} \tag{1.4}$$

where $\lambda_{2p-1}(q)$ *is the* q*-binomial coefficient*

$$\begin{Bmatrix} p+1 \\ 2 \end{Bmatrix}_q := (1-q^{p+1})(1-q^p)/((1-q^2)(1-q))$$

and $\lambda_{2p}(q) = \ q\lambda_{2p-1}(q)$ *for all* $p \geq 1$.

The polynomials $(\bar{c}_n(q))_{n\geq 1}$ are q-refinements of the normalized median Genocchi numbers: $\bar{c}_n(1) = h_{n-1}$ for all $n \geq 1$. They are named *q-extended normalized median Genocchi numbers*. In §1.3.1, we give a combinatorial interpretation of $\bar{c}_n(q)$ by slightly adjusting the definition of normalized Genocchi permutations.

In [Fei11, Fei12], Feigin introduced a q-analog of the normalized median Genocchi number h_n with the Poincar polynomial $P_{\mathcal{F}_n^a}(q)$ of the degenerate flag variety $\mathcal{F}_n^a$ (whose Euler characteristic is $P_{\mathcal{F}_n^a}(1) = h_n$), and gave a combinatorial interpretation of $P_{\mathcal{F}_n^a}(q)$ through Dellac configurations.

Theorem 1.5 (Feigin, 2012). *For all* $n \geq 0$*, the polynomial* $P_{\mathcal{F}_n^a}(q)$ *is generated by DC*(n)*:*

$$P_{\mathcal{F}_n^a}(q) = \sum_{C\in DC(n)} q^{2fal(C)}.$$

The degree of the polynomial $P_{\mathcal{F}_n^a}(q)$ being $n(n+1)$ (for algebraic considerations, or because every Dellac configuration $C \in \mathrm{DC}(n)$ has at most $\binom{n}{2}$ falls, see §1.3.1), Feigin introduced the following q-analog of h_n:

$$\tilde{h}_n(q) = q^{\binom{n}{2}} P_{\mathcal{F}_n^a}(q^{-1/2}) = \sum_{C\in \mathrm{DC}(n)} q^{\binom{n}{2}-\mathrm{fal}(C)}, \tag{1.5}$$

and proved the following theorem by using the geometry of quiver Grassmannians (see [IFR12]) and Flajolet's theory of continued fractions [Fla80].

Theorem 1.6 (Feigin, 2012)**.** *The generating function* $\sum_{n\geq 0}\tilde{h}_n(q)t^n$ *has the continued fraction expansion of Formula (1.4).*

Corollary 1.7 (Feigin, 2012)**.** *For all* $n \geq 0$*, we have* $\tilde{h}_n(q) = \bar{c}_{n+1}(q)$*.*

This raises two questions.

1. Prove combinatorially Corollary 1.7 by constructing a bijection between Dellac configurations and some appropriate model of $\bar{c}_n(q)$ which preserves the statistics.
2. Prove combinatorially Theorem 1.6 within the framework of Flajolet's theory of continued fractions by defining a combinatorial model of $\tilde{h}_n(q)$ related to Dyck paths (see [Fla80]), and constructing a statistic-preserving bijection between Dellac configurations and that new model.

The aim of this chapter is to answer above two questions.

We answer the first one in §1.3. In §1.3.1, we define a combinatorial model of $\bar{c}_n(q)$ through *normalized Dumont permutations*, and we provide general results about Dellac configurations. In §1.3.2, we enounce and prove Theorem 1.23, which connects Dellac configurations to normalized Dumont permutations through a stastistic-preserving bijection, and implies immediatly Corollary 1.7.

We answer the second question in §1.4. In §1.4.1, we recall the definition of a Dyck path and some results of Flajolet's theory of continued fractions. In §1.4.2, we define *Dellac histories*, which consist of Dyck paths weighted with pairs of integers, and we show that their generating function has the continued fraction expansion of Formula (1.4). In §1.4.3, we enounce and prove Theorem 1.48, which connects Dellac configurations to Dellac histories through a statistic-preserving bijection, thence proving Theorem 1.6 combinatorially.

1.3 Connection between Dellac configurations and Dumont permutations

In §1.3.1, we define *normalized Dumont permutations* of order $2n$, whose set is denoted by $\mathcal{D}'_n$, and we prove that they generate $\bar{c}_n(q)$ with respect to the statistic st defined in Formula (1.1), then we define the label of a Dellac configuration and a *switching* transformation on the set $\mathrm{DC}(n)$.

In §1.3.2, we enounce Theorem 1.23 and we prove it. To do so, we first give two maps $\phi :\ \mathrm{DC}(n) \rightarrow \mathcal{D}'_{n+1}$ and $\tilde{\phi} :\ \mathcal{D}_{n+1} \rightarrow\ \mathrm{DC}(n)$, and we prove that ϕ and $\tilde{\phi}_{|\mathcal{D}'_{n+1}}$ are inverse maps. Then, we show that Equation (1.6) is true for all $C \in \mathrm{DC}(n)$, by showing that it is true for some particular $C^0 \in \mathrm{DC}(n)$, then by connecting C^0 to every other $C \in \mathrm{DC}(n)$ thanks to a switching transformation, which happens to preserve Equation (1.6).

1.3.1 Preliminaries

Combinatorial interpretation of $\bar{c}_n(q)$.

Definition 1.8. A *normalized Dumont permutation* of order $2n$ is a permutation $\sigma \in \mathcal{D}_n$ such that, for all $j \in [n-1]$, the two integers $\sigma^{-1}(2j)$ and $\sigma^{-1}(2j+1)$ have the same parity if and only if $\sigma^{-1}(2j) > \sigma^{-1}(2j+1)$. Let $\mathcal{D}'_n \subset \mathcal{D}_n$ be the set of these permutations.

Proposition 1.9. *For all $n \geq 1$, we have $\bar{c}_n(q) = \sum_{\sigma \in \mathcal{D}'_n} q^{st(\sigma)}$.*

Proof. Let $j \in [n-1]$ and $\sigma \in \mathcal{D}_n$. Recall that

$$\mathrm{st}(\sigma) = n^2 - \sum_{i=1}^{n} \sigma(2i) - \mathrm{inv}(\sigma^o) - \mathrm{inv}(\sigma^e).$$

It is easy to see that the composition $\sigma' = (2j, 2j+1) \circ \sigma$ of σ with the transposition $(2j, 2j+1)$ is still a Dumont permutation, and that if σ fits the condition $C(j)$ defined as

$$"\sigma^{-1}(2j) > \sigma^{-1}(2j+1) \Leftrightarrow\ \sigma^{-1}(2j) \text{ and } \sigma^{-1}(2j+1) \text{ have the same parity}",$$

then $\mathrm{st}(\sigma') = \mathrm{st}(\sigma) + 1$.

Now, if we denote by $\mathcal{D}_n^j \subset \mathcal{D}_n$ the subset of permutations that fit the condition $C(j)$, then $\mathcal{D}_n$ is the disjoint union

$$\mathcal{D}_n^j \sqcup \left((2j, 2j+1) \circ \mathcal{D}_n^j\right),$$

where $(2j, 2j+1) \circ \mathcal{D}_n^j$ is the set defined as $\{(2j, 2j+1) \circ \sigma,\ \sigma \in \mathcal{D}_n^j\}$. Since $\mathrm{st}((2j, 2j+1) \circ \sigma) = \mathrm{st}(\sigma) + 1$ for all $\sigma \in \mathcal{D}_n^j$, Formula (1.2) of Theorem 1.3 becomes

$$C_n(1, q) = (1+q) \sum_{\sigma \in \mathcal{D}_n^j} q^{\mathrm{st}(\sigma)}.$$

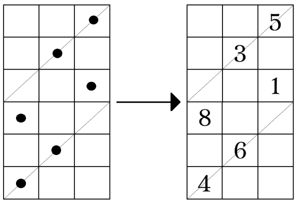


Figure 1.3: Label of a Dellac configuration $C \in \mathrm{DC}(3)$.

This yields immediatly:

$$C_n(1,q) = (1+q)^{n-1} \sum_{\sigma \in \bigcap_{j=1}^{n-1} \mathcal{D}_n^j} q^{\mathrm{st}(\sigma)} = (1+q)^{n-1} \sum_{\sigma \in \mathcal{D}'_n} q^{\mathrm{st}(\sigma)}.$$

The proposition then follows from Formula (1.3). ☐

Label of a Dellac configuration

Definition 1.10. Let $C \in \mathrm{DC}(n)$. For all $i \in [n]$, the dot of the i-th line of C (from bottom to top) is labelled by the integer $e_i = 2i+2$, and the dot of the $(n+i)$-th line is labelled by the integer $e_{n+i} = 2i-1$ (see Figure 1.3 for an example).

From now on, we will assimilate each dot of a Dellac configuration into its label.

Definition 1.11 (Particular dots)**.** Let $C \in \mathrm{DC}(n)$. For all $j \in [n]$, we define $i_1^C(j) < i_2^C(j)$ such that the two dots of the j-th column of C (from left to right) are $e_{i_1^C(j)}$ and $e_{i_2^C(j)}$. When there is no ambiguity, we write $e_{i_1(j)}$ and $e_{i_2(j)}$ instead of $e_{i_1^C(j)}$ and $e_{i_2^C(j)}$.

Afterwards, for all $i \in [n]$, we define the integers $p_C(i)$ and $q_C(i)$ such that $e_{p_C(i)}$ and $e_{n+q_C(i)}$ are respectively the i-th even dot and i-th odd dot of the sequence

$$\left(e_{i_1(1)}, e_{i_2(1)}, e_{i_1(2)}, e_{i_2(2)}, \dots, e_{i_1(n)}, e_{i_2(n)}\right).$$

For example, in Figure 1.3, the two dots $e_{i_1(2)}$ and $e_{i_2(2)}$ of the second column are respectively $6 = e_2 = e_{p_C(3)}$ and $3 = e_5 = e_{3+q_C(1)}$.

Remark 1.12. For all $i \in [2n]$, if the dot e_i appears in the j_i-th column of C, then, by Definition 1.2, we have $j_i \leq i \leq j_i + n$. As a result, the first j columns of C always contain the j even dots

$$e_1, e_2, \ldots, e_j,$$

and the only odd dots they may contain are

$$e_{n+1}, e_{n+2}, \ldots, e_{n+j}.$$

Likewise, the last $n - j + 1$ columns of C always contain the $n - j + 1$ odd dots

$$e_{n+j}, e_{n+j+1}, \ldots, e_{2n},$$

and the only even dots they may contain are

$$e_j, e_{j+1}, e_{j+2}, \ldots, e_n.$$

Remark 1.13. Let $C \in \mathrm{DC}(n)$ and $j \in [n]$. If the j-th column of C contains the even dot $e_{i \leq n} = 2i + 2$, then, since $j \leq i$, we have $e_i \in \{2j + 2, 2j + 4, \ldots, 2n + 2\}$. Similarly, if the j-th column of C contains the odd dot $e_{i>n} = 2(i - n) - 1$, since $i \leq j + n$, we have $e_i \in \{1, 3, \ldots, 2j - 1\}$. As a result, the three following conditions are equivalents :

$$\begin{array}{lr} (i) & e_{i_1^C(j)} > e_{i_2^C(j)}; \\ (ii) & i_1^C(j) \leq n < i_2^C(j); \\ (iii) & e_{i_1^C(j)} \text{ and } e_{i_2^C(j)} \text{ have different parities.} \end{array}$$

Definition 1.14 (Particular configurations)**.** For all $n \geq 1$, we denote by $C_0(n)$ (respectively $C_1(n)$) the Dellac configuration of size n such that $(e_{i_1(j)}, e_{i_2(j)}) = (e_{2j-1}, e_{2j})$ (resp. $(e_{i_1(j)}, e_{i_2(j)}) = (e_j, e_{n+j})$) for all $j \in [n]$.

For example $C_0(3)$ and $C_1(3)$ are the two configurations depicted in Figure 1.4. It is obvious that $C_0(n)$ is the unique Dellac configuration of size n with 0 fall, and that $\mathrm{fal}(C_1(n)) = \binom{n}{2}$. We can also prove by induction on $n \geq 1$ that every Dellac configuration $C \in \mathrm{DC}(n)$ has at most $\binom{n}{2}$ falls with equality if and only if $C = C_1(n)$.

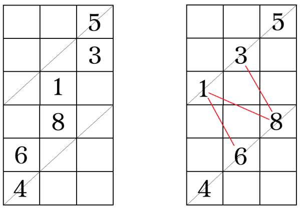


Figure 1.4: $C_0(3)$ (on the left) and $C_1(3)$ (on the right).

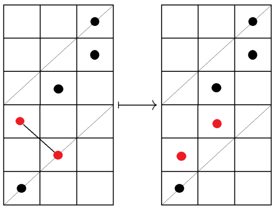


Figure 1.5: $C \in \mathrm{DC}(3) \mapsto Sw^2(C) \in \mathrm{DC}(3)$.

Refinements of the fal statistic on DC(n)

Definition 1.15. Let $C \in \mathrm{DC}(n)$ and $i \in [2n]$. We define the number $l_C(e_i)$ (resp. $r_C(e_i)$) as the number of falls of C between the dot e_i and any dot $e_{i'}$ with $i' > i$ (resp. $i' < i$).

For example, if $C = C_1(3)$ (see Figure 1.4), then $l_C(6) = r_C(3) = 1$ and $r_C(1) = l_C(8) = 2$.

Switching of a Dellac configuration

In the following definition, we provide a tool which transforms a Dellac configuration of DC(n) into a slightly modified tableau, which is not necessarily a Dellac configuration and consequently brings the notion of *switchability* (still being a Dellac configuration after a switching transformation).

Definition 1.16. Let $C \in \mathrm{DC}(n)$ and $i \in [2n-1]$. We denote by $Sw^i(C)$ the tableau obtained by switching the two consecutive dots e_i and e_{i+1} (*i.e.*, inserting e_i in e_{i+1}'s column and e_{i+1} in e_i's column). If the tableau $Sw^i(C)$ is still a Dellac configuration, we say that C is *switchable* at i.

In Figure 1.5, we give an example $C \in \mathrm{DC}(3)$ switchable at 2.

It is easy to verify the following facts.

Fact 1.17. If $C \in \mathrm{DC}(n)$ is switchable at i, then $|\mathrm{fal}((Sw^i(C))) - \mathrm{fal}(C)| \leq 1$.

Fact 1.18. A Dellac configuration $C \in \mathrm{DC}(n)$ is switchable at $i \in [2n-1]$ if and only if C and i satisfy one of the two following conditions:

(1) $i \leq n$ and if e_{i+1} is in the j_{i+1}-th column of C, then $j_{i+1} < i+1$;
(2) $i > n$ and if e_i is in the j_i-th column of C, then $j_i > i - n$.

In particular :

Fact 1.19. If C is switchable at i, then $Sw^i(C)$ is still switchable at i and $Sw^i(Sw^i(C)) = C$.

Fact 1.20. If e_i and e_{i+1} are in the same column of C, then C is switchable at i and $C = Sw^i(C)$.

Fact 1.21. If (e_i, e_{i+1}) is a fall of C, then C is switchable at i and $\mathrm{fal}(Sw^i(C)) = \mathrm{fal}(C) - 1$ (like in Figure 1.5).

Fact 1.22. A Dellac configuration $C \in \mathrm{DC}(n)$ is always switchable at n.

1.3.2 Construction of a statistic-preserving bijection

In this part, we intend to prove the following result.

Theorem 1.23. *There exists a bijection $\phi : DC(n) \rightarrow \mathcal{D}'_{n+1}$ such that the equality*

$$st(\phi(C)) = \binom{n}{2} - fal(C) \tag{1.6}$$

is true for all $C \in DC(n)$.

In the following, we define $\phi : \mathrm{DC}(n) \rightarrow \mathcal{D}'_{n+1}$ and in order to prove that it is bijective, we construct $\tilde{\phi} : \mathcal{D}_{n+1} \rightarrow \mathrm{DC}(n)$ such that ϕ and $\tilde{\phi}_{|\mathcal{D}'_{n+1}}$ are inverse maps.

Bijections

Definition 1.24 (definition of ϕ)**.** We define $\phi : \mathrm{DC}(n) \rightarrow \mathfrak{S}_{2n+2}$ by mapping $C \in \mathrm{DC}(n)$ to the permutation $\phi(C) \in \mathfrak{S}_{2n+2}$ defined by

$$\phi(C)^{-1} = 2e_{i_2(1)}e_{i_1(1)}e_{i_2(2)}e_{i_1(2)} \dots e_{i_2(n)}e_{i_1(n)}(2n+1),$$

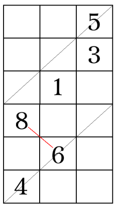


Figure 1.6: $C \in \mathrm{DC}(3)$.

where we recall that $e_{i_1(j)}$ and $e_{i_2(j)}$ are respectively the lower and upper dots of the j-th column of C for all $j \in [n]$.

In other words, the permutation $\sigma = \phi(C)$ is defined by

$$\begin{cases} (\sigma(2), \sigma(2n+1)) = (1, 2n+2), \\ \left(\sigma\left(e_{i_1(j)}\right), \sigma\left(e_{i_2(j)}\right)\right) = (2j+1, 2j) \text{ for all } j \in [n]. \end{cases} \tag{1.7}$$

Example 1.25. If $C \in \mathrm{DC}(3)$ is the Dellac configuration depicted in Figure 1.6, we obtain $\phi(C)^{-1} = 28416537$.

Proposition 1.26. *For all $C \in DC(n)$, the permutation $\phi(C)$ is a normalized Dumont permutation.*

Proof. Let $\sigma = \phi(C)$.

It is a Dumont permutation : $(\sigma(2), \sigma(2n+1)) = (1, 2n+2)$ and for all $i \in \{2, 3, \ldots, n-1\}$, if the dot $2i = e_{i-1}$ is in the j-th column of C (resp. if the dot $2i+1 = e_{n+1+i}$ is in the j'-th column of C), then

$$\sigma(2i) = \sigma(e_{i-1}) \leq 2j+1 < 2i$$

because $j \leq i-1$ (respectively

$$\sigma(2i+1) = \sigma(e_{n+1+i}) \geq 2j' > 2i+1$$

because $n+1+i \leq j'+n$).

It is also normalized according to Remark 1.13. $\square$

Definition 1.27 (definition of $\tilde{\phi}$)**.** Let $\mathcal{T}_n$ be the set of rectangular tableaux of size $n \times 2n$ whose each row contains one dot and each column contains two dots.

We define $\tilde{\phi} : \mathcal{D}_{n+1} \to \mathcal{T}_n$ by mapping $\sigma \in \mathcal{D}_{n+1}$ to the tableau $\tilde{\phi}(\sigma) \in \mathcal{T}_n$ whose j-th column contains the two dots labelled by $\sigma^{-1}(2j)$ and $\sigma^{-1}(2j+1)$ for all $j \in [n]$.

Proposition 1.28. *For all $\sigma \in \mathcal{D}_{n+1}$, the tableau $\tilde{\phi}(\sigma)$ is a Dellac configuration.*

Proof. Let $j \in [n]$ and $i \in [2n]$ such that $\tilde{\phi}(\sigma)$ contains a dot in the box (j, i) (*i.e.*, the j-th column of $\tilde{\phi}(\sigma)$ contains the dot e_i). By definition, we have

$$2j \leq \sigma(e_i) \leq 2j + 1.$$

If $i \leq n$, then $e_i = 2i + 2$ and $2j \leq \sigma(2i + 2) < 2i + 2$ thence $j \leq i < j + n$. Else $e_i = 2(i - n) - 1$ and $2j + 1 \geq \sigma(2(i - n) - 1) > 2(i - n) - 1$ thence $j \geq i - n > 0 \geq j - n$.

In either case we obtain $j \leq i \leq j + n$ so $\tilde{\phi}(\sigma) \in \mathrm{DC}(n)$. □

Example 1.29. Consider the permutation $\sigma = 41726583 \in \mathcal{D}_4$. From $\sigma^{-1} = 24816537$, we obtain the Dellac configuration $\tilde{\phi}(\sigma)$ illustrated in Figure 1.6.

It is easy to verify that $\phi \circ \tilde{\phi}|\, \mathcal{D}'_{n+1} = Id_{\mathcal{D}'_{n+1}}$ and $\tilde{\phi} \circ \phi = Id_{\mathrm{DC}(n)}$.

Remark 1.30. There is a natural interpretation in terms of group action : in the proof of Proposition 1.9, we show that the subgroup of $\mathfrak{S}_{2n+2}$ generated by the n permutations $(2, 3), (4, 5), \ldots, (2n, 2n + 1)$, freely operates by left multiplication on $\mathcal{D}_{n+1}$, and that each orbit of that action contains exactly one normalized Dumont permutation. Also, the orbits are indexed by elements of $\mathrm{DC}(n)$: two permutations σ_1 and $\sigma_2 \in \mathcal{D}_{n+1}$ are in the same orbit if and only if $\tilde{\phi}(\sigma_1) = \tilde{\phi}(\sigma_2)$, and for all $\sigma \in \mathcal{D}_{n+1}$, the permutation $\phi(\tilde{\phi}(\sigma))$ is the unique normalized Dumont permutation in the orbit of σ.

Example 1.31. In Examples 1.25 and 1.29, we have $\tilde{\phi}(\phi(C)) = C$ and $\phi(\tilde{\phi}(\sigma)) = (2, 3) \circ \sigma$.

Alternative algorithm

Definition 1.32. Let $(y_1, y_2, \ldots, y_{2n})$ be the sequence $(3, 2, 5, 4, \ldots, 2n + 1, 2n)$.

For all $C \in \mathrm{DC}(n)$, we define a permutation $\tau_C \in \mathfrak{S}_{2n}$ by $\phi(C)(e_i) = y_{\tau_C(i)}$ for all $i \in [2n]$.

Lemma 1.33. *Let $C \in DC(n)$ and $(p,q) \in [2n]^2$ such that $p < q$. The pair (e_p, e_q) is a fall of C if and only if (p,q) is an inversion of τ_C, i.e., if $\tau_C(p) > \tau_C(q)$.*

Proof. Recall that if the dot e_i is located in the j-th column of C, then $\phi(C)(e_i) = 2j$ or $2j+1$. Consequently, since $y_i = i$ if i is even, and $y_i = i+2$ if i is odd, then $\tau_C(i) = 2j$ or $2j-1$.

Now let $1 \leq p < q \leq 2n$, and let (j_p, j_q) such that the dot e_p (resp. e_q) is located in the j_p-th column (resp. j_q-th column) of C. If (e_p, e_q) is a fall of C, *i.e.*, if $j_p > j_q$, then $\tau_C(p) \geq 2j_p - 1 > 2j_q \geq \tau_C(q)$ and (p,q) is an inversion of τ_C.

Reciprocally, if $\tau_C(p) > \tau_C(q)$, then $2j_p \geq \tau_C(p) > \tau_C(q) \geq 2j_q - 1$, hence $j_p \geq j_q$. Now suppose that $j_p = j_q =: j$. It means that e_p and e_q are the lower dot and the upper dot of the j-th column respectively, which translates into $y_{\tau_C(p)} = \phi(C)(e_p) = 2j+1$ and $y_{\tau_C(q)} = \phi(C)(e_q) = 2j$. Consequently, we obtain $\tau_C(p) = 2j-1$ and $\tau_C(q) = 2j$, which is in contradiction with $\tau_C(p) > \tau_C(q)$. So $j_p > j_q$ and (e_p, e_q) is a fall of C. $\square$

Proposition 1.34. *Let $C \in DC(n)$. For all $i \in [2n]$, we have $\tau_C(i) = i + l_C(e_i) - r_C(e_i)$.*

Proof. From Lemma 1.33, we know that

$$\begin{cases} l_C(e_i) = |\{k > i : \tau_C(k) < \tau_C(i)\}|, \\ r_C(e_i) = |\{k < i : \tau_C(k) > \tau_C(i)\}|. \end{cases}$$

Thus, the lemma follows from the well-known equality

$$\pi(i) = i + |\{k > i : \pi(k) < \pi(i)\}| - |\{k < i : \pi(k) > \pi(i)\}|$$

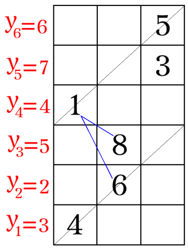
for all permutation $\pi \in \mathfrak{S}_m$ and for all integer $m \geq 1$. $\square$

This immediatly yields the following result.

Corollary 1.35 (Alternative algorithm for the map $\phi : \mathrm{DC}(n) \to \mathcal{D}'_{n+1}$)**.** *For all $C \in DC(n)$, the permutation $\sigma = \phi(C) \in \mathcal{D}'_{n+1}$ is defined by*

$$\begin{cases} \sigma(2) & = 1, \\ \sigma(2n+1) & = 2n+2, \\ \sigma(e_i) & = y_{i + l_C(e_i) - r_C(e_i)} \text{ for all } i \in [2n]. \end{cases}$$

Example 1.36. Consider the following Dellac configuration $C \in \mathrm{DC}(3)$.



By Corollary 1.35, we obtain $\phi(C) = 21736584$. This is coherent with the algorithm given in Definition 1.24, which confirms that $\phi(C)^{-1} = 21486537$.

Switchability and Dumont permutations

We have built a bijection $\phi : \mathrm{DC}(n) \to \mathcal{D}'_{n+1}$. To demonstrate Formula (1.6), we will use the notion of switchability defined in §1.3.1, by showing that if Formula (1.6) is true for some particular configuration C^0, and if C^1 is a configuration connected to C^0 by a switching transformation, then Formula (1.6) is also true for C^1. We will also need Lemma 1.37 and Proposition 1.38 to prove (in Proposition 1.39) that any two Dellac configurations are connected by a sequence of switching transformations.

Lemma 1.37. *Let $\sigma \in \mathcal{D}_{n+1}$ and $i \in [2n-1]$. We denote by σ' the composition $\sigma \circ (e_i, e_{i+1})$ of the transposition (e_i, e_{i+1}) with the permutation σ.*

The Dellac configuration $\tilde{\phi}(\sigma)$ is switchable at i if and only if σ' is still a Dumont permutation, and in that case $\tilde{\phi}(\sigma') = Sw^i(\tilde{\phi}(\sigma))$.

Proof. Let T be the tableau $Sw^i(\tilde{\phi}(\sigma))$. If T is a Dellac configuration, one can check that $\sigma' \in \mathcal{D}_{n+1}$ thanks to Fact 1.18.

Reciprocally, if σ' is a Dumont permutation, we may consider the Dellac configuration $\tilde{\phi}(\sigma')$. For all $j \in [n]$, let $\left(e_{i_1(j)}, e_{i_2(j)}\right)$ (with $i_1(j) < i_2(j)$) be the two dots of the j-th column of $\tilde{\phi}(\sigma)$, and $\left(e_{i'_1(j)}, e_{i'_2(j)}\right)$ (with $i'_1(j) < i'_2(j)$) the two dots of the j-th column of $\tilde{\phi}(\sigma')$. Then, we have

$$\begin{aligned} e_{i'_1(j)} = \quad & \sigma'^{-1}(2j+1) = \quad (e_i, e_{i+1}) \circ \sigma^{-1}(2j+1) = \quad (e_i, e_{i+1})\left(e_{i_1(j)}\right) \\ e_{i'_2(j)} = \quad & \sigma'^{-1}(2j) = \quad (e_i, e_{i+1}) \circ \sigma^{-1}(2j) = \quad (e_i, e_{i+1})\left(e_{i_2(j)}\right) \end{aligned}$$

for all j, which exactly translates into $\tilde{\phi}(\sigma') = Sw^i(\tilde{\phi}(\sigma)) = T$. □

The following result is easy.

Proposition 1.38. *In the setting of Lemma 1.37, if* $\tilde{\phi}(\sigma)$ *is switchable at* i*, then the following conditions are equivalent.*

1. $\tilde{\phi}(\sigma') \neq \tilde{\phi}(\sigma)$;
2. *the two dots* e_i *and* e_{i+1} *are not in the same column of* $\tilde{\phi}(\sigma)$;
3. $fal(\tilde{\phi}(\sigma')) - fal(\tilde{\phi}(\sigma)) = \pm 1$;
4. $\phi(\tilde{\phi}(\sigma)) \circ (e_i, e_{i+1}) \in \mathcal{D}'_{n+1}$;
5. $\phi(\tilde{\phi}(\sigma')) = \phi(\tilde{\phi}(\sigma)) \circ (e_i, e_{i+1})$.

Proposition 1.39. *Let* $(C_1, C_2) \in DC(n)^2$*. There exists a finite sequence of switching transformations from* C_1 *to* C_2*, i.e., a sequence* $(C^0, C^1, \dots, C^m)$ *in* $DC(n)$ *for some* $m \geq 0$ *such that* $(C^0, C^m) = (C_1, C_2)$ *and such that* $C^k = Sw^{i_{k-1}}(C^{k-1})$ *for some index* $i_{k-1} \in [2n]$*, for all* $k \in [m]$.

Proof. From Fact 1.19, it is sufficient to prove that for all $C \in \mathrm{DC}(n)$, there exists a finite sequence of switching transformations from C to $C_0(n)$, which is the unique Dellac configuration of size n with 0 fall (see Definition 1.14).

If $C = C_0(n)$, the statement is obvious. Else, let $C^0 = C$. From Lemma 1.33, for all $i \in [2n]$, the pair (e_i, e_{i+1}) is a fall of C^0 if and only if the integer i is a descent of τ_{C^0}, *i.e.*, if $\tau_{C^0}(i) > \tau_{C^0}(i+1)$. Now, from Corollary 1.35, the permutation $\tau_{C_0(n)}$ is the identity map Id of $\mathfrak{S}_{2n+2}$. Consequently, since $C^0 \neq C_0(n)$, we have $\tau_{C^0} \neq Id_{\mathfrak{S}_{2n}}$, so τ_{C^0} has at least one descent. Let i_0 be one of those descents, and let $C^1 = Sw^{i_0}(C^0) \in \mathrm{DC}(n)$. Since (e_{i_0}, e_{i_0+1}) is a fall of C^0, in particular e_{i_0} and e_{i_0+1} are not in the same column, so, from Proposition 1.38, we have $\phi(C^1) = \phi(C^0) \circ (e_{i_0}, e_{i_0+1})$, hence $\tau_{C^1} = \tau_{C^0} \circ (i_0, i_0+1)$. Consequently, since i_0 is a descent of τ_{C^0}, it is not a descent of τ_{C^1}. Iterating the process with C^1, and by induction, we build a finite sequence of switching transformations $(C^0, C^1, \dots, C^m)$ such that τ_{C^m} has no descent, *i.e.*, such that $\tau_{C^m} = Id = \tau_{C_0(n)}$, which implies $C^m = C_0(n)$. □

Example 1.40. In Figure 1.7, we give a graph whose vertices are the $h_3 = 7$ elements of DC(3), and in which two Dellac configurations are connected by an edge if they are connected by a switching transformation.

Proof of the statistic preservation formula (1.6)

To finish the proof of Theorem 1.23, it remains to prove that Formula (1.6) is true for all $C \in \mathrm{DC}(n)$.

This is done in Appendix A.1.

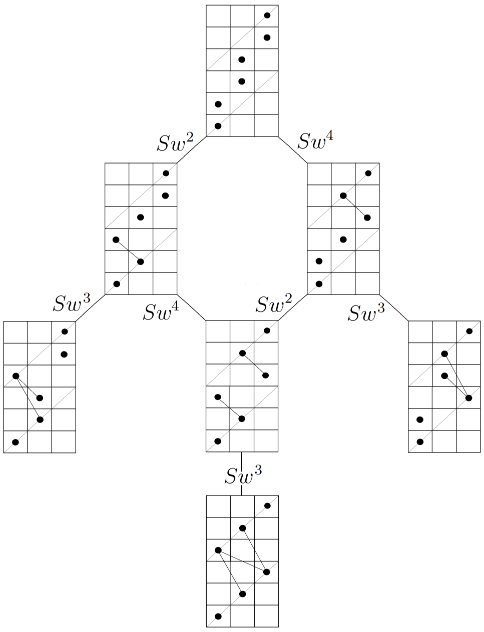


Figure 1.7: The switching transformations of DC(3).

Remark 1.41. In [HZ99a], Han and Zeng proved that $\bar{c}_n(q)$ is divisible by $1+q$ if n is odd, but requested a combinatorial proof of this statement. Now, if n is odd, one can prove that every Dellac configuration $C \in \mathrm{DC}(n-1)$ is switchable at some even integer, which yields a natural involution $\mathcal{I}$ on $\mathrm{DC}(n-1)$ such that $\mathrm{inv}(\mathcal{I}(C)) = \mathrm{fal}(C) \pm 1$ for all C. This proves combinatorially the divisibility of $\bar{c}_n(q)$ by $1+q$ in view of Theorem 1.23.

1.4 Dellac histories

1.4.1 Weighted Dyck paths

Recall (see [Fla80]) that a *Dyck path* γ of length $2n$ is a sequence of points $(p_0, p_1, \dots, p_{2n})$ in $\mathbb{N}^2$ such that $(p_0, p_{2n}) = ((0,0),(2n,0))$, and for all $i \in [2n]$, the step (p_{i-1}, p_i) is either an *up step* $(1,1)$ or a *down step* $(1,-1)$. We denote by $\Gamma(n)$ the set of Dyck paths of length $2n$. Furthermore, let $\mu = (\mu_n)_{n\geq 1}$ be a sequence of elements of a ring. A *weighted Dyck path* is a Dyck path $\gamma = (p_i)_{0\leq i\leq 2n} \in \Gamma(n)$ whose each up step has been weighted by 1,

and each down step (p_{i-1}, p_i) from height h (*i.e.*, such that $p_{i-1} = (i-1, h)$) has been weighted by μ_h.
The weight

$$\omega_\mu(\gamma) \tag{1.8}$$

of the weighted Dyck path γ is the product of the weights of all steps.

Remark 1.42. If $\gamma = (p_i)_{0 \leq i \leq 2n} \in \Gamma(n)$, then $p_i = (i, n_u(i) - n_d(i))$ where $n_u(i)$ and $n_d(i)$ are defined as the numbers of up steps and down steps on the left of p_i respectively (in particular $n_u(i) + n_d(i) = i$). Consequently, since the final point of γ is $p_{2n} = (2n, 0)$, the path γ has exactly n up steps and n down steps, and for all $j \in [n]$, the points p_{2j-1} and p_{2j} are at heights respectively odd and even.

Definition 1.43 (Labelled steps)**.** Let $\gamma = (p_i)_{0 \leq i \leq 2n} \in \Gamma(n)$. For all $i \in [n]$, we denote by $s_i^u(\gamma)$ (resp. $s_i^d(\gamma)$) the i-th up step (resp. down step) of γ. When there is no ambiguity, we write s_i^u and s_i^d instead of $s_i^u(\gamma)$ and $s_i^d(\gamma)$.

Remark 1.44. If $s_i^u(\gamma) = (p_{2j-2}, p_{2j-1})$ or (p_{2j-1}, p_{2j}) where $p_{2j-2} = (2j-2, 2k)$ for some $k \geq 0$, then, following Remark 1.42, we know that $2k = n_u(2j-2) - n_d(2j-2) = 2n_u(2j-2) - (2j-2)$, and by definition of $s_i^u(\gamma)$ it is necessary that $n_u(2j-2) = i-1$, and we obtain $2k = 2(i-j)$ hence $i = j+k$. In the same context, if $s_i^d(\gamma) = (p_{2j-1}, p_{2j})$ or (p_{2j-2}, p_{2j-1}), then we obtain $i = j-k$ by an analogous reasoning.

1.4.2 Dellac histories

Definition 1.45. A *Dellac history* of length $2n$ is a pair (γ, ξ) where $\gamma = (p_i)_{0 \leq i \leq 2n} \in \Gamma(n)$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ where ξ_i is a pair of nonnegative integers $(n_1(i), n_2(i))$ with the following conditions. Let $j \in [n]$ be such that the i-th down step s_i^d of γ is one the two steps (p_{2j-2}, p_{2j-1}) and (p_{2j-1}, p_{2j}), and let $2k$ be the height of p_{2j-2}. There are three cases.

1. If $s_i^d = (p_{2j-2}, p_{2j-1})$ such that (p_{2j-1}, p_{2j}) is an up step (see Figure 1.8,(1)), then
$$k \geq n_1(i) > n_2(i) \geq 0,$$
and we attach a weight $\omega_i = q^{2k - n_1(i) - n_2(i)}$ to s_i^d.
2. If $s_i^d = (p_{2j-1}, p_{2j})$ such that (p_{2j-2}, p_{2j-1}) is an up step (see Figure 1.8,(2)), then
$$0 \leq n_1(i) \leq n_2(i) \leq k,$$
and we attach a weight $\omega_i = q^{2k - n_1(i) - n_2(i)}$ to s_i^d.

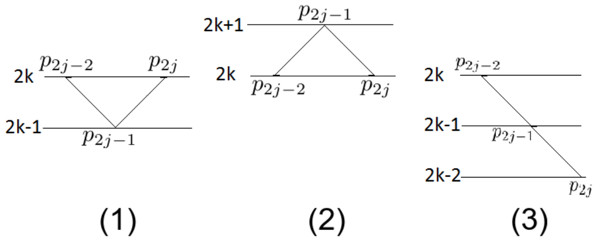


Figure 1.8: Three cases of down steps in a Dyck path.

3. If (p_{2j-2}, p_{2j-1}) and (p_{2j-1}, p_{2j}) are both down steps (see Figure 1.8,(3)), we can suppose that $s_i^d = (p_{2j-2}, p_{2j-1})$ and $s_{i+1}^d = (p_{2j-1}, p_{2j})$, then

$$k-1 \geq n_1(i) \geq n_2(i) \geq 0,$$

and we attach a weight $\omega_i = q^{2k-1-n_1(i)-n_2(i)}$ to s_i^d, also

$$0 \leq n_1(i+1) \leq n_2(i+1) \leq k-1,$$

and we attach a weight $\omega_{i+1} = q^{2k-2-n_1(i+1)-n_2(i+1)}$ to s_{i+1}^d.

The *weight* $\omega(\gamma, \xi)$ of the history (γ, ξ) is the product of the weights of all down steps. We denote by $\mathrm{DH}(n)$ the set of Dellac histories of length $2n$.

Prior to connecting Dellac histories to weighted Dyck paths, one can easily verify the two following results.

Lemma 1.46. *For all $p \geq 1$, the polynomial*

$$\sum_{0 \leq n_1 \leq n_2 \leq p-1} q^{2p-2-n_1-n_2}$$

is the q-binomial coefficient $\left\{ {p+1 \atop 2} \right\}_q = (1-q^{p+1})(1-q^p)/((1-q^2)(1-q))$.

Proposition 1.47. *For all $\gamma_0 \in \Gamma(n)$, we have the equality*

$$\sum_{(\gamma_0, \xi) \in DH(n)} \omega(\gamma_0, \xi) = \omega_{\lambda(q)}(\gamma_0)$$

where $\omega_{\lambda(q)}$ has been defined in (1.8), and where $\lambda(q) = (\lambda_n(q))_{n \geq 1}$ is the sequence defined in Theorem 1.4.

Proof. By Definition 1.45, the polynomial $\sum_{(\gamma_0,\xi)\in\mathrm{DH}(n)}\omega(\gamma_0,\xi)$ is the product of the weights of all down steps of γ_0, where the weight of a down step $s_i^d(\gamma_0)$ is a sum of monomials over pairs of nonnegative integers with conditions depending on the three cases 1.,2. or 3. in which $s_i^d(\gamma_0)$ may be found.

If $s_i^d(\gamma_0)$ is a down step (p_{2j-1},p_{2j}) from height $2k-1$ in the case 3., then it is weighted by the polynomial

$$\sum_{0\leq n_1\leq n_2\leq k-1} q^{2k-2-n_1-n_2},$$

which, in view of Lemma 1.46, equals $\left\{{k+1 \atop 2}\right\}_q = \lambda_{2k-1}(q)$.

If $s_i^d(\gamma_0)$ is a down step (p_{2j-2},p_{2j-1}) from height $2k$, still in the case 3., then it is weighted by the polynomial

$$\sum_{k-1\geq n_1\geq n_2\geq 0} q^{2k-1-n_1-n_2} = q\left\{{k+1 \atop 2}\right\}_q = \lambda_{2k}(q).$$

If $s_i^d(\gamma_0)$ is a down step (p_{2j-1},p_{2j}) from height $2k+1$ in the case 2., then it is weighted by the polynomial

$$\sum_{0\leq n_1\leq n_2\leq k} q^{2k-n_1-n_2} = \left\{{k+2 \atop 2}\right\}_q = \lambda_{2k+1}(q).$$

Finally, if $s_i^d(\gamma_0)$ is a down step (p_{2j-2},p_{2j-1}) from height $2k$ in the case 1., then it is weighted by the polynomial

$$\sum_{k\geq n_1>n_2\geq 0} q^{2k-n_1-n_2},$$

which, by setting $m_1 = n_1 - 1$ and $m_2 = n_2$, equals

$$\sum_{k-1\geq m_1\geq m_2\geq 0} q^{2k-1-m_1-m_2} = q\left\{{k+1 \atop 2}\right\} = \lambda_{2k}(q)$$

in view of Lemma 1.46. □

Following Proposition 1.47, we have

$$\sum_{(\gamma,\xi)\in\mathrm{DH}(n)}\omega(\gamma,\xi) = \sum_{\gamma\in\Gamma(n)}\omega_{\lambda(q)}(\gamma)$$

for all $n \geq 0$. Therefore, from a well-known result due to Flajolet [Fla80], the generating function $\sum_{n\geq 0}\left(\sum_{(\gamma,\xi)\in\mathrm{DH}(n)}\omega(\gamma,\xi)\right)t^n$ is the continued fraction expansion of Formula (1.4). Consequently, to demonstrate Theorem 1.6, it suffices to prove that $\tilde{h}_n(q) = \sum_{(\gamma,\xi)\in \mathrm{DH}(n)} \omega(\gamma,\xi)$, which is a straight corollary of the following theorem.

Theorem 1.48. *There exists a bijective map $\psi : DC(n) \to DH(n)$ such that*

$$\omega(\psi(C)) = q^{\binom{n}{2}-fal(C)} \tag{1.9}$$

for all $C \in DC(n)$.

1.4.3 Proof of Theorem 1.48

In this part, we give preliminaries and connections between Dellac configurations and Dyck paths. Then, we define the map $\psi : \mathrm{DC}(n) \to \mathrm{DH}(n)$ and we demonstrate the statistic preservation formula (1.9). Finally, we prove that ψ is bijective by giving a map $\tilde{\psi} : \mathrm{DH}(n) \to \mathrm{DC}(n)$ which happens to be ψ^{-1}.

Preliminaries on Dellac configurations

Definition 1.49. Let $C \in \mathrm{DC}(n)$. If $i \leq n$, we denote by $l_C^e(e_i)$ the number of falls of C towards e_i from any even dot $e_{i'\leq n}$ with $i' > i$. In the same way, if $i > n$, we denote by $r_C^o(e_i)$ the number of falls of C from e_i towards any odd dot $e_{i'>n}$ with $i' < i$.

Definition 1.50. Let $C \in \mathrm{DC}(n)$ and $j \in [n]$. We define the *height* $h(j)$ of the integer j as the number $n_e(j) - n_o(j)$ where $n_e(j)$ (resp. $n_o(j)$) is the number of even dots (resp. odd dots) in the first $j-1$ columns of C (with $n_e(1) = n_o(1) = 0$).

Remark 1.51. Since the first $j-1$ columns of C contain exactly $2j-2$ dots and, from Remark 1.12, always contain the $j-1$ even dots $e_1, e_2, \ldots, e_{j-1}$, there exists $k \in \{0, 1, \ldots, j-1\}$ such that $n_e(j) = j-1+k$ and $n_o(j) = j-1-k$. In particular $h(j) = 2k$.

Lemma 1.52. *Let $C \in DC(n)$, let $j \in [n]$ and $k \geq 0$ such that $h(j) = 2k$ in view of Remark 1.51. If the j-th column of C contains two odd dots, there exists $j' < j$ such $h(j'+1) = 2k$ and such that the j'-th column of C contains two even dots.*

Proof. From Remark 1.51, we have $n_e(j) = j - 1 + k$ and $n_o(j) = j - 1 - k$. Since the only j odd dots that the first j columns may contain are $e_{n+1}, e_{n+2}, \ldots, e_{n+j-1}, e_{n+j}$, and since the j-th column already contains two odd dots, the first $j-1$ columns contain at most $j-2$ odd dots. In other words, since they contain $n_o(j) = j - 1 - k$ odd dots, we obtain $k \geq 1$. Thus $h(j) = 2k > 0$. Since $h(1) = 0$, there exists $j' \in [j-1]$ such that $h(j'+1) = 2k$ and $h(j') < 2k$. Obviously $h(j'+1) - h(j') \in \{-2, 0, 2\}$, so $h(j') = 2k - 2$ and the j'-th column of C contains two even dots. □

Map $\psi : \mathbf{DC}(n) \to \mathbf{DH}(n)$

Definition 1.53 (definition of ψ)**.** Let $C \in \mathrm{DC}(n)$, we define $\psi(C)$ as (γ, ξ), where $\gamma = (p_i)_{0 \leq i \leq 2n}$ (which is a path in $\mathbb{Z}^2$ whose initial point p_0 is defined as $(0,0)$) and $\xi = (\xi_1, \ldots, \xi_n)$ (which is a sequence of pairs of nonnegative integers) are provided by the following algorithm. For $j = 1$ to n, let $e_{i_1(j)}$ and $e_{i_2(j)}$ (with $i_1(j) < i_2(j)$) be the two dots of the j-th column of C.

1. If $i_2(j) \leq n$, then (p_{2j-2}, p_{2j-1}) and (p_{2j-1}, p_{2j}) are defined as up steps.
2. If $i_1(j) \leq n < i_2(j)$, let $i \in [n]$ such that $i-1$ down steps have already been defined. We define ξ_i as $(l_C^e(e_{i_1(j)}), r_C^o(e_{i_2(j)}))$. Afterwards,
   (a) if $l_C^e(e_{i_1(j)}) > r_C^o(e_{i_2(j)})$, we define (p_{2j-2}, p_{2j-1}) as a down step and (p_{2j-1}, p_{2j}) as an up step (see Figure 1.8,(1));
   (b) if $l_C^e(e_{i_1(j)}) \leq r_C^o(e_{i_2(j)})$, we define (p_{2j-2}, p_{2j-1}) as an up step and (p_{2j-1}, p_{2j}) as a down step (see Figure 1.8,(2)).
3. If $n < i_1(j)$, let $i \in [n]$ such that $i-1$ down steps have already been defined. We define (p_{2j-2}, p_{2j-1}) and (p_{2j-1}, p_{2j}) as down steps (see Figure 1.8,(3)). Afterwards, let $k \geq 0$ such that $p_{2j-2} = (2j-2, 2k)$. Obviously, the number $n_u(2j-2) = j - 1 + k$ of up steps (resp. the number $n_d(2j-2) = j - 1 - k$ of down steps) that have already been defined is the number $n_e(j)$ of even dots (resp. the number $n_o(j)$ of odd dots) in the first $j-1$ columns of C, thence $h(j) = 2k$. From Lemma 1.52, there exists $j' < j$ such that $h(j'+1) = 2k$ (which means $p_{2j'} = (2j', 2k)$) and such that the j'-th column of C contains two even dots, which means $(p_{2j'-2}, p_{2j'-1})$ and $(p_{2j'-1}, p_{2j'})$ are two consecutive up steps (see Figure 1.9). Now, we consider the maximum $j_m < j$ of the integers j' that verify this property, and we consider the two dots

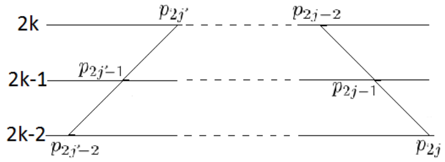


Figure 1.9: Two consecutive up steps and down steps at the same level.

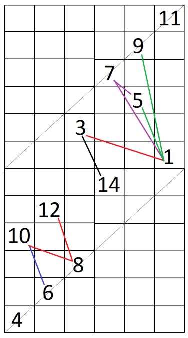


Figure 1.10: $C \in \mathrm{DC}(6)$.

$e_{i_1(j_m)}$ and $e_{i_2(j_m)}$ (with $i_1(j_m) < i_2(j_m)$) of the j_m-th column of C. Finally, we define ξ_i and ξ_{i+1} as

$$\xi_i = (l^e_C(e_{i_1(j_m)}), l^e_C(e_{i_2(j_m)})),$$
$$\xi_{i+1} = (r^o_C(e_{i_1(j)}), r^o_C(e_{i_2(j)})).$$

Example 1.54. The Dellac configuration $C \in \mathrm{DC}(6)$ of Figure 1.10 yields the data $\psi(C) = (\gamma, \xi)$, which is in fact a Dellac history, depicted in Figure 1.11 (since $\psi(C)$ is a Dellac history, we have indicated the weight ω_i of the i-th down step s^d_i of γ for all $i \in [6]$, see Definition 1.45).

Remark 1.55. If $\psi(C) = (\gamma, \xi)$, let $j \in [n]$, the number of up steps (resp. down steps) among the first $2j$ steps of γ, is in fact the number of even dots (resp. odd dots) in the first j columns of C. With precision, for all $i \in [n]$, the even dot $e_{p_C(i)}$ and the odd dot $e_{n+q_C(i)}$ (see Definition 1.11) give birth to the i-th up step and the i-th down step of γ respectively. In particular, the path γ has n up steps and n down steps, so $p_{2n} = (2n, 0)$. To prove that γ is a Dyck path, we still have to check that it never goes below the horizontal line $y = 0$.

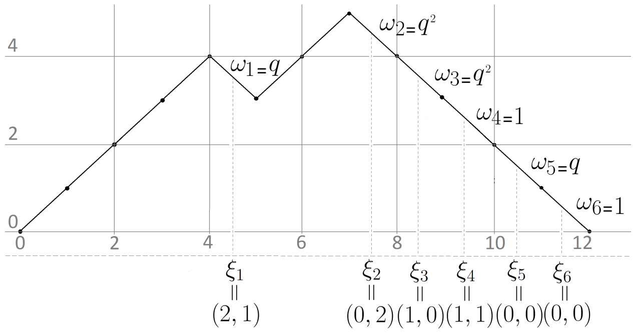


Figure 1.11: $\psi(C) \in \mathrm{DH}(6)$.

Remark 1.56. In the context (3) of Definition 1.53, if $h(j) = 2k$ (*i.e.*, if $p_{2j-2} = (2j-2, 2k)$), then the maximum j_m of the integers $j' < j$ such that $h(j'+\ 1) = 2k$ and such that the j'-th column contains two even dots, is such that the two steps (p_{2j_m-2}, p_{2j_m-1}) and $(p_{2j_m-1,2j_m})$ are the last two consecutive up steps from level $2k-2$ towards level $2k$ in γ.

Proposition 1.57. *Let $C \in DC(n)$ and $(\gamma, \xi) = \psi(C)$. The path γ is a Dyck path.*

Proof. From Remark 1.55, it suffices to prove that $\gamma = (p_0, p_1, \dots, p_{2n})$ never goes below the line $y = 0$. If we suppose the contrary, there exists $i_0 \in \{0, 1, \dots, 2n-1\}$ such that $p_{i_0} = (i_0, 0)$ and (p_{i_0}, p_{i_0+1}) is a down step. From Remark 1.42, we know that $p_{i_0} = (i_0, 0) = (i_0, 2n_u(i_0) - i_0)$, so $i_0 = 2n_u(i_0)$. Let $j_0 = n_u(i_0) + 1 \in [n]$. In the first $j_0 - 1$ columns of C, from Remark 1.55, there are $n_u(i_0) = j_0 - 1$ even dots and $n_d(i_0) = j_0 - 1$ odd dots. Consequently, since those first $j_0 - 1$ columns always contain the $j_0 - 1$ even dots $e_1, e_2, \dots, e_{j_0-1}$ and cannot contain any other odd dot than $e_{n+1}, e_{n+2}, \dots, e_{n+j_0-1}$ (see Remark 1.12), the $2j_0 - 2$ dots they contain are precisely $e_1, e_2, \dots, e_{j_0-1}$ and $e_{n+1}, e_{n+2}, \dots, e_{n+j_0-1}$. Therefore, the only two dots that the j_0-th column may contain are e_{j_0} and e_{n+j_0}. But then, it forces $l_C^e(e_{j_0})$ and $r_C^o(e_{n+j_0})$ to equal 0. In particular $l_C^e(e_{j_0}) \leq r_C^o(e_{n+j_0})$. Following the case $2(b)$ of Definition 1.53, it means (p_{i_0}, p_{i_0+1}) is defined as an up step, which is absurd by hypothesis. □

Proposition 1.58. *For all $C \in DC(n)$, the data $\psi(C)$ is a Dellac history of length $2n$.*

Proof. Let $\psi(C) = (\gamma, \xi) = ((p_0, p_1, \dots, p_{2n}), (\xi_1, \xi_2, \dots, \xi_n))$. We know that $\gamma \in \Gamma(n)$. It remains to prove that ξ fits the appropriate inequalities described in Definition 1.45. Let $j \in [n]$ and let $e_{i_1(j)}$ and $e_{i_2(j)}$ (with $j \leq i_1(j) < i_2(j) \leq j + n$) be the two dots of the j-th column of C.

- If (p_{2j-1}, p_{2j}) is the down step s_i^d in the context $2(a)$ of Definition 1.53, then $\xi_i = (n_1, n_2) = (l_C^e(e_{i_1(j)})), r_C^o(e_{i_2(j)}))$ with $l_C^e(e_{i_1(j)}) > r_C^o(e_{i_2(j)})$. Here, the appropriate inequality to check is $k \geq n_1 > n_2$ (this is the context 1. of Definition 1.45). Since the first $j-1$ columns of C contain $j-1+k$ even dots, including the $j-1$ dots $e_1, e_2, \dots, e_{j-1}$ (with $j-1 < i_1(j)$), there is no fall from any of these dots to $e_{i_1(j)}$. Consequently, in the first $j-1$ columns of C, there are at most $(j-1+k)-(j-1) = k$ even dots e_i with $n \geq i > i_1(j)$, thence $n_1 = l_C^e(e_{i_1(j)}) \leq k$.
- Similarly, if (p_{2j-2}, p_{2j-1}) is the down step s_i^d set in the context $2(b)$ of Definition 1.53, then we have $\xi_i = (n_1, n_2) = (l_C^e(e_{i_1(j)})), r_C^o(e_{i_2(j)}))$, with $l_C^e(e_{i_1(j)}) \leq r_C^o(e_{i_2(j)})$. Now, the appropriate equality to check is $n_1 \leq n_2 \leq k$ (this is the context 2. of Definition 1.45). The first j columns of C contain $j-k$ odd dots and the $i_2(j)-n$ lines from the $(n+1)$-th line to the $i_2(j)$-th line contain $i_2(j)-n$ odd dots, so, in the $n-j$ last columns, the number of odd dots e_i with $n < i < i_2(j)$ is at most $(i_2(j)-n)-(j-k) = k + (i_2(j)-j-n) \leq k$, thence $n_2 = r_C^o(e_{i_2(j)}) \leq k$.
- Finally, if (p_{2j-2}, p_{2j-1}) and (p_{2j-1}, p_{2j}) are two consecutive down steps s_i^d and s_{i+1}^d in the context 3. of Definition 1.53, then

  $$\begin{aligned} \xi_i &= (l_C^e(e_{i_1(j_m)}), l_C^e(e_{i_2(j_m)})), \\ \xi_{i+1} &= (r_C^o(e_{i_1(j)}), r_C^o(e_{i_2(j)})) \end{aligned}$$

  and the two inequalities to check (this is the context 3. of Definition 1.45) are:

  $$k-1 \geq l_C^e(e_{i_1(j_m)}) \geq l_C^e(e_{i_2(j_m)}), \tag{1.10}$$
  $$r_C^o(e_{i_1(j)}) \leq r_C^o(e_{i_2(j)}) \quad \leq k-1. \tag{1.11}$$

  - Proof of (1.10): since $i_1(j_m) < i_2(j_m)$, obviously $l_C^e(e_{i_1(j_m)}) \geq l_C^e(e_{i_2(j_m)})$. Afterwards, since p_{2j_m-2} is at the level $h(j_m) = 2k-2$, there are $j_m - 1 + (k-1) = j_m + k - 2$ even dots in the first $j_m - 1$ columns of C. Since the first $j_m - 1$ rows of C contain the $j_m - 1$ even dots $e_1, e_2, \dots, e_{j_m-1}$, the first $j_m - 1$

columns of C contain at most $(j_m + k - 2) - (j_m - 1) = k - 1$ even dots e_i with $n \geq i > i_1(j_m)$, thence $l^e_C(e_{i_1(j_m)}) \leq k - 1$.

- Proof of (1.11): since $i_1(j) < i_2(j)$, obviously $r^o_C(e_{i_1(j)}) \leq r^o_C(e_{i_2(j)})$. Afterwards, since p_{2j} is at the level $h(j+1) = 2k - 2$, there are $j - (k-1) = j - k + 1$ odd dots in the first j columns of C. Since the j rows, from the $(n+1)$-th row to the $(n+j)$-th row of C, contain j odd dots, the $n - j$ last columns of C contain at most $j - (j - k + 1) = k - 1$ odd dots e_i with $n < i < i_2(j_m)$, thence $r^o_C(e_{i_2(j)}) \leq k - 1$.

So $\psi(C)$ is a Dellac history of length n. □

Proof of the statistic preservation formula (1.9)

Let $C \in \mathrm{DC}(n)$ and $\psi(C) = (\gamma, \xi)$ with $\gamma = (p_0, p_1, \ldots, p_{2n})$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_{2n})$. By definition, we have $\omega(\psi(C)) = \Pi_{i=1}^n \omega_i$ where ω_i is the weight of the i-th down step s^d_i of γ. In the contexts 1. or 2. of Definition 1.45, we have

$$\omega_i = q^{2k - l^e_C\left(e_{i_1(j)}\right) - r^o_C\left(e_{i_2(j)}\right)}. \tag{1.12}$$

Since p_{2j-2} is at the level $h(j) = 2k$, the first $j - 1$ columns of C contain $j - 1 - k$ odd dots. Consequently, following Definition 1.53, the step s^d_i is the $(j-k)$-th down step of γ, *i.e.*, the integer i equals $j - k$. Also, since the first j columns of C contain $j + k$ even dots, the last $n - j$ columns of C (from the $(j+1)$-th column to the n-th column) contain $n - (j + k) = n - j - k = i - k$ even dots. As a result, we obtain the equality

$$r_C(e_{i_2(j)}) = r^o_C(e_{i_2(j)}) + i - k. \tag{1.13}$$

In view of (1.13), Equality (1.12) becomes $\omega_i = q^{n-i-\left(l^e_C\left(e_{i_1(j)}\right) + r_C\left(e_{i_2(j)}\right)\right)}$. With the same reasoning, if s^d_i and s^d_{i+1} are two consecutive down steps in the context 3. of Definition 1.45, then by commuting factors of ω_i and ω_{i+1}, we obtain the equality

$$\omega_i \omega_{i+1} = \left(q^{n-i-\left(l^e_C\left(e_{i_1(j_m)}\right) + r_C\left(e_{i_2(j_m)}\right)\right)}\right)\left(q^{n-(i+1)-\left(l^e_C\left(e_{i_1(j)}\right) + r_C\left(e_{i_2(j)}\right)\right)}\right).$$

From $\omega(\psi(C) = \Pi_{i=1}^n \omega_i$, it follows that

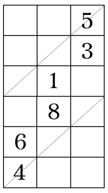
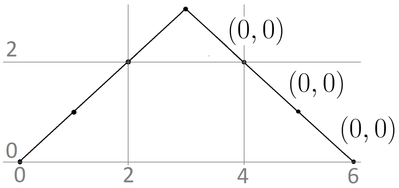
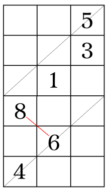
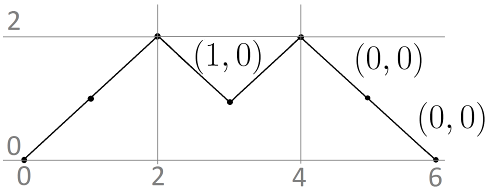
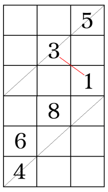
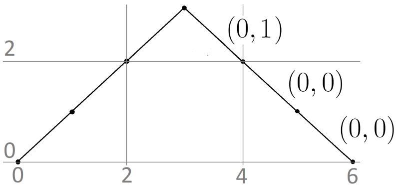
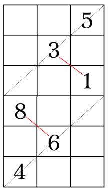
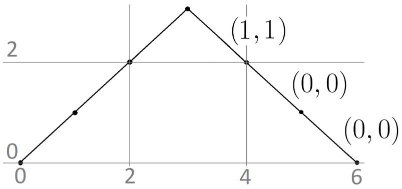
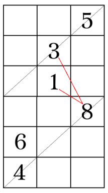
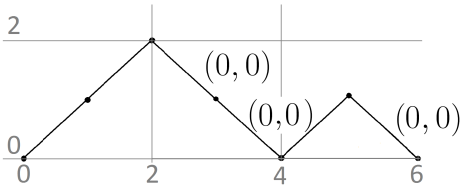
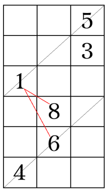
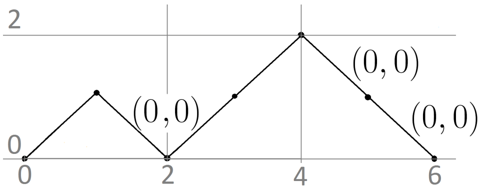
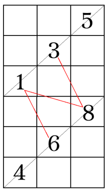
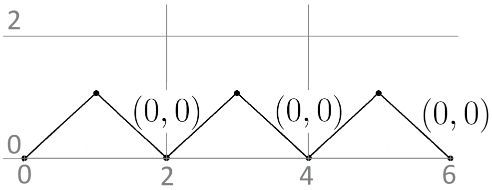
$$\omega(\psi(C)) = q^{\left(\sum_{i=1}^n n-i\right) - \left(\sum_{i \leq n} l^e_C(e_i) + \sum_{i > n} r_C(e_i)\right)}. \tag{1.14}$$

Now, it is easy to see that $\mathrm{fal}(C) = \sum_{i \leq n} l^e_C(e_i) + \sum_{i>n} r_C(e_i)$. In view of the latter remark, Formula (1.14) becomes Formula (1.9). $\square$

Proof of the bijectivity of $\psi : \mathbf{DC}(n) \to \mathbf{DH}(n)$

To end the proof of Theorem 1.48, it remains to show that ψ is bijective. To this end, we construct (in Definition A.2) a map $\tilde{\psi} : \mathrm{DH}(n) \to \mathrm{DC}(n)$ and we prove in Lemma A.6 that ψ and $\tilde{\psi}$ are inverse maps. This is done in Appendix A.2.

As an illustration of the whole chapter, the table depicted in the next page makes explicit the statistic-preserving bijections between the $h_3 = 7$ objects of DC(3), $\mathcal{D}'_4$ and DH(3).

$C \in \mathrm{DC}(3)$	$\phi(C) \in \mathcal{D}'_4$	$\psi(C) \in \mathrm{DH}(3)$
	41736285	
	41736582	
	71436285	
	71436582	
	51436287	
	21736584	
	21436587	

Chapter 2

A bijection between the irreducible k-shapes and the surjective pistols of height $k-1$

2.1 Abstract

This chapter constructs a bijection between irreducible k-shapes and surjective pistols of height $k-1$, which carries the "free k-sites" to the fixed points of surjective pistols. It confirms a conjecture of Hivert and Mallet (FPSAC 2011) that the number of irreducible k-shapes is the Genocchi number G_{2k}, and, with precision, that the irreducible k-shapes (with $k \geq 2$) generate the Gandhi polynomial $Q_{k-1}(x)$ with respect to the statistic of free k-sites.

2.2 Introduction

This chapter largely follows [Big15a].

Recall that a partition is a a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$. By abuse of definition, we consider that a partition may be empty (in which case $m = 0$). The integers λ_i are called the *parts* of the partition λ. Let $p = \lambda_1$. For all $i \in [n]$, if the integer i appears q_i times among the parts of λ, we use the notation $\lambda = p^{q_p} \dots 2^{q_2} 1^{q_1}$, and if $q_i = 0$ for some i, we allow ourselves not to write i^{p_i} in this notation. For example, the partition $\lambda = (4, 2, 2, 1)$ equals $4^1 3^0 2^2 1^1 = 4^1 2^2 1^1$. If λ and μ are two partitions, we define their union $\lambda \cup \mu$

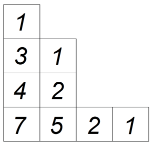


Figure 2.1: Ferrers diagram $[(4, 2, 2, 1)]$.

as the sequence obtained by inserting the parts of μ between the parts of λ so that the final result is still a partition. For example, the union of the partitions $\lambda = (4, 2, 2, 1)$ and $\mu = (6, 3, 2)$ is the partition $\lambda \cup \mu = (6, 4, 3, 2, 2, 2, 1)$.

A convenient way to visualize a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ is to consider its Ferrers diagram (denoted by $[\lambda]$), which is composed of cells organized in left-justified rows such that the i-th row (from bottom to top) contains λ_i cells. The *hook length* of a cell c is defined as the number of cells located to its right in the same row (including c itself) or above it in the same column. If the hook length of a cell c equals h, we say that c is hook lengthed by the integer h. For example, the Ferrers diagram of the partition $\lambda = (4, 2, 2, 1)$ is represented in Figure 2.1, in which every cell is labelled by its hook length.

We will sometimes assimilate partitions into their Ferrers diagrams.

The study of k-shapes arises naturally in the combinatorics of k-Schur functions (see [LLMS13]). Recall that the regular Schur functions s_λ, indexed by the partitions λ, are symmetric functions which form a basis of the space of symmetric functions Sym, and which may be defined by

$$s_\lambda = \sum_T \mathbf{x}^T$$

where the sum is over all semi standard young tableaux of shape λ (tableaux obtained by labelling the Ferrers diagram of λ with positive integers that increase from left to right and from bottom to top) and where $\mathbf{x}^T$ is the monomial $\prod_{i \geq 1} x_i^{\mu_i}$ where μ_i is the number of occurences of the integer i among the labels of T for all i. Another basis of Sym lies in the *complete homogeneous symmetric functions* h_λ, indexed by the partitions and defined by

$$h_\lambda = \prod_{i=1}^{m} h_{\lambda_i}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and h_n is the sum of all monomials $x_1^{\alpha_1} x_2^{\alpha_2} \ldots$ such that $\sum \alpha_i = n$.

Now, let k be a positive integer and $\mathrm{Sym}^{(k)}$ the subspace of Sym generated by the homogeneous symmetric functions h_λ indexed by the k-bounded partitions, *i.e.* the partitions λ whose parts do not exceed k. The k-Schur functions [LM05] $s_\lambda^{(k)}$ are symmetric functions that form a basis of $\mathrm{Sym}^{(k)}$, in which they play the same role as the regular Schur functions s_λ in the space Sym. They may be defined by

$$h_\lambda = s_\lambda^{(k)} + \sum_\mu K_{\mu,\lambda}^{(k)} s_\mu^{(k)}$$

for all k-bounded partition λ, where $K_{\mu,\lambda}^{(k)}$ is the number of tableaux named *k-tableaux* related to the k-bounded partitions λ and μ. The k-shapes (see Definition 2.1) appear thereafter in the expansion of the k-Schur functions in terms of the regular Schur functions, the coefficients of the latter expansion being related to the enumeration of some paths in the poset of k-shapes [LLMS13].

In a 2011 FPSAC paper [HM11], Hivert and Mallet showed that the generating function of all k-shapes is a rational function whose numerator $P_k(t)$ is defined in terms of what they called irreducible k-shapes. The sequence of numbers of irreducible k-shapes $(P_k(1))_{k\geq 1}$ seemed to be the sequence of Genocchi numbers $(G_{2k})_{k\geq 1} = (1, 1, 3, 17, 155, 2073, \ldots)$, which we recall are the positive integers that can be defined by $G_{2k} = Q_{k-1}(1)$ for all $k \geq 2$ (see [Car71, RS73]) where $Q_k(x)$ is the k-th Gandhi polynomial [Gan70], defined by the recursion $Q_1(x) = 1$ and

$$Q_k(x) = (x+1)^2 Q_{k-1}(x+1) - x^2 Q_{k-1}(x). \tag{2.1}$$

Hivert and Mallet defined a statistic $\mathrm{fr}(\lambda)$ counting the so-called free k-sites on the partitions λ in the set of irreducible k-shapes IS_k, and conjectured that

$$Q_{k-1}(x) = \sum_{\lambda \in \mathrm{IS}_k} x^{\mathrm{fr}(\lambda)} \tag{2.2}$$

for all $k \geq 2$. We recall that $Q_{k-1}(x)$ is generated by the surjective pistols of height $k-1$ with respect to the statistic of fixed points [Dum74], *i.e.*,

$$Q_{k-1}(x) = \sum_{f \in \mathrm{SP}_{k-1}} x^{\mathrm{fix}(f)-1} \tag{2.3}$$

where the set of surjective pistols of height $k-1$, denoted by SP_{k-1}, is the set of surjective maps $f : [2k-2] \to \{2, 4, \ldots, 2k-2\}$ such that $f(j) \geq$

j for all $j \in [2k-2]$, and where $\text{fix}(f)$ is the number of fixed points of $f \in \text{SP}_{k-1}$, that is, the number of integers $j \in [2k-2]$ such that $f(j) = j$. Recall that we assimilate every surjective pistol $f \in SP_{k-1}$ into the sequence $(f(1), f(2), \dots, f(2k-2))$, and that SP_{k-1} is in bijection with the set of tableaux made of $k-1$ left-justified rows of length $2, 4, 6, \dots, 2k-2$ (from bottom to top, thence giving the tableau the shape of a pistol) such that each row contains at least one dot and each column contains exactly one dot. Indeed, a simple bijection consists in mapping such a tableau T to the surjective pistol f defined by $f(j) = 2(\lceil j/2 \rceil + z_j)$ where the j-th column of T contains a dot in its $(1+z_j)$-th cell (from top to bottom) for all $j \in [2k-2]$. For example, if $f = (2,4,4,8,8,6,8,8) \in SP_4$, the tableau corresponding to f is depicted in Figure 2.2.

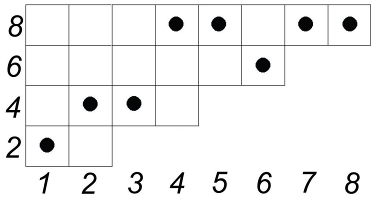


Figure 2.2: Tableau of $f = (2,4,4,8,8,6,8,8) \in SP_4$.

The goal of this chapter is to construct a bijection between the irreducible k-shapes and the surjective pistols of height $k-1$, such that the free k-sites of an irreducible k-shape are carried to the fixed points of the corresponding surjective pistol. In view of Formula (2.3), this bijection will imply the conjectured Formula (2.2).

The rest of this chapter is organized as follows. In §2.3, we give some background about skew partitions and k-shapes (in § 2.3.1), then we focus on irreducible k-shapes (in § 2.3.2) and enounce Conjecture 2.9 raised by Mallet (which implies Formula 2.2), and the main result of this chapter, Theorem 2.10, whose latter conjecture is a straight corollary. In §2.4, we give preliminaries of the proof of Theorem 2.10 by introducing the notion of partial k-shapes. In §2.5, we demonstrate Theorem 2.10 by defining two inverse maps φ (in § 2.5.1) and $\tilde{\varphi}$ (in § 2.5.2) which connect irreducible k-shapes and surjective pistols together while keeping track of the two statistics. Finally, in §2.6, we explore the corresponding interpretations of some generalizations of the Gandhi polynomials, generated by the surjective pistols with respect to refined statistics, on the irreducible k-shapes.

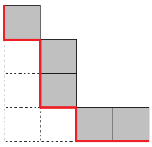


Figure 2.3: Skew partition $\lambda\backslash\mu$.

2.3 Definitions and main result

2.3.1 Skew partitions and k-shapes

If two partitions $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_q)$ (with $q \leq p$) are such that $\mu_i \leq \lambda_i$ for all $i \leq q$, then the Ferrers diagram $[\mu]$ appears naturally in the bottom left-hand side of $[\lambda]$ and we define the *skew partition* $s = \lambda\backslash\mu$ as the diagram $[\lambda]\backslash[\mu]$. For example, if $\lambda = (4, 2, 2, 1)$ and $\mu = (2, 1, 1)$, then $\lambda\backslash\mu$ is the diagram depicted in Figure 2.3.

For every skew partition s, we name *row shape* (respectively *column shape*) of s, and we denote by $rs(s)$ (resp. $cs(s)$), the sequence of the lengths of the rows from bottom to top (resp. the sequence of the heights of the columns from left to right) of s. Those sequences are not necessarily partitions. For example, if s is the skew partition depicted in Figure 2.3, then $rs(s) = (2, 1, 1, 1)$ and $cs(s) = (1, 2, 1, 1)$ (in particular $cs(s)$ is not a partition). The lower border of a skew partition s is the set of polygonal paths made of the bottom left-hand side vertical and horizontal edges of the cells of s with no cell beneath it or on the left of it. The left edge (respectively bottom edge) of such a cell is then called a *south step* (resp. *east step*) of the lower border of s. If the lower border of s is a continuous polygonal path, *i.e.*, if it is not fragmented into several pieces, we also define a canonical partition $< s >$ obtained by inserting cells in the empty space beneath every column and on the left of every row of s. For example, in Figure 2.3, the lower border of $s = \lambda/\mu$ is the polygonal path drawn in red, and $< s >$ is simply the original partition $\lambda = (4, 2, 2, 1)$. Now, consider a positive integer k. For every partition λ, it is easy to see that the diagram composed of the cells of $[\lambda]$ whose hook length does not exceed k, is a skew partition, which we name *k-boundary* of λ and denote by $\partial^k(\lambda)$. Incidentally, we name *k-rim* of λ the lower border of $\partial^k(\lambda)$, and we denote by $rs^k(\lambda)$ (respectively $cs^k(\lambda)$) the sequence $rs(\partial^k(\lambda))$ (resp. the sequence $cs(\partial^k(\lambda))$. For example, the 2-boundary of the partition $\lambda = (4, 2, 2, 1)$ depicted in Figure 2.1, is in fact the skew partition $s = \lambda/\mu$ of Figure 2.3. Note that if the k-rim of λ is a continuous polygonal path, then the partition $< \partial^k(\lambda) >$ is simply λ.

Definition 2.1 (Lam et al [LLMS13])**.** A k-shape is a partition λ such that the sequences $rs^k(\lambda)$ and $cs^k(\lambda)$ are also partitions.

For example, the partition $\lambda = (4, 2, 2, 1)$ depicted in Figure 2.1 is not a 2-shape since $cs^2(\lambda) = (1, 2, 1, 1)$ is not a partition, but it is a k-shape for any $k \geq 4$ (for instance $cs^5(\lambda) = (3, 3, 1, 1)$ and $rs^5(\lambda) = (3, 2, 2, 1)$ are partitions, so λ is a 5-shape, see Figure 2.4). Note that the k-rim of a k-shape λ is always a continuous polygonal path and that $\lambda =< \partial^k(\lambda) >$. Consequently, we will sometimes assimilate a k-shape into its k-boundary.

2.3.2 Irreducible k-shapes

This part largely follows Section 3 of [HM11]. It deals first with an operation which maps a k-shape and a $k-$ or $(k-1)$-rectangle (namely, a partition whose Ferrers diagram is a rectangle whose largest hook length is k or $k-1$) to a new k-shape. Irreducible k-shapes will then be defined as k-shapes that cannot be defined in such a way.

Definition 2.2 ([LLM03])**.** A k-rectangle is a partition of the form u^v for some positive integers u and v such that $u + v = k + 1$. The Ferrers diagram of such a partition is a rectangle made of v rows of length u. In particular, the greatest hook length of this diagram is the hook length of the cell in its bottom left side corner, which equals $u + v - 1 = k$.

Definition 2.3. Let λ be a k-shape and let u and v be two positive integers. We denote by $H_u^k(\lambda)$ (respectively $V_v^k(\lambda)$) the set of the cells of the skew partition $\partial^k(\lambda)$ that are contained in a row of length u (resp. in a column of height v).

For example, consider the 5-shape $\lambda = (4, 2, 2, 1)$. The sets $(H_u^5(\lambda))_{u\geq 1}$ and $(V_v^5(\lambda))_{v\geq 1}$ are outlined in Figure 2.4 (in this example the set $V_2^5(\lambda)$ is empty). Note that for every k-shape λ and for every pair of positive integers (u, v), if the set $H_u^k(\lambda) \cap V_v^k(\lambda)$ is not empty, then there exists a cell in $V_v^k(\lambda)$ whose hook length is at least $u + v - 1$. Consequently, every cell of $\partial^k(\lambda)$ appears in $H_u^k(\lambda) \cap V_v^k(\lambda)$ for some pair (u, v) such that $u + v \leq k + 1$.

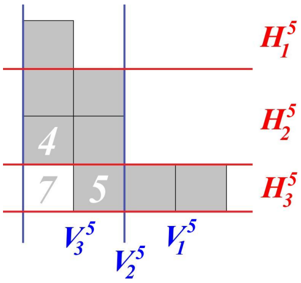


Figure 2.4: Skew partition $\partial^5((4,2,2,1))$.

Proposition 2.4 ([HM11])**.** *Let λ be a k-shape and u^v be a k- or $(k-1)$-rectangle. Then, the k-rim of λ crosses the space delimited by the set $H_u^k(\lambda) \cap V_v^k(\lambda)$.*

For example, in the 5-shape $\lambda = (4,2,2,1)$ depicted in Figure 2.4, the 5-rim of λ crosses the square delimited by the set $H_2^5(\lambda) \cap V_3^5(\lambda)$, the rectangle delimited by the set $H_3^5(\lambda) \cap V_3^5(\lambda)$, and the segment delimited by the set $H_3^5(\lambda) \cap V_2^5(\lambda)$ (which is empty).

Proposition/Definition 2.5 ([HM11])**.** Let λ be a k-shape and u^v be a k- or $(k-1)$-rectangle. Then, there exists a unique k-shape, which we denote by $\lambda + u^v$, such that

$$rs^k(\lambda + u^v) = rs^k(\lambda) \cup u^v,$$
$$cs^k(\lambda + u^v) = cs^k(\lambda) \cup v^u.$$

The principle of construction of $\lambda + u^v$ is to insert the rectangle u^v in the skew partition $\partial^k(\lambda)$ as follows : we first choose any point p of the k-rim of λ such that the Cartesian coordinates of p are integers (thus p is a left-side corner of a cell of the skew partition $\partial^k(\lambda)$) and such that p is in the space delimited by the set $H_u^k(\lambda) \cap V_v^k(\lambda)$ (there exists at least one such point p in view of Proposition 2.4). Then, we decompose the skew partition $\partial^k(\lambda)$ into three regions : the set F of the cells located to the right of p and above it, the set A of the cells located to the left of F, and the set B of the cells located below F (see Figure 2.5). We keep in mind that $F \subset H_u^k(\lambda) \cap V_v^k(\lambda)$, which means we can consider its complement $\bar{F}$ in the rectangle u^v, which is delimited by dotted lines in Figure 2.5. Then, the k-shape $\lambda + u^v$ is the partition whose k-boundary is like depicted in Figure 2.6, where F_2 is a copy of F. This construction does not depend on the choice of p.

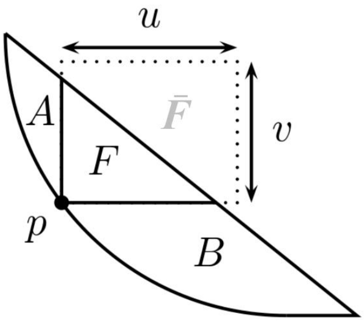


Figure 2.5: Decomposition of $\partial^k(\lambda)$.

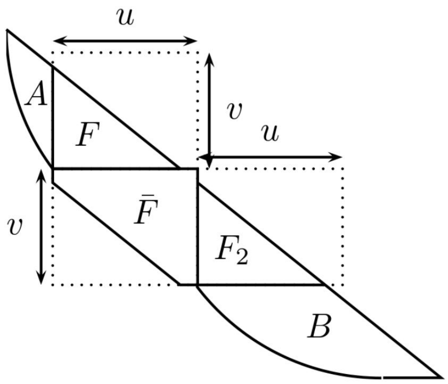


Figure 2.6: k-boundary of the k-shape $\lambda + u^v$.

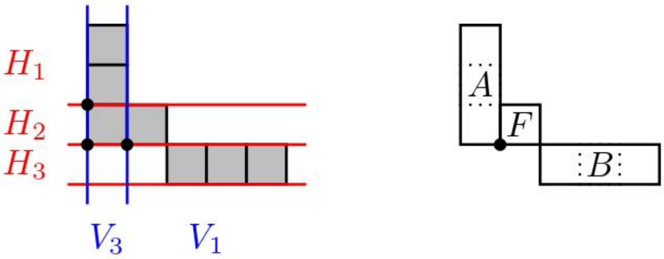
For example, consider the 4-shape $\lambda = (5,2,1,1)$ and the 4-rectangle 2^3, the construction of the 4-shape $\lambda + 2^3$ is depicted in Figure 2.7.

Definition 2.6 ([HM11]). An *irreducible* k-shape is a k-shape that cannot be obtained from another k-shape by inserting a k- or $(k-1)$-rectangle as described in Definition 2.5. The set of irreducible k-shapes is denoted by IS_k.

The following proposition gives another characterisation of the irreducible k-shapes, which we will use from now.

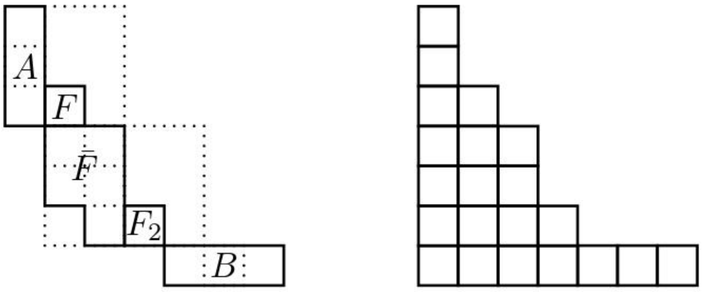
Proposition 2.7 ([HM11]). *Let λ be a k-shape, and let u^v be a k- or $(k-1)$-rectangle. The following are equivalent :*

1. *there exists a k-shape μ such that $\lambda = \mu + (u^v)$,*
2. *there exist two points (x_1, y_1) and (x_2, y_2) of the k-rim of λ lying in $H_u^k(\lambda) \cap V_v^k(\lambda)$ such that $x_2 - x_1 \geq u$,*



(a) The initial 4-shape $\lambda = (5,2,1,1)$ with the possible insertion points for the 4-rectangle 2^3.

(b) Decomposition of $\partial^4(\lambda)$ with respect to the insertion point $p = (1,1)$.



(c) The skew partition resulting from the insertion process.

(d) The final 4-shape $(7,4,3,3,2,1,1)$.

Figure 2.7: Construction of $(5,2,1,1) + 2^3$.

3. *there exist two points* (x_1, y_1) *and* (x_2, y_2) *of the* k*-rim of* λ *lying in* $H_u^k(\lambda) \cap V_v^k(\lambda)$ *such that* $y_1 - y_2 \geq v$.

Proof. The statement $(i) \Rightarrow (ii)$ is obvious from Figure 2.6. Suppose now that (ii) is true. Let $q = (x_1, y_1)$ and $r = (x_2, y_2)$ be two points of the k-rim of λ lying in $H_u^k(\lambda) \cap V_v^k(\lambda)$ such that $x_2 - x_1 \geq u$. We can suppose that x_1, x_2, y_1, y_2 are integers, *i.e.*, that q and r are corners of cells of the skew partition $\partial^k(\lambda)$. We can also suppose that $x_2 - x_1 = u$. Then, we decompose $\partial^k(\lambda)$ into five regions (see Figure 2.8):

1. the set $G \subset H_u^k(\lambda) \cap V_v^k(\lambda)$ of the cells located simultaneously to the right of q and below it, and to the left of s and above it;
2. the set $F \subset V_v^k(\lambda)$ of the cells located above G;

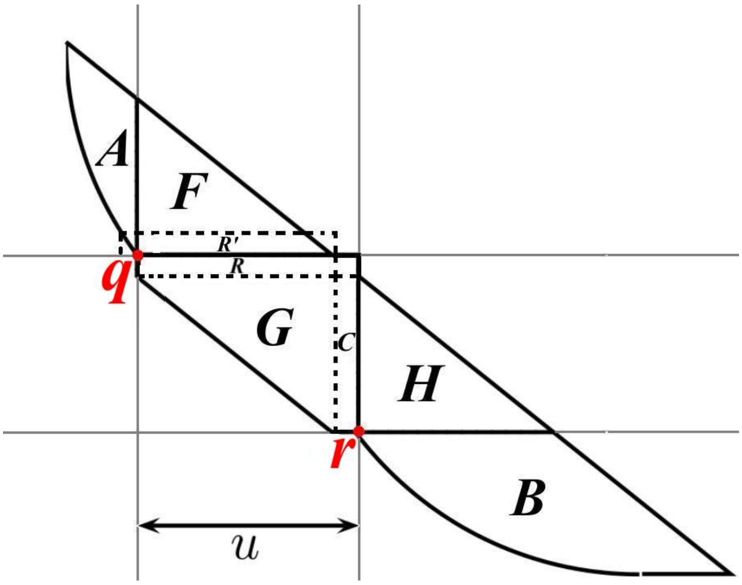


Figure 2.8: Decomposition of $\partial^k(\lambda)$.

3. the set $H \subset H^k_u(\lambda)$ of the cells located to the right of G;
4. the set A of the cells located to the left of F;
5. the set B of the cells located below H.

Let R and C be respectively the first row (from top to bottom) and the last column (from left to right) of the region G. The top left-side corner of R is the point q, and its length is u. Likewise, the bottom right-side corner of C is the point r, and its height is v. Now, let R' be the row above R. By definition of a k-shape, the length of R' is at most the length u of R. With no loss of generality, we can suppose that the first cell (from left to right) of R' appears in the region A, *i.e.*, since the length of R' is at most u, that there is no cell of the column C in the region F. Consequently, the integer $y_1 - y_2$ is the height of the column C, *i.e.* $y_1 - y_2 = v$ (see Figure 2.9), so $(ii) \Rightarrow (iii)$. The proof of $(iii) \Rightarrow (ii)$ is analogous. Finally, always under the assumption that (ii) (or (iii)) is true and that $\partial^k(\lambda)$ is decomposed as depicted in Figure 2.9, it is obvious from this latter picture that the regions $F \subset V^k_v(\lambda)$ and $H \subset H^k_u(\lambda)$ are copies of the complement of G in the rectangle u^v. So, by Definition 2.5, we obtain $\lambda = \mu + u^v$ where μ is the k-shape whose k-boundary $\partial^k(\mu)$ is depicted in Figure 2.10 (this is indeed a k-shape because the hook lengths of the cells of the regions A, F and B in Figure

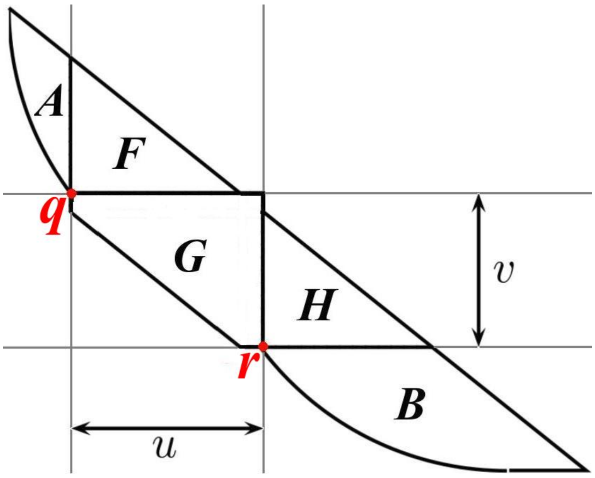


Figure 2.9: Decomposition of $\partial^k(\lambda)$.

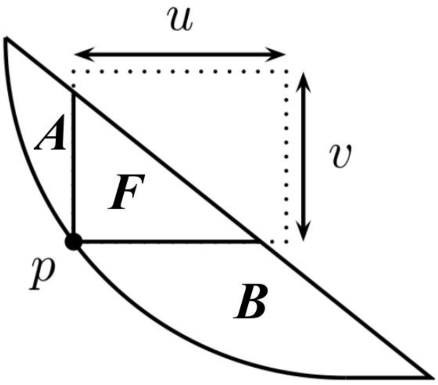


Figure 2.10: k-boundary $\partial^k(\mu)$.

2.10 are exactly the same as in Figure 2.9 : this is obvious for A and F, and this is true for B because H is a copy of F). □

Thus, from Proposition 2.7, a k-shape λ is irreducible if and only if the intersections $H_i^k(\lambda) \cap V_{k-i}^k(\lambda)$ and $H_j^k(\lambda) \cap V_{k+1-j}^k(\lambda)$ contain respectively at most $i-1$ and $j-1$ east steps of the k-rim of λ for all $i \in [k-1]$ and $j \in [k]$.

For example, the 5-shape $\lambda = (4,2,2,1)$ (see Figure 2.11) is irreducible: the sets $H_i^5(\lambda) \cap V_{5-i}^5(\lambda)$ and $H_j^5(\lambda) \cap V_{6-j}^5(\lambda)$ are empty if $i \neq 2$ and $j \neq 3$, and the two sets $H_2^5(\lambda) \cap V_3^5(\lambda)$ and $H_3^5(\lambda) \cap V_3^5(\lambda)$ contain respectively $1 < 2$ and $1 < 3$ east steps of the k-rim of λ.

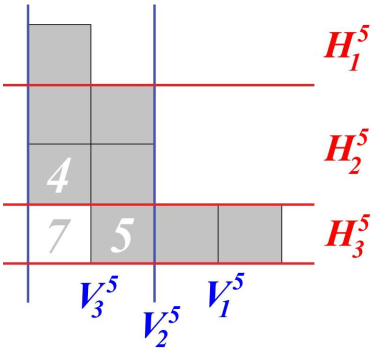


Figure 2.11: Skew partition $\partial^5((4,2,2,1))$.

In general, it is easy to see that for any k-shape λ to be irreducible, the sets $H_1^k(\lambda) \cap V_k^k(\lambda)$ and $H_k^k(\lambda) \cap\ V_1^k(\lambda)$ must be empty, and by definition the set $H_1^k(\lambda) \cap V_{k-1}^k(\lambda)$ must contain no east step of the k-rim of λ. In particular, for $k = 1$ or 2 there is only one irreducible k-shape: the empty partition.

Definition 2.8 (Hivert and Mallet [HM11, Mal11]). Let λ be an irreducible k-shape with $k \geq 3$. For all $i \in [k-2]$, we say that the integer i is a *free k-site* of λ if the set $H_{k-i}^k(\lambda) \cap V_{i+1}^k(\lambda)$ is empty. We define $\overrightarrow{\mathrm{fr}}(\lambda)$ as the vector $(t_1, t_2, \ldots, t_{k-2}) \in \{0,1\}^{k-2}$ where $t_i = 1$ if and only if i is a free k-site of λ. We also define $\mathrm{fr}(\lambda)$ as $\sum_{i=1}^{k-2} t_i$ (the number of free k-sites of λ).

For example, the irreducible 5-shape $\lambda = (4,2,2,1)$ depicted in Figure 2.11 is such that $\overrightarrow{\mathrm{fr}}(\lambda) = (1,0,1)$.

In order to prove the conjecture of Formula 2.2, and in view of Theorem 0.4, Hivert and Mallet proposed to construct a bijection $\tilde{\varphi} : \mathrm{IS}_k \to \mathrm{SP}_{k-1}$ such that $\mathrm{fix}(\tilde{\varphi}(\lambda)) = \mathrm{fr}(\lambda) + 1$ for all λ. Mallet [Mal11] refined the conjecture by introducing a vectorial version of the statistic of fixed points: for all $f \in \mathrm{SP}_{k-1}$, we define $\overrightarrow{\mathrm{fix}}(f)$ as the vector $(t_1, \ldots, t_{k-2}) \in \{0,1\}^{k-2}$ where $t_i = 1$ if and only if $f(2i) = 2i$ (in particular $\sum_i t_i = \mathrm{fix}(f) - 1$).

Conjecture 2.9 (Mallet [Mal11]). *Let $k \geq 3$. For all vector $\overrightarrow{v} = (v_1, \ldots, v_{k-2}) \in \{0,1\}^{k-2}$, the number of irreducible k-shapes λ such that $\overrightarrow{fr}(\lambda) = \overrightarrow{v}$ is the number of surjective pistols $f \in SP_{k-1}$ such that $\overrightarrow{fix}(f) = \overrightarrow{v}$.*

The main result of this chapter is the following theorem, which implies Conjecture 2.9.

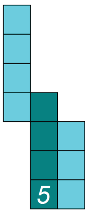


Figure 2.12: Partial 6-shape s.

Theorem 2.10. *There exists a bijection* $\varphi : SP_{k-1} \to IS_k$ *such that*

$$\overrightarrow{fr}(\varphi(f)) = \overrightarrow{fix}(f)$$

for all $f \in SP_{k-1}$.

We intend to demonstrate Theorem 2.10 in §2.4 and §2.5.

2.4 Partial k-shapes

Definition 2.11 (labelled skew partitions, partial k-shapes and saturation property)**.** A *labelled skew partition* is a skew partition s whose columns are labelled by the integer 0 or 1. If $cs(s)$ is a partition (*i.e.*, if the heights of the columns of s decrease from left to right) and if the hook length of every cell c of s doesn't exceed $k - b$ where b is the label of the column that contains c, we say that s is a *partial k-shape*. In that case, if C_0 is a column labelled by 0 which is rooted in a row R_0 (*i.e.*, whose bottom cell is located in R_0) whose greatest hook length is k, we say that C_0 is saturated. For all $i \in [k-1]$, if every column of s whose height is $i+1$ and whose label is 0 is saturated, we say that s is saturated in i. If s is saturated in i for all i, we say that s is saturated.

We represent labelled skew partitions by painting in dark blue columns labelled by 0, and in light blue columns labelled by 1. For example, the skew partition depicted in Figure 2.12 is a partial 6-shape, which is not saturated because its unique column labelled by 0 is rooted in a row whose greatest hook length is 5 instead of 6.

Definition 2.12 (Sum of partial k-shapes with rectangles)**.** Let s be a partial k-shape, and let $j \geq 1$ with the condition that the height of every column of

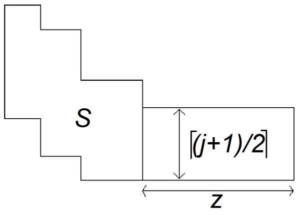


Figure 2.13: $\tilde{s}$, obtained by gluing the rectangle $\lceil (j+1)/2 \rceil^z$ to s.

s is at least $\lceil (j+2)/2 \rceil$ (since the heights of the columns of a partial k-shape may equal at most k by definition, we also have $j \leq 2k-1$). Let z be a nonnegative integer, and define the integer $t(j)$ by

$$t(j) = j \bmod 2.$$

We consider the labelled skew partition $\tilde{s}$ obtained by gluing right to the last column of s, the amount of z columns of height $\lceil (j+1)/2 \rceil$ (see Figure 2.13) labelled by the integer $t(j)$ (if s is empty then the result is simply the rectangle made of the z columns of height $\lceil (j+1)/2 \rceil$ and label $t(j)$, this rectangle being itself the empty partition if $z = 0$).

The following algorithm aims at transforming $\tilde{s}$ into a partial k-shape. It consists in lifting the columns of $\tilde{s}$ following the conditions depicted in the next algorithm. Note that, whenever we refer to the bottom cell of a column, we refer to a cell which is prone to moving.

Algorithm : as long as there exists a column of $\tilde{s}$ that is concerned by one of the three following rules, we consider the first one (from right to left) of these colums, denoted by C, and we apply the corresponding rules by decreasing order of priority. We will prove that this algorithm is finite after enunciating the rules.

1. Let C' be the column of $\tilde{s}$ located to the right of C. If C' is labelled by 0, if C has not the same label or height as C', and if C and C' are rooted in a same row, then we *lift* every column rooted in the same row as C and on the left of C (including C itself), *i.e.*, we erase the bottom cells of these columns and we draw a cell on the top of each of them (see Figure 2.14). In particular, the bottom cell c' of C' becomes a corner in $\tilde{s}$ (*i.e.*, a cell of $\tilde{s}$ with no other cell beneath it or on the left of it).
2. If the hook length h of the bottom cell c of C is such that $h > k+1-t(p)$, then we lift every column rooted in the same row as C and on the left

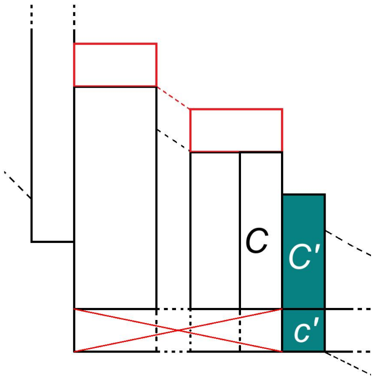


Figure 2.14: Rule 1.

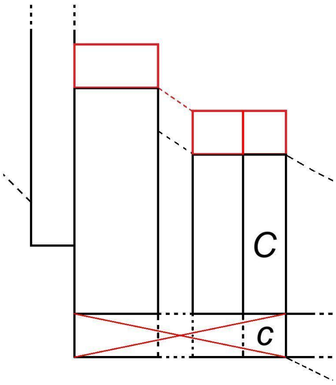


Figure 2.15: Rule 2.

of C, including C itself (*i.e.*, we erase the bottom cells of these columns and we draw a cell on the top of each of them, see Figure 2.15) as long as the hook length h of c is such that $h > k+1-t(p)$. These columns may then be lifted several times.

3. The goal of this rule is to prevent the saturated columns of s from not being saturated any more in the final version of $\tilde{s}$ at the end of the algorithm. We suppose that C is a column of $\tilde{s}$ labelled by 0 such that C was saturated in s but is no more saturated in $\tilde{s}$. Following Rule 1 of the present algorithm, the first column (from left to right) in which C is rooted, which we denote by C', has the same height and label as C, and by definition C' is not saturated in $\tilde{s}$ since C is not. Consequently, the bottom cell c' of C' is a corner of $\tilde{s}$ whose hook length h' equals

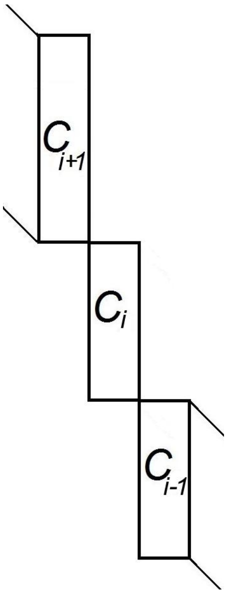


Figure 2.16: N-th step of the algorithm.

$k-l < k$ for some $l \geq 1$. Consider now (in $\tilde{s}$) the last column (from left to right) C_0 that intersects the row R in which C and C' are rooted, and consider also the l columns $C_1, C_2, \ldots, C_l$ (from left to right) located on the right of C_0. Then, for i from 1 to l, we lift C_i so that its top cell is at the same level as the row R (potentially lifting some columns located to its left so that its bottom cell is not located higher than in the row in which C_{i-1} is rooted, which is necessary for $\tilde{s}$ to remain a skew partition). Thus, in the resulted version of $\tilde{s}$, the hook length h' of c' equals $k-l+l=k$, which implies that both C and C' are now saturated.

We prove that this algorithm is finite : we can construct a partial k-shape $s_{\max}$ whose columns $C_1, C_2, \ldots$ (from right to left) are the columns of $\tilde{s}$ (in the same order) and for which none of the three rules are requested : it consists in placing the bottom right-hand side corner of C_{i+1} at the same place as the top left-hand side corner of C_i for all i (see Figure 2.16), then in lifting (from left to right) every column in order to saturate the columns that are saturated in s. By definition of the three rules, the level of the column C_i in $\tilde{s}$ cannot exceed the level of C_i in $s_{\max}$ for all i, so the algorithm is finite.

We define the $t(j)$-sum of the partial k-shape s with the rectangle made of z columns of height $\lceil (j+1)/2 \rceil$ and label $t(j)$ as the final version of $\tilde{s}$. It is a partial k-shape by construction. We denote such a sum by

$$s \oplus_{t(j)}^{k} \lceil (j+1)/2 \rceil^{z}.$$

For example, with $k = 4$, $j = 2$ (so $t(j) = 0$) and $z = 2$, consider the partial 4-shape s depicted in Figure 2.17. The height of every column of s is at least $\lceil (j+2)/2 \rceil = 2$. To compute the 0-sum $s \oplus_{1}^{4} 2^{2}$ of s with the rectangle composed of $z = 2$ columns of height $\lceil (j+1)/2 \rceil = 2$ and label $t(j) = 0$, we first glue the latter rectangle to the right of the last column of s (see Figure 2.18). In Figure 2.18, the Rule 1 forces the third column (from left to right) of $\tilde{s}$ to be lifted up to one cell, providing the picture depicted in Figure 2.19. Then, the Rule 2 forces the second and third columns to be lifted up to one cell, producing the picture depicted in Figure 2.20, where the Rule 2 must be applied again to lift the first column up to two cells (see Figure 2.21). But then, the Rule 3 forces the second column to be lifted up to one cell in order to preserve the saturation of the first column, which produces the partial 4-shape depicted in Figure 2.22.

Remark 2.13. The Rule 3 of Definition 2.12 guarantees that any saturated column of s is still saturated in $s \oplus_{t(j)}^{k} \lceil (j+1)/2 \rceil^{z}$. In particular, if s is saturated in $i \in [k-2]$, then $s \oplus_{t(j)}^{k} \lceil (j+1)/2 \rceil^{z}$ is also saturated in i.

Lemma 2.14. *Let s be a partial k-shape obtained by adding rectangles to the empty partition, i.e., such that s is the result of a sequence of sums*

$$\begin{aligned} s_0 &= s_0 \oplus_{t(j_0)}^{k} \lceil (j_0+1)/2 \rceil^{z_0}, \\ s_2 &= s_1 \oplus_{t(j_1)}^{k} \lceil (j_1+1)/2 \rceil^{z_1}, \\ &\vdots \\ s &= s_m \oplus_{t(j_m)}^{k} \lceil (j_m+1)/2 \rceil^{z_m} \end{aligned}$$

where s_0 is the empty partition. Let $j \in [2k-4]$ such that every column of s is at least $\lceil (j+2)/2 \rceil$ cells high, and let $z \in \{0, 1, \ldots, k-1-\lceil j/2 \rceil\}$. We consider two consecutive columns of $s \oplus_{t(j)}^{k} \lceil (j+1)/2 \rceil^{z}$, which we denote by C_1 (on the left) and C_2 (on the right), with the same height and the same label but not the same level, and such that C_1 has been lifted by the Rule 2 of Definition 2.12 (note that it cannot be by the Rule 1 because C_1 is not the first column from right to left to have its height and label). If C_2 has been lifted at the same level as C_1 in $s \oplus_{t(j)}^{k} \lceil (j+1)/2 \rceil^{z}$, then it is not by the Rule 2 of Definition 2.12.

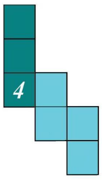


Figure 2.17: Partial 4-shape s.

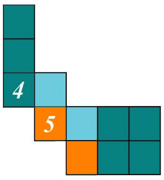


Figure 2.18: Glueing of 2^2 to s.

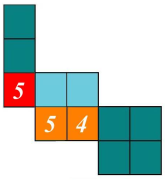


Figure 2.19: Lifting following Rule 1.

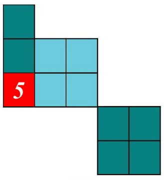


Figure 2.20: Lifting following Rule 2.

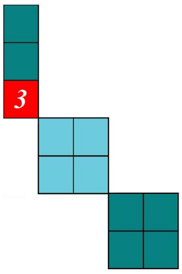


Figure 2.21: Lifting following Rule 2.

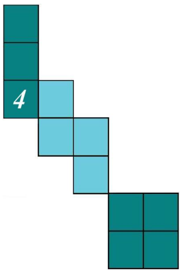


Figure 2.22: Lifting following Rule 3.

Proof. See Appendix B.1. ☐

Lemma 2.15. *Let s be a partial k-shape in the context of Lemma 2.14, and let $j \geq 1$ such that the height of every column of s is at least $\lceil (j+2)/2 \rceil$, and such that the quantity of integers $i \in [k-2]$ in which s is not saturated is at most $\lceil j/2 \rceil$. If s is not saturated in $i_0 \in [k-2]$, then there exists a unique integer $z \in [k-1-\lceil j/2 \rceil]$ such that the partial k-shape $s \oplus_{t(j)}^{k} \lceil (j+1)/2 \rceil^{z}$ is saturated in i_0.*

Proof. See Appendix B.2 ☐

2.5 Proof of Theorem 2.10

We first construct the two key maps in the first two subsections.

2.5.1 Map $\varphi : \mathbf{SP}_{k-1} \to \mathbf{IS}_k$

Definition 2.16 (map φ)**.** Let $f \in \mathrm{SP}_{k-1}$. We define $s^{2k-3}(f)$ as the empty partition. For j from $2k-4$ down to 1, let $i \in [k-1]$ such that $f(j) = 2i$, and suppose that the hypothesis $H(j+1)$ defined as "the height of every column of $s^{j+1}(f)$ is at least $\lceil (j+2)/2 \rceil$, and the number of integers i' in which $s^{j+1}(f)$ is not saturated is at most $\lceil j/2 \rceil$" is true (in particular $H(2k-3)$ is true so we can initiate the algorithm).

1. If $f(2i) > 2i$, if $j = \min\{j' \in [2k-4] : f(j') = 2i\}$ and if the partial k-shape $s^{j+1}(f)$ is not saturated in i, then we define $s^j(f)$ as
$$s^{j+1}(f) \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^{z_j(f)}$$
where $z_j(f)$ is the unique element of $[k-1-\lceil j/2 \rceil]$ such that the partial k-shape $s^{j+1}(f) \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^{z_j(f)}$ is saturated in i (see Lemma 2.15 in view of Hypothesis $H(j+1)$).
2. Else, we define $s^j(f)$ as
$$s^{j+1}(f) \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^{z_j(f)}$$
where $f(j) = 2(\lceil j/2 \rceil + z_j(f))$ (note that $z_j(f) \in \{0, 1, \ldots, k-1-\lceil j/2 \rceil\}$ by definition of a surjective pistol).

In either case, the height of every column of $s^j(f)$ is at least $\lceil (j+1)/2 \rceil$. Also, suppose there exist at least $\lceil (j-1)/2 \rceil + 1$ different integers $i' \in [k-2]$ in which $s^j(f)$ is not saturated. For each of these i', there must exist some $j' < j$ such that $f(j') = 2i'$ (otherwise we would have
$$\min\{j' \in [2k-4] : f(j') = 2i'\} \geq j$$
which forces i' to be saturated in $s^j(f)$ by the Rule 2 of the present algorithm). Since each of those i' is such that $i' + 1 \geq \lceil (j+2)/2 \rceil$ (following $H(j+1)$), this implies there are at least $\lceil (j-1)/2 \rceil + 1$ integers $j' \leq j-1$ such that $f(j') \geq 2\lceil j/2 \rceil$. Also, since f is surjective, there exist at least $\lceil j/2 \rceil - 1$ integers $j'' \leq j-1$ such that $f(j') \leq 2(\lceil j/2 \rceil - 1)$. Consequently, we obtain $(\lceil (j-1)/2 \rceil + 1) + (\lceil j/2 \rceil - 1) \leq j-1$, which is false because $\lceil (j-1)/2 \rceil + \lceil j/2 \rceil = j$. So the hypothesis $H(j)$ is true and the algorithm goes on. Ultimately, we define $\varphi(f)$ as the partition $< s^1(f) >$.

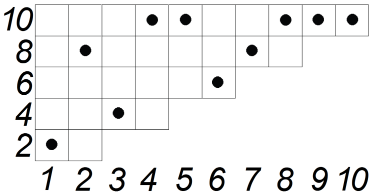


Figure 2.23: Surjective pistol $f = (2, 8, 4, 10, 10, 6, 8, 10, 10, 10) \in \mathrm{SP}_5$.

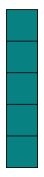
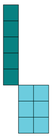
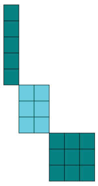
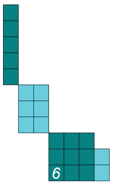
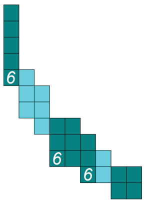
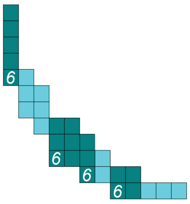
j	$8, 7, 6$	5	4	3	2	1
$s^j(f)$						

Figure 2.24: Sequence $(s^j(f))_{j \in [8]}$.

Proposition 2.17. *For all $f \in SP_{k-1}$, the partition $\lambda = \varphi(f)$ is an irreducible k-shape such that $\partial^k(\lambda) = s^1(f)$ and $\overrightarrow{fr}(\lambda) = \overrightarrow{fix}(f)$.*

For example, consider the following surjective pistol oh height 5 : the map $f = (2, 8, 4, 10, 10, 6, 8, 10, 10, 10) \in \mathrm{SP}_5$, whose tableau is depicted in Figure 2.23. Apart from 10, the only fixed point of f is 6, so $\overrightarrow{\mathrm{fix}}(f) = (0, 0, 1, 0)$.

Definition 2.16 provides the sequence $(s^8(f), s^7(f), \dots, s^1(f))$ depicted in Figure 2.24 (note that $s^8(f) = s^7(f) = s^6(f)$ because $z_7(f) = z_6(f) = 0$).

Thus, we obtain $s^1(f) = \partial^6(\lambda)$ where λ is the partial 6-shape $\varphi(f) =< s^1(f) >$. In particular, the sequences $rs^6(\lambda) = (5, 4, 4, 3, \dots, 1)$ and $cs^6(\lambda) = (5, 3, 3, 3, \dots, 1)$ are partitions, so λ is a 6-shape. Finally, we can see in Figure 2.25 that λ is irreducible and $\overrightarrow{\mathrm{fr}}(\lambda) = (0, 0, 1, 0) = \overrightarrow{\mathrm{fix}}(f)$.

See also Appendix B.5 for the detailed computation of another example : $\varphi(f)$ where $f = (4, 2, 6, 8, 6, 10, 8, 10, 10, 10) \in \mathrm{SP}_5$.

We split the proof of Proposition 2.17 into Lemmas 2.18, 2.19 and 2.20.

Lemma 2.18. *For all $f \in SP_{k-1}$, we have $\partial^k(\varphi(f)) = s^1(f)$.*

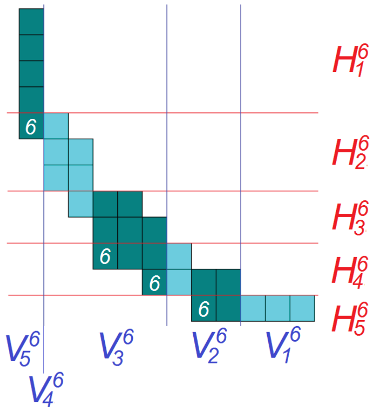


Figure 2.25: 6-boundary $\partial^6(\lambda) = s^1(f)$ of the irreducible 6-shape $\lambda = \varphi(f)$.

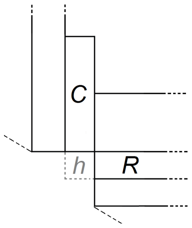


Figure 2.26: Anticorner of $s^1(f)$.

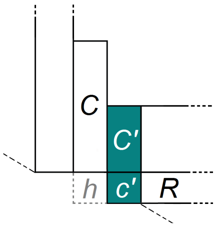


Figure 2.27: Lifting by the Rule 1

Proof. By construction, the skew partition $s^1(f)$ is a saturated partial k-shape (the saturation is guaranteed by Hypothesis $H(1)$ of Definition 2.16). As a partial k-shape, the hook length of every cell of $s^1(f)$ doesn't exceed k. Consequently, to prove that $\partial^k(\varphi(f)) = s^1(f)$, we only need to show that the hook length h of every *anticorner* of $s^1(f)$ (namely, a cell of $\varphi(f)$ glued simultaneously to the left of a row R of $s^1(f)$ and beneath a column C of $s^1(f)$, see Figure 2.26) is such that $h > k$.
Anticorners of $s^1(f)$ have been created by lifting columns by one of the three Rules 1,2 or 3 of Definition 2.12. Let x (resp. y) be the length of the row R (resp. the height of the column C).

1. If C has been lifted by the Rule 2, then $x + y > k$ (if C is labelled by 0) or $x + y > k - 1$ (if C is labelled by 1). In either case, we obtain $h = 1 + x + y > k$.
2. If C has been lifed by the Rule 1, then the first cell (from left to right) of the row R is a corner, whose hook length is denoted by h', and it is

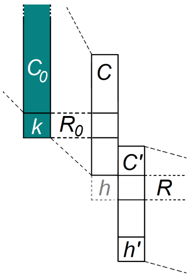


Figure 2.28: $s^1(f)$.

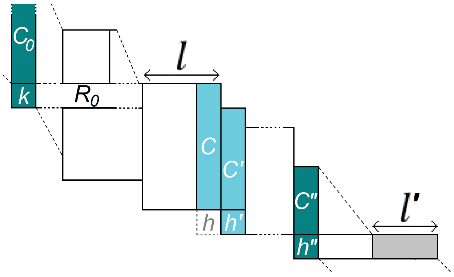


Figure 2.29: $s^j(f)$.

the bottom cell of a column C' labelled by 0 (see Figure 2.27). Let y' be the height of C'. Since C' is saturated, the hook length $h' = x + y' - 1$ of its bottom cell equals k. Consequently, since $y \geq y'$ by construction, we obtain $h = x + y + 1 > k$.

3. Else C has been lifted by the Rule 3 Let C' be the column following C in $s^1(f)$, and C_0 the saturated column of $s^1(f)$ such that C is the column which contains the last cell (from left to right) of the row R_0 in which C_0 is rooted (see Figure 2.28).
   Let $j \in [2k-4]$ such that C_0 is saturated in the partial k-shape $s^{j+1}(f)$ and such that C_0 loses momentarily its saturation from $s^{j+1}(f)$ to $s^j(f)$. By definition of the Rule 3 of Definition 2.12, it is easy to see that the column C' already exists in $s^{j+1}(f)$, that the columns C and C' have the same height and the same label 1, and that there exists a column C'' labelled by 0 in $s^j(f)$ which lifts the column C' by the Rule 1 of Definition 2.12, so that the partial k-shape $s^j(f)$ is like depicted in Figure 2.29. Also, the hook length h' of the bottom cell of C' equals $k-1$ in $s^{j+1}(f)$, so that the hook length h' equals $k-1-l'$ in $s^j(f)$ where $l' \geq 1$ is the number of gray cells introduced by Figure 2.29. Now, by construction, every column labelled by 0 in $s^1(f)$ must be saturated. In the case of C'', whose bottom cell is a corner, the hook length h'' of the latter bottom cell must be k, which is not the case yet in $s^j(f)$ because otherwise C'' would have lifted C'' by the Rule 2 of Definition 2.12 instead of the Rule 1 from $s^{j+1}(f)$ to $s^j(f)$. Consequently, in order to saturate C'' from $s^j(f)$ to $s^1(f)$, *i.e.*, to make h'' equal k, it is necessary to lift the l' gray cells. Indeed, otherwise, the column C'' would become saturated by lifting some columns $C_1, C_2, \ldots, C_m$ whose top cells would be glued to the right of the last gray cells, meaning those columns have

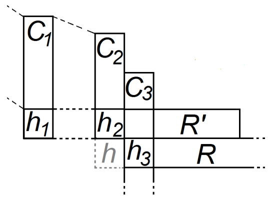


Figure 2.30: $s^1(f)$.

> the same height y and the same label t as the l' columns whose top cells are the gray cells. Obviously, since C''' is not saturated yet at that moment, the columns $C_1, C_2, \ldots, C_m$ are not lifted by the Rule 3 of Definition 2.12. In view of Lemma 2.14, these columns must have been lifted by the Rule 2, which implies every column of height y and label t are lifted to the same level as the last gray cell. But then it would lift the column C''' by the Rule 2 of Definition 2.12 instead of saturating it (because C''' was necessarily lifted by the columns of height y and label t in $s^{j+1}(f)$ and it could only be by the Rule 2), which is absurd. So the l' gray cells are necessarily lifted in $s^1(f)$, which forces the hook length h' to equal, in $s^1(f)$, what it equaled in $s^{j+1}(f)$, *i.e.* the integer $k-1$. In particular, in $s^1(f)$, the hook length h equals $h' + 2 = k + 1 > k$.

Anticorners of $s^1(f)$ being hook lengthed by integers exceeding k, we obtain $s^1(f) = \partial^k(\varphi(f))$. □

Lemma 2.19. *For all $f \in SP_{k-1}$, the partition $\lambda = \varphi(f)$ is a k-shape.*

Proof. From Lemma 2.18 we know that $s^1(f) = \partial^k(\lambda)$, and since $s^1(f)$ is a partial k-shape by construction, the sequence $cs^k(\lambda) = cs(s^1(f))$ is a partition. To prove that λ is a k-shape, it remains to show that the sequence $rs^k(\lambda) = rs(s^1(f))$ is a partition. Let R and R' be two consecutive rows (from bottom to top) of $s^1(f)$. If the first cell (from left to right) of R' is not a corner (*i.e.*, in this case, if there exists a cell beneath it, which is necessarily the first cell of R by construction) then the length of R obviously equals or exceeds the length of R'. Otherwise, the rows R and R' are like depicted in Figure 2.30, in which we introduce three columns from left to right C_1, C_2 and C_3, such that C_1 is the column whose bottom cell is the first cell of R' (which is a corner), C_3 is the column which contains the first cell of R (which is not necessarily a corner), and C_2 is the column which

preceeds C_3. Let x, x' be the respective lengths of R, R' and $y_1 \geq y_2 \geq y_3$ the respective heights of the columns C_1, C_2, C_3. We also introduce h_1 and h_2 the respective hook lengths of the bottom cells of C_1 and C_2, h_3 the hook length of the first cell of R, and h the hook length of the cell of $\varphi(f)$ beneath C_2 (this cell doesn't belong to $s^1(f)$). Since $s^1(f) = \partial^k(\lambda)$, we know that $y_2 + x + 1 = h > k$ hence $k \leq y_2 + x$. Also, we have $h_1 \leq k$, so $x' = h_1 - y_1 + 1 \leq k - y_1 + 1 \leq k - y_2 + 1 \leq x + 1$. Now suppose that $x' = x + 1$. Then $h_1 = y_1 + x' - 1 = y_1 + x \geq y_2 + x \geq k$, so $h_1 = k$ and $y_1 = y_2$. Consequently C_1 and C_2 are two columns of height $y_1 = y_2$, and C_1 is labelled by 0 because $h_1 = k$. Also, since columns of height y_1 and labelled by 1 are on the left of columns of height y_1 and labelled by 0 (like C_1) by construction of $\varphi(f)$ (see Definition 2.16), then C_2 is also labelled by 0.
Now, since $y_2 + x = k$, the column C_2 cannot have been lifted by the Rule 2 of Definition 2.12. Since $h_3 \leq y_3 + x - 1 \leq y_1 + x' - 2 = k - 1$, it cannot have been lifted by the Rule 1 (because otherwise the first cell of R would have been the bottom cell of a column C_3 labelled by 0, forcing h_3 to equal k because every column labelled by 0 is saturated in $s^1(f)$). So C_2 must have been lifted by the Rule 3, which is absurd because it would imply that its label is 1, which is not the case. Consequently, it is necessary that $x \geq x'$, thence $rs^k(\lambda)$ is a partition and λ is a k-shape. □

Lemma 2.20. *For all $f \in SP_{k-1}$, the k-shape $\lambda = \varphi(f)$ is irreducible and $\overrightarrow{fr}(\lambda) = \overrightarrow{fix}(f)$.*

Proof. For all $i \in [k-2]$, let n_i (resp. m_i) be the number of east steps of the k-rim of λ inside the set $H^k_{k-i}(\lambda) \cap V^k_{i+1}(\lambda)$ (resp. inside the set $H^k_{k-i}(\lambda) \cap V^k_i(\lambda)$). Recall that λ is irreducible if and only if $(n_i, m_i) \in \{0, 1, \ldots, k-1-i\}^2$ for all $i \in [k-2]$. Consider $i_0 \in [k-2]$. The number n_{i_0} is precisely the number of saturated columns of height $i_0 + 1$ of the partial k-shape $s^1(f) = \partial^k(\lambda)$. Since $s^1(f)$ is saturated by construction, this number is the quantity $z_{2i_0}(f) < k - i_0$ according to Definition 2.16. This statement being true for any $i_0 \in [k-2]$, in particular, if $i_0 > 1$, there are $n_{i_0-1} = z_{2i_0-2}(f)$ columns of height i_0 and label 0 in $s^1(f)$, so the quantity m_{i_0} is precisely the number $z_{2i_0-1}(f) < k - i_0$ of columns of height i_0 and label 1. Also, the columns of height 1 are necessarily labelled by 1, so $m_1 = z_1(f) < k - 1$. Consequently, the k-shape λ is irreducible. Finally, for all $i \in [k-2]$, we have the equivalence $f(2i) = 2i \Leftrightarrow z_{2i}(f) = 0$. Indeed, if $f(2i) = 2i$ then by definition $z_{2i}(f) = f(2i)/2 - i = 0$. Reciprocally, if

$f(2i) > 2i$, then either $z_{2i}(f)$ is defined by the Rule 2 of Definition 2.16, in which case $z_{2i}(f) > 0$, or $z_{2i}(f) = f(2i)/2 - i > 0$. Therefore, the equivalence is true and exactly translates into $\overrightarrow{\text{fix}}(f) = \overrightarrow{\text{fr}}(\lambda)$. ☐

2.5.2 Map $\tilde{\varphi} : \mathbf{IS}_k \to \mathbf{SP}_{k-1}$

Definition 2.21. Let λ be an irreducible k-shape. For all $i \in [k-2]$, we denote by $x_i(\lambda)$ the number of east steps of the k-rim of λ inside the set $H^k_{k-i}(\lambda) \cap V^k_{i+1}(\lambda)$, and by $y_i(\lambda)$ the number of east steps inside the set $V^k_i(\lambda) \backslash H^k_{k+1-i}(\lambda) \cap V^k_i(\lambda) = \bigsqcup_{j=1}^{k-i} H^k_j(\lambda) \cap V^k_i(\lambda)$. Finally, for all $j \in [2k-4]$, we set

$$z_j(\lambda) = \begin{cases} y_i(\lambda) & \text{if } j = 2i-1, \\ x_i(\lambda) & \text{if } j = 2i. \end{cases}$$

For example, if λ is the irreducible 6-shape represented in Figure 2.25, then $(z_j(\lambda))_{j\in[8]} = (3, 2, 1, 3, 2, 0, 0, 1)$. Note that in general, if λ is an irreducible k-shape and $(t_1, t_2, \ldots, t_{k-2}) = \overrightarrow{\text{fr}}(\lambda)$, then $t_i = 1$ if and only if $x_i(\lambda) = 0$ for all i.

Lemma 2.22. *For all $\lambda \in IS_k$ and for all $j \in [2k-4]$, we have*

$$z_j(\lambda) \in \{0, 1, \ldots, k-1-\lceil j/2 \rceil\}.$$

Proof. See Appendix B.3 ☐

Definition 2.23. Let $\lambda \in \mathrm{IS}_k$. We define a sequence $(s^j(\lambda))_{j\in[2k-3]}$ of partial k-shapes by $s^{2k-3}(\lambda) = \varnothing$ and

$$s^j(\lambda) = s^{j+1}(\lambda) \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^{z_j(\lambda)}.$$

Lemma 2.24. *We have $s^1(\lambda) = \partial^k(\lambda)$ for all $\lambda \in IS_k$.*

Proof. See Appendix B.4 ☐

Note that the statement of Lemma 2.24 is obvious if $\lambda = \varphi(\lambda)$ for some surjective pistol $f \in \mathrm{SP}_{k-1}$, because in that case $s^j(\lambda) = s^j(f)$ for all j.

Definition 2.25 (Map $\tilde{\varphi}$)**.** Let $\lambda \in \mathrm{IS}_k$. We define $m(\lambda) \in \{0, 1, \ldots, k-2\}$ and

$$1 \leq i_1(\lambda) < i_2(\lambda) < \ldots i_m(\lambda) \leq k-2$$

such that

$$\{i_1(\lambda), i_2(\lambda), \ldots, i_{m(\lambda)}(\lambda)\} = \{i \in [k-2] : x_i(\lambda) > 0\}$$

(this set is empty when $m(\lambda) = 0$). For all $p \in [m(\lambda)]$, let

$$j_p(\lambda) = \max\{j \in [2i_p(\lambda) - 1] : s^j(\lambda) \text{ is saturated in } i_p(\lambda)\}.$$

Let $L(\lambda) = [2k-4]$. For j from 1 to $2k-4$, if $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$, and if there is no $j' \in L(\lambda)$ such that $j' < j$ and $\lceil j'/2 \rceil + z_{j'} = i_p(\lambda)$, then we set $L(\lambda) := L(\lambda) \backslash \{j_p(\lambda)\}$. Now we define $\tilde{\varphi}(\lambda) \in \mathbb{N}^{[2k-2]}$ as the following: the integers $\tilde{\varphi}(\lambda)(2k-2)$ and $\tilde{\varphi}(\lambda)(2k-3)$ are defined as $2k-2$; afterwards, let $j \in [2k-4]$.

- If $j \in L(\lambda)$ then $\tilde{\varphi}(\lambda)(j)$ is defined as $2(\lceil j/2 \rceil + z_j(\lambda))$.
- Else there exists a unique $p \in [m(\lambda)]$ such that $j = j_p(\lambda)$, and we define $\tilde{\varphi}(\lambda)(j)$ as $2i_p(\lambda)$.

Proposition 2.26. *For all $\lambda \in IS_k$, the map $\tilde{\varphi}(\lambda)$ is a surjective pistol of height $k-1$, such that $\overrightarrow{fix}(\tilde{\varphi}(\lambda)) = \overrightarrow{fr}(\lambda)$.*

For example, consider the irreducible 6-shape λ of Figure 2.25, such that $(z_j(\lambda))_{j \in [8]} = (3, 2, 1, 3, 2, 0, 0, 1)$. In particular $(x_1(\lambda), x_2(\lambda), x_3(\lambda), x_4(\lambda)) = (2, 3, 0, 1)$ so $m(\lambda) = 3$ and $(i_1(\lambda), i_2(\lambda), i_3(\lambda)) = (1, 2, 4)$. Moreover, by considering the sequence of partial 6-shapes $(s^8(\lambda), \ldots, s^1(\lambda))$, which is in fact (because $\lambda = \varphi(f)$ where f is the surjective pistol of Figure 2.23) the sequence $(s^8(f), \ldots, s^1(f))$ depicted in Figure 2.24, we obtain $(j_2(\lambda), j_3(\lambda), j_1(\lambda)) = (3, 2, 1)$. Applying the algorithm of Definition 2.25 on $L(\lambda) = [8]$, we quickly obtain $L(\lambda) = \{4, 5, 6, 7, 8\}$. Consequently, if $g = \tilde{\varphi}(\lambda)$, then automatically $g(10) = g(9) = 10$, afterwards

$$\begin{aligned} g(1), g(2), g(3)) &= (g(j_1(\lambda)), g(j_3(\lambda), g(j_2(\lambda)) \\ &= (2i_1(\lambda), 2i_3(\lambda), 2i_2(\lambda)) \\ &= (2, 8, 4) \end{aligned}$$

because $j_p(\lambda) \notin L(\lambda)$ for all $p \in [3]$, and $g(j) = 2(\lceil j/2 \rceil + z_j(\lambda))$ for all $j \in L(\lambda)$. Finally, we obtain $g = (2, 8, 4, 10, 10, 6, 8, 10, 10, 10) = f$ (and

$\overrightarrow{\text{fix}}(g) = \overrightarrow{\text{fr}}(\lambda))$.

Proof of Proposition 2.26. Let $\lambda \in \text{IS}_k$ and $f = \tilde{\varphi}(\lambda)$. We know that $f(2k-2) = f(2k-3) = 2k-4$. Consider $j \in [2k-4]$.

1. If $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$ and if $j \notin L(\lambda)$, then $f(j) = 2i_p(\lambda)$. By definition $2i_p(\lambda) > j_p(\lambda)$, so $2k-2 \geq f(j) > j$.
2. Else $f(j) = 2(\lceil j/2 \rceil + z_j(\lambda))$, so $2k-2 \geq f(j) \geq j$ following Lemma 2.22.

Consequently f is a map $[2k-2] \to \{2, 4, \ldots, 2k-2\}$ such that $f(j) \geq j$ for all $j \in [2k-2]$. Now, we prove that f is surjective. We know that $2k-2 = f(2k-2)$. Let $i \in [k-2]$.

- If $i = i_p(\lambda)$ for some $p \in [m(\lambda)]$, then either $j_p(\lambda) \notin L(\lambda)$, in which case $2i = f(j_p(\lambda))$, or there exists $j < j_p(\lambda)$ in $L(\lambda)$ such that $\lceil j/2 \rceil + z_j = i$, in which case $2i = f(j)$.
- Else $z_{2i}(\lambda) = 0$ and $s^{2i}(\lambda) = s^{2i+1}(\lambda) \oplus_1^k (i+1)^{z_j(\lambda)} = s^{2i+1}(\lambda)$, which implies that $2i$ cannot equal any $j_p(\lambda)$. Consequently $2i \in L(\lambda)$, thence $f(2i) = 2(i + z_{2i}(\lambda)) = 2i$.

Therefore $f \in \text{SP}_{k-1}$. Finally, for all $i \in [k-2]$, we have just proved that $z_{2i}(\lambda) = 0$ implies $f(2i) = 2i$. Reciprocally, if $f(2i) = 2i$, then necessarily $2i \in L(\lambda)$ (otherwise $2i$ would be $j_p(\lambda)$ for some p and $f(2i)$ would be $2i_p(\lambda) > j_p(\lambda) = 2i$), meaning $2i = f(2i) = 2(i + z_{2i}(\lambda))$ thence $z_{2i}(\lambda) = 0$. The equivalence $z_{2i}(\lambda) = 0 \Leftrightarrow f(2i) = 2i$ for all $i \in [k-2]$ exactly translates into $\overrightarrow{\text{fr}}(\lambda) = \overrightarrow{\text{fix}}(f)$. □

2.5.3 Proof of Theorem 2.10

At this stage, we know that φ is a map $\text{SP}_{k-1} \to \text{IS}_k$ which carries the statistic $\overrightarrow{\text{fix}}$ to the statistic $\overrightarrow{\text{fr}}$. The bijectivity of φ is a consequence of the following proposition.

Proposition 2.27. *The maps* $\varphi : SP_{k-1} \to IS_k$ *and* $\tilde{\varphi} : IS_k \to SP_{k-1}$ *are inverse maps.*

Lemma 2.28. *Let* $(f, \lambda) \in SP_{k-1} \times IS_k$ *such that* $\lambda = \varphi(f)$ *or* $f = \tilde{\varphi}(\lambda)$. *Let* $p \in [m(\lambda)]$ *and* $j^p(\lambda) := \min\{j \in [2k-4] : f(j) = 2i_p(\lambda)\}$. *The two following assertions are equivalent.*

1. $j_p(\lambda) \notin L(\lambda)$.
2. $j_p(\lambda) = j^p(\lambda)$.

Proof. Let $f \in \mathrm{SP}_{k-1}$ and $\lambda = \varphi(f)$. In particular, we have $s^j(\lambda) = s^j(f)$ and $z_j(\lambda) = z_j(f)$ for all $j \in [2k-4]$. For all $p \in [m(\lambda)]$, by Definition 2.16 the partial k-shape $s^{j^p(\lambda)}(f) = s^{j^p(\lambda)}(\lambda)$ is necessarily saturated in $i_p(\lambda)$, thence $j_p(\lambda) \geq j^p(\lambda)$.

1. If $j_p(\lambda) \notin L(\lambda)$, suppose that $j_p(\lambda) > j^p(\lambda)$. Then, the partial k-shape $s^{j^p(\lambda)+1}(f) = s^{j^p(\lambda)+1}(\lambda)$ is saturated in $i_p(\lambda)$, meaning the integer $z_{j^p(\lambda)}(f) = z_{j^p(\lambda)}(\lambda)$ is defined as $f(j^p(\lambda))/2 - \lceil j^p(\lambda)/2 \rceil = i_p(\lambda) - \lceil j^p(\lambda)/2 \rceil$. Consequently, since $j_p(\lambda) \notin L(\lambda)$ and $j^p(\lambda) < j_p(\lambda)$, the integer $j^p(\lambda)$ cannot belong to $L(\lambda)$ either. So $j^p(\lambda) = j_{p_1}(\lambda)$ for some $p_1 \neq p$ because $j_p(\lambda) \neq j^p(\lambda)$. Also, since $f(j_{p_1}(\lambda)) = 2i_p(\lambda) \neq 2i_{p_1(\lambda)}$, then $j_{p_1}(\lambda) > j^{p_1}(\lambda)$ (and $j_{p_1}(\lambda) = j^p(\lambda) \notin L(\lambda)$). By iterating, we build an infinite decreasing sequence $(j^{p_n}(\lambda))_{n\geq 1}$ of distinct elements of $[2k-4]$, which is absurd. Therefore, it is necessary that $j_p(\lambda) = j^p(\lambda)$.
2. Reciprocally, if $j_p(\lambda) = j^p(\lambda)$, suppose that $j_p(\lambda) \in L(\lambda)$. Then, there exists $j \in L(\lambda)$ such that $j < j_p(\lambda)$ and $z_j(\lambda) = i_p(\lambda) - \lceil j/2 \rceil$. Let $i \in [k-1]$ such that $f(j) = 2i$ (since $j < j_p(\lambda) = j^p(\lambda)$, we know that $i \neq i_p(\lambda)$). Suppose $s^j(f)$ is defined by the Rule 2 of Definition 2.16. In particular $i = i_{p_1}(\lambda)$ for some $p_1 \in [m(\lambda)]$ (because $f(2i) > 2i$ which implies $z_{2i}(\lambda) = z_{2i}(f) > 0$), and $j = j^{p_1}(\lambda)$ and $s^{j+1}(f)$ is not saturated in i. Then, by definition the partial k-shape $s^j(f)$ is the first partial k-shape to be saturated in $i_{p_1}(\lambda)$ in the sequence $(s^{2k-4}(f) = s^{2k-4}(\lambda), \ldots, s^1(f) = s^1(\lambda))$, meaning $j = j_{p_1}(\lambda)$. To sum up, the integer $j = j^{p_1}(\lambda) = j_{p_1}(\lambda)$ doesn't belong to $L(\lambda)$, and $p_1 \neq p$ because $f(j) = 2i_{p_1}(\lambda)$ and $j < j^p(\lambda) = j_p(\lambda)$. By iterating, we build an infinite decreasing sequence $(j^{p_n}(\lambda))_{n\geq 1}$ of elements of $[2k-4]$, which is absurd. So $s^{j+1}(f)$ is necessarily defined by the Rule 1 of Definition 2.16, meaning $z_j(f) = f(j)/2 - \lceil j/2 \rceil$). Since $z_j(f) = z_j(\lambda) = i_p(\lambda) - \lceil j/2 \rceil$, we obtain $f(j) = 2i_p(\lambda)$, which is in contradiction with $j_p(\lambda) = j^p(\lambda) > j$. As a conclusion, it is necessary that $j_p(\lambda) \notin L(\lambda)$.

Now let $\lambda \in \mathrm{IS}_k$ and $f = \tilde{\varphi}(\lambda)$. We consider $p \in [m(\lambda)]$.

1. If $j_p(\lambda) \notin L(\lambda)$, suppose that $j_p(\lambda) \neq j^p(\lambda)$. Then, by definition $f(j_p(\lambda)) = 2i_p(\lambda)$, meaning $j_p(\lambda) > j^p(\lambda)$. Suppose now that $j^p(\lambda) \in L(\lambda)$, then $2i_p(\lambda) = f(j^p(\lambda)) = 2(\lceil j^p(\lambda)/2 \rceil + z_{j^p(\lambda)}(\lambda))$. As a result, we obtain $z_{j^p(\lambda)}(\lambda) = i_p(\lambda) - \lceil j^p(\lambda)/2 \rceil$, which is in contradiction with $j_p(\lambda) \notin L(\lambda)$. So $j^p(\lambda) \notin L(\lambda)$, which implies $j^p(\lambda) = j_{p_1}(\lambda)$ for some $p_1 \neq p$, and necessarily $j_{p_1}(\lambda) \neq j^{p_1}(\lambda)$ since $f(j_{p_1}(\lambda)) = 2i_p(\lambda) \neq$

$2i_{p_1}(\lambda)$. By iterating, we build a sequence $(j^{p_n}(\lambda))_{n\geq 1}$ of distinct elements of $[2k-4]$, which is absurd. So $j_p(\lambda) = j^p(\lambda)$.

2. Reciprocally, if $j_p(\lambda) = j^p(\lambda)$, suppose that $j_p(\lambda) \in L(\lambda)$. Then, there exists $j \in L(\lambda)$ such that $j < j_p(\lambda)$ and $z_j(\lambda) = i_p(\lambda) - \lceil j/2 \rceil$. Let $i \in [2k-4]$ such that $f(j) = 2i$. Because $j^p(\lambda) > j$, we have $i \neq i_p(\lambda)$. And since $j \in L(\lambda)$, we obtain $2i = f(j) = 2(\lceil j/2 \rceil + z_j(\lambda)) = 2i_p(\lambda)$, which is absurd. So $j_p(\lambda) \notin L(\lambda)$. □

Proof of Proposition 2.27. Let $f \in \mathrm{SP}_{k-1}$ and $\lambda = \varphi(f)$ and $g = \tilde{\varphi}(\lambda)$. Let $j \in [2k-4]$ and $i \in [k-1]$ such that $f(j) = 2i$.

1. If $s^j(f)$ is defined by the Rule 2 of Definition 2.16, then there exists $p \in [m(\lambda)]$ such that $i = i_p(\lambda)$ and $j = j^p(\lambda) = j_p(\lambda)$. Consequently, in view of Lemma 2.28 with $\lambda = \varphi(f)$, we know that $j \notin L(\lambda)$, implying $g(j) = g(j_p(\lambda)) = 2i_p(\lambda) = 2i = f(j)$.
2. If $s^j(f)$ is defined by the Rule 1 of Definition 2.16, then $z_j(f) = f(j)/2 - \lceil j/2 \rceil = i - \lceil j/2 \rceil$. Now it is necessary that $j \in L(\lambda)$: otherwise $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$, and from Lemma 2.28 we would have $j = j_p(\lambda) = j^p(\lambda)$, which is impossible because we are by the Rule 1 of Definition 2.16. So $j \in L(\lambda)$, implying $g(j) = 2(\lceil j/2 \rceil + z_j(\lambda)) = 2(\lceil j/2 \rceil + z_j(f)) = 2i = f(j)$.

As a conclusion, we obtain $g = f$ so $\tilde{\varphi} \circ \varphi$ is the identity map of SP_{k-1}. Reciprocally, let $\mu \in \mathrm{IS}_k$ and $h = \tilde{\varphi}(\mu)$. We are going to prove by induction that $s^j(\mu) = s^j(h)$ for all $j \in [2k-3]$. By definition $s^{2k-3}(\mu) = s^{2k-3}(h) = \varnothing$. Suppose that $s^{j+1}(\mu) = s^{j+1}(h)$ for some $j \in [2k-4]$.

1. If $s^j(h)$ is defined by the Rule 2 of Definition 2.16, then there exists $p \in [m(\lambda)]$ such that $h(j) = 2i_p(\mu)$, such that $j = j^p(\mu)$ and such that $s^{j+1}(h)$ is not saturated in $i_p(\mu)$. Since the partial k-shape $s^{j+1}(\mu) = s^{j+1}(h)$ is not saturated in $i_p(\mu)$, by definition $j \geq j_p(\mu)$. Suppose that $j > j_p(\mu)$. Since $j = j^p(\mu)$, we know from Lemma 2.28 (with $\lambda = \mu$ and $f = \tilde{\varphi}(\lambda) = h$) that $j_p(\mu) \in L(\mu)$. It means there exists $j' < j_p(\mu) < j$ such that $j' \in L(\mu)$ and $\lceil j'/2 \rceil + z_{j'}(\mu) = i_p(\mu)$, implying $h(j') = 2i_p(\mu) = h(j)$, which contradicts $j = j^p(\mu)$. So $j = j_p(\mu)$, therefore $s^j(\mu)$ is saturated in $i_p(\mu)$. But since we are by the Rule 2 of Definition 2.16, the partial k-shape $s^j(h)$ is defined as $s^{j+1}(h) \oplus^k_{t(j)} (\lceil (j+1)/2 \rceil^{z_j(h)}$ where $z_j(h)$ is the unique integer $z \in [k-1-\lceil j/2 \rceil]$ such that $s^{j+1}(h) \oplus^k_{t(j)} (\lceil (j+1)/2 \rceil^z$ is saturated in $i_p(\mu)$. Since the partial k-shape

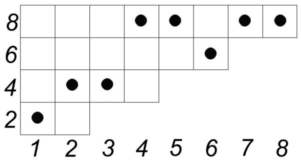


Figure 2.31: Tableau of $f = (2,4,4,8,8,6,8,8) \in \mathrm{SP}_4$.

$s^j(\mu) = s^{j+1}(\mu) \oplus^k_{t(j)} (\lceil (j+1)/2\rceil^{z_j(\mu)} = s^{j+1}(h) \oplus^k_{t(j)} (\lceil (j+1)/2\rceil^{z_j(\mu)}$ is saturated in $i_p(\mu)$, we obtain $z_j(\mu) = z_j(h)$ and $s^j(\mu) = s^j(h)$.

2. If $s^j(h)$ is defined by the Rule 1 of Definition 2.16, then $s^j(h) = s^{j+1}(\mu) \oplus^k_{t(j)} \lceil (j+1)/2\rceil^{z_j(h)}$ with $z_j(h) = h(j)/2 - \lceil j/2\rceil$. Now either $h(j) = 2(\lceil j/2\rceil + z_j(\mu))$, in which case we obtain $z_j(h) = z_j(\mu)$, or $h(j) = 2i_p(\mu)$ for some $p \in [m(\mu)]$ such that $j = j_p(\mu) \notin L(\mu))$. In view of Lemma 2.28, it means $j = j^p(\mu)$, which cannot happen because otherwise we would be by the Rule 2 of Definition 2.16. So $z_j(h) = z_j(\mu)$ and $s^j(h) = s^j(\mu)$.

By induction, we obtain $s^1(\mu) = s^1(h)$, thence $\mu = \varphi(h)$. Consequently, the map $\varphi \circ \tilde{\varphi}$ is the identity map of IS_k. □

2.6 Extensions

Dumont and Foata [DF76] introduced a refinement of the Gandhi polynomials $(Q_k(x))_{k\geq 1}$ through the polynomial sequence $(F_k(x,y,z))_{k\geq 1}$ defined by $F_1(x,y,z) = 1$ and

$$F_{k+1}(x,y,z) = (x+y)(x+z)F_k(x+1,y,z) - x^2 F_k(x,y,z).$$

Note that $F_k(x,1,1) = Q_k(x)$ for all $k \geq 1$ in view of Definition 0.1. Now, for all $k \geq 2$ and $f \in \mathrm{SP}_k$, let $\max(f)$ be the number of *maximal* points of f (integers $j \in [2k-2]$ such that $f(j) = 2k$) and $\mathrm{pro}(f)$ the number of *prominent* points (integers $j \in [2k-2]$ such that $f(i) < f(j)$ for all $i \in [j-1]$). For example, if f is the surjective pistol $(2,4,4,8,8,6,8,8) \in \mathrm{SP}_4$ depicted in Figure 2.31, then the maximal points of f are $\{4,5\}$, and its prominent points are $\{2,4\}$. Dumont and Foata gave a combinatorial interpretation of $F_k(x,y,z)$ in terms of surjective pistols.

Theorem 2.29 ([DF76]). *For all $k \geq 2$, the Dumont-Foata polynomial $F_k(x,y,z)$ is symmetrical and generated by SP_k:*

$$F_k(x,y,z) = \sum_{f \in SP_k} x^{\max(f)} y^{fix(f)} z^{pro(f)}.$$

In 1996, Han [Han96] gave another interpretation by introducing the statistic $\operatorname{sur}(f)$ defined as the number of *surfixed* points of $f \in \mathrm{SP}_k$ (integers $j \in [2k-2]$ such that $f(j) = j+1$; for example, the surfixed points of the surjective pistol $f \in \mathrm{SP}_4$ of Figure 2.31 are $\{1,3\}$).

Theorem 2.30 (Han [Han96]). *For all $k \geq 2$, the Dumont-Foata polynomial $F_k(x,y,z)$ has the following combinatorial interpretation:*

$$F_k(x,y,z) = \sum_{f \in SP_k} x^{\max(f)} y^{fix(f)} z^{sur(f)}.$$

Theorem 0.4 then appears as a particular case of Theorem 2.29 or Theorem 2.30 by setting $x = z = 1$ (and by applying the symmetry of $F_k(x,y,z)$). It has been proved by Gessel and Zeng that the Dumont-Foata polynomials are the moments of orthogonal polynomials named *continuous dual Hahn polynomials.*

Furthermore, for all $f \in \mathrm{SP}_k$ and $j \in [2k-2]$, we say that j is a *lined* point of f if there exists $j' \in [2k-2]\backslash\{j\}$ such that $f(j) = f(j')$. We define $\operatorname{mo}(f)$ (resp. $\operatorname{me}(f)$) as the number of odd (resp. even) maximal points of f, and $\operatorname{fl}(f)$ (resp. $\operatorname{fnl}(f)$) as the number of lined (resp. non lined) fixed points of f, and $\operatorname{sl}(f)$ (resp. $\operatorname{snl}(f)$) as the number of lined (resp. non lined) surfixed points of f. Dumont [Dum95] defined generalized Dumont-Foata polynomials $(\Gamma_k(x,y,z,\bar{x},\bar{y},\bar{z}))_{k\geq 1}$ by

$$\Gamma_k(x,y,z,\bar{x},\bar{y},\bar{z}) = \sum_{f \in \mathrm{SP}_k} x^{\mathrm{mo}(f)} y^{\mathrm{fl}(f)} z^{\mathrm{snl}(f)} \bar{x}^{\mathrm{me}(f)} \bar{y}^{\mathrm{fnl}(f)} \bar{z}^{\mathrm{sl}(f)}.$$

This is a refinement of Dumont-Foata polynomials in view of the equality $\Gamma_k(x,y,z,x,y,z) = F_k(x,y,z)$. Dumont conjectured the following induction formula:
$\Gamma_1(x,y,z,\bar{x},\bar{y},\bar{z}) = 1$ and

$$\begin{aligned}\Gamma_{k+1}(x,y,z,\bar{x},\bar{y},\bar{z}) &= (x+\bar{z})(y+\bar{x})\Gamma_k(x+1,y,z,\bar{x}+1,\bar{y},\bar{z}) \\ &\quad + (x(\bar{y}-y) + \bar{x}(z-\bar{z}) - x\bar{x})\Gamma_k(x,y,z,\bar{x},\bar{y},\bar{z}). \quad (2.4)\end{aligned}$$

This was proven independently by Randrianarivony [Ran94] and Zeng [Zen96]. See also [JV11] for a new combinatorial interpretation of $\Gamma_k(x, y, z, \bar{x}, \bar{y}, \bar{z})$.

Now, let $f \in \mathrm{SP}_{k-1}$ and $\lambda = \varphi(f) \in \mathrm{IS}_k$. For all $j \in [2k-4]$, we say that j is a *chained* k-site of λ if $j \notin L(\lambda)$. Else, we say that it is an *unchained* k-site. In view of Lemma 2.28, an integer $j \in [2k-4]$ is a chained k-site if and only if $j = j_p(\lambda) = j^p(\lambda)$ for some $p \in [m(\lambda)]$, in which case $f(j) = 2i_p(\lambda)$ (the integer j is forced to be mapped to $2i_p(\lambda)$). If j is an unchained k-site, by definition $f(j) = 2(\lceil j/2 \rceil + z_j(\lambda))$. Consequently, every statistic of Theorems 2.29, 2.30 and Formula 2.4 has its own equivalent among irreducible k-shapes. However, the objects counted by these statistics are not always easily pictured or formalized. We only give the irreducible k-shapes version of Theorem 2.30.

Recall that for all $i \in [k-2]$, the integer $2i$ is a fixed point of f if and only if $2i$ is a free k-site of λ, which is also equivalent to $z_{2i}(\lambda) = 0$. We extend the notion of free k-site to any $j \in [2k-4]$: the integer j is said to be a free k-site if $z_j(\lambda) = 0$. Note that the free k-sites of λ are necessarily unchained because $z_j(\lambda) = 0$ implies $s^j(\lambda) = s^{j+1}(\lambda)$ thence $j \neq j_p(\lambda)$ for all $p \in [m(\lambda)]$. We denote by $\mathrm{fro}(\lambda)$ the quantity of odd free sites of λ. We denote by $\mathrm{ful}(\lambda)$ the quantity of *full* k-sites of λ (namely, unchained k-sites $j \in L(\lambda)$ such that $z_j(\lambda) = k - 1 - \lceil j/2 \rceil$), and by $\mathrm{sch}(\lambda)$ the quantity of *surchained* k-sites (chained k-sites $j \in [2k-4]$ such that $j = j_p(\lambda)$ for some $p \in [m(\lambda)]$ such that $2i_p(\lambda) = j+1$). Theorem 2.30 can now be reformulated as follows.

Theorem 2.31. *For all $k \geq 3$, the Dumont-Foata polynomial $F_{k-1}(x, y, z)$ has the following combinatorial interpretation:*

$$F_{k-1}(x, y, z) = \sum_{\lambda \in IS_k} x^{ful(\lambda)} y^{fr(\lambda)} z^{fro(\lambda)+sch(\lambda)}.$$

Proof. First of all, maximal points of $f \in \mathrm{SP}_{k-1}$ are full k-sites of $\lambda = \varphi(f) \in \mathrm{IS}_k$: if $f(j) = 2k-2$ then $z_j(f)$ is necessarily defined by the Rule 1 of Definition 2.16, thence $z_j(\lambda) = z_j(f) = f(j)/2 - \lceil j/2 \rceil = k - 1 - \lceil j/2 \rceil$, and $j \in L(\lambda)$ because otherwise $f(j)$ would equal $2i_p(\lambda) < 2k-2 = f(j)$ for some $p \in [m(\lambda)]$. So j is a full k-site of λ. Reciprocally, if $j \in L(\lambda)$ is such that $z_j(\lambda) = k - 1 - \lceil j/2 \rceil$, then $f(j) = 2(\lceil j/2 \rceil + z_j(\lambda)) = 2k-2$ so j is a maximal point of f. Afterwards, the set of surfixed points of f is the union set of the odd free k-sites and surchained k-sites of λ: if $f(j) = j+1$, then $j = 2i - 1$ for some $i \in [k-1]$ and either $j \in L(\lambda)$, in which case $f(j) = 2(i + z_j(\lambda)) = 2i$ hence $z_j(\lambda) = 0$ and j is an odd free k-site, or $j = j_p(\lambda) = j^p(\lambda)$ for some $p \in [m(\lambda)]$ such that $2i_p(\lambda) = 2i = j+1$, *i.e.*, the integer j is a surchained k-site. Reciprocally, if j is an odd free k-site then $f(j) = 2(\lceil j/2 \rceil + z_j(\lambda)) = 2(\lceil j/2 \rceil) = j+1$, and if j is a surchained k-site then in particular $f(j) = 2i_p(\lambda) = j+1$ for some $p \in [m(\lambda)]$. As a conclusion, the result comes from Theorem 2.10. $\square$

As for now, constructing a combinatorial interpretation of the Dumont-Foata polynomials that makes their symmetry obvious is still an open problem.

Chapter 3

A new bijection relating q-Eulerian polynomials

3.1 Abstract

On $\mathfrak{S}_n$, we construct a bijection which maps the 3-vector of statistics $(\mathrm{maj} - \mathrm{exc}, \mathrm{des}, \mathrm{exc})$ to a 3-vector $(\widetilde{\mathrm{maj}_2, \mathrm{des}_2}, \mathrm{inv}_2)$ associated with the q-Eulerian polynomials introduced by Shareshian and Wachs in *Chromatic quasisymmetric functions, arXiv:1405.4269(2014) (to appear in Advances in Math).*

3.2 Introduction

This chapter faithfully follows [Big15b].

We first give a reminder about the combinatorial class of permutations. Recall that for all permutation $\sigma \in \mathfrak{S}_n$, the numbers $\mathrm{des}(\sigma)$ and $\mathrm{exc}(\sigma)$ count the number of descents and excedances of σ respectively, *i.e.*, the elements of the respective sets

$$\begin{aligned} \mathrm{DES}(\sigma) &= \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}, \\ \mathrm{EXC}(\sigma) &= \{i \in [n-1] : \sigma(i) > i\}. \end{aligned}$$

It is due to MacMahon [Mac15] and Riordan [RS73] that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} = A_n(t),$$

where $A_n(t)$ is the n-th Eulerian polynomial [Eul55], which may be defined by the generating function

$$\sum_{n\geq 0} A_n(t)\frac{x^n}{n!} = \frac{t-1}{t-exp((t-1)x)} \tag{3.1}$$

(see [Sta11, page 34]). A statistic equidistributed with des or exc is said to be *Eulerian*. The statistic ides defined by $\mathrm{ides}(\sigma) = \mathrm{des}(\sigma^{-1})$ obviously is Eulerian. It is also easy to see that the number $\mathrm{asc}(\sigma)$, which counts the number of ascents of $\sigma \in \mathfrak{S}_n$, *i.e.*, the elements of the set

$$\mathrm{ASC}(\sigma) = \{i \in [n-1] : \sigma(i) < \sigma(i+1)\},$$

defines an Eulerian statistic.

Afterwards, recall that the number $\mathrm{inv}(\sigma)$ counts the number of inversions of σ, *i.e.*, the elements of the set

$$\mathrm{INV}(\sigma) = \{(i,j) \in [n]^2 : i < j, \sigma(i) > \sigma(j)\},$$

and that the q-factorial $[n]_q! = \prod_{i=1}^{n} \sum_{k=0}^{i-1} q^k$ is generated by $\mathfrak{S}_n$ with respect to the statistic inv, *i.e.*, that

$$[n]_q! = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}$$

(see for example [Mac15]). MacMahon [Mac15] also defined the major index maj defined by

$$\mathrm{maj}(\sigma) = \sum_{i\in\mathrm{DES}(\sigma)} i$$

for all $\sigma \in \mathfrak{S}_n$, and proved that it is equidistributed with inv. In general, any statistic equidistribued with inv is said to be *Mahonian*.

In this chapter, we study the combinatorics of the permutations with respect to statistics which refine those defined above : for all $\sigma \in \mathfrak{S}_n$, the numbers $\mathrm{des}_2(\sigma)$, $\mathrm{asc}_2(\sigma)$ and $\mathrm{inv}_2(\sigma)$ count the *2-descents*, *2-ascents* and *2-inversions* of σ respectively, *i.e.*, the elements of the respective sets

$$\begin{aligned}\mathrm{DES}_2(\sigma) &= \{i \in [n-1] : \sigma(i) > \sigma(i+1)+1\},\\ \mathrm{ASC}_2(\sigma) &= \{i \in [n-1] : \sigma(i) < \sigma(i+1)+1\},\\ \mathrm{INV}_2(\sigma) &= \{(i,j) \in [n]^2 : i < j, \sigma(i) = \sigma(j)+1\}.\end{aligned}$$

The *2-major index* maj_2 is then defined by

$$\mathrm{maj}_2(\sigma) = \sum_{i \in \mathrm{DES}_2(\sigma)} i$$

for all $\sigma \in \mathfrak{S}_n$.

By using quasisymmetric function techniques, Shareshian and Wachs [SW10, SW14] and Hance and Li [HL12] proved the equalities

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{exc}(\sigma)}. \tag{3.2}$$

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{amaj}_2(\sigma)} y^{\widetilde{\mathrm{asc}_2}(\sigma)} z^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{exc}(\sigma)} \tag{3.3}$$

respectively, where, similarly as for maj_2, the statistic amaj_2 is defined by

$$\mathrm{amaj}_2(\sigma) = \sum_{i \in \mathrm{ASC}_2(\sigma)} i,$$

and where

$$\widetilde{\mathrm{asc}_2}(\sigma) = \begin{cases} \mathrm{asc}_2(\sigma) & \text{if } \sigma(1) = 1, \\ \mathrm{asc}_2(\sigma) + 1 & \text{if } \sigma(1) \neq 1. \end{cases}$$

Definition 3.1. Let $\sigma \in \mathfrak{S}_n$ and $(p, q) \in [n]^2$ such that $p < q$. A sequence of 2-inversions from p to q is a finite sequence

$$((i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)) \in (\mathrm{INV}_2(\sigma))^k$$

(for some $k \geq 1$) such that :

- $i_1 = p$;
- $j_l = i_{l+1}$ for all $l \in [k-1]$;
- $j_k = q$.

The *length* of such a sequence is defined as k.

Definition 3.2. Let $\sigma \in \mathfrak{S}_n$. We consider the smallest 2-descent $d_2(\sigma)$ of σ such that $\sigma(i) = i$ for all $i \in [d_2(\sigma) - 1]$ (if there is no such 2-descent, we define $d_2(\sigma)$ as 0 and $\sigma(0)$ as $+\infty$).

Now, let $d_2'(\sigma) > d_2(\sigma)$ be the smallest 2-descent of σ greater than $d_2(\sigma)$ (if there is no such 2-descent, we define $d_2'(\sigma)$ as n).

We define an inductive property $\mathcal{P}(d_2(\sigma))$ by :

1. for all $(i,j) \in \mathrm{INV}_2(\sigma)$ such that $d_2(\sigma) < i < d'_2(\sigma)$ and such that there is no sequence of 2-inversions from $d_2(\sigma)$ to i, we have $\sigma(d_2(\sigma)) < \sigma(i)$;
2. if $(d'_2(\sigma), j) \in \mathrm{INV}_2(\sigma)$ for some j, then either :
   – $\sigma(d_2(\sigma)) < \sigma(d'_2(\sigma))$,
   – or there exists a sequence of 2-inversions of σ from $d_2(\sigma)$ to $d'_2(\sigma)$,
   – or $d'_2(\sigma)$ has the property $\mathcal{P}(d'_2(\sigma))$ (where the role of $d_2(\sigma)$ is played by $d'_2(\sigma)$ and that of $d'_2(\sigma)$ by $d''_2(\sigma)$ where $d''_2(\sigma) > d'_2(\sigma)$ is the smallest 2-descent of σ greater than $d'_2(\sigma)$, defined as n if there is no such 2-descent).

This property is well-defined because $(n,j) \notin \mathrm{INV}_2(\sigma)$ for all $j \in [n]$.

Finally, we define a statistic $\widetilde{\mathrm{des}_2}$ by :

$$\widetilde{\mathrm{des}_2}(\sigma) = \begin{cases} \mathrm{des}_2(\sigma) & \text{if the property } \mathcal{P}(d_2(\sigma)) \text{ is true,} \\ \mathrm{des}_2(\sigma) + 1 & \text{otherwise.} \end{cases}$$

For example, consider the permutation $\sigma = 7153426 \in \mathfrak{S}_7$, whose set of 2-descents is $\mathrm{DES}_2(\sigma) = \{1,3,5\}$, and whose set of 2-inversions is $\mathrm{INV}_2(\sigma) = \{(1,7),(3,5),(4,6)\}$. We have $d_2(\sigma) = 1$ and $d'_2(\sigma) = 3$. The requirement (1) of Definition 3.2 is fulfiled because there is no beginning of a 2-inversion of σ between $d_2(\sigma) = 1$ and $d'_2(\sigma) = 3$. However, the requirement (2) is not fulfiled because $\sigma(d_2(\sigma)) > \sigma(d'_2(\sigma))$, and there is no sequence of 2-inversions of σ from $d_2(\sigma)$ to $d'_2(\sigma)$, and $d'_2(\sigma)$ doesn't have the property $\mathcal{P}(d'_2(\sigma))$: the requirement (1) of $\mathcal{P}(d'_2(\sigma))$ is not fulfiled because $d'_2(\sigma) < 4 < d''_2(\sigma) = 5$ and 4 is the beginning of the 2-inversion $(4,6) \in \mathrm{INV}_2(\sigma)$ such that there is no sequence of 2-inversions from $d'_2(\sigma)$ to 4, and $\sigma(d'_2(\sigma)) > \sigma(4)$. So $\widetilde{\mathrm{des}_2}(\sigma) = \mathrm{des}_2(\sigma) + 1 = 4$.

In the present chapter, we prove the following theorem.

Theorem 3.3. *There exists a bijection* $\Psi : \mathfrak{S}_n \to \mathfrak{S}_n$ *such that*

$$(maj_2(\sigma), \widetilde{des_2}(\sigma), inv_2(\sigma)) = (maj(\Psi(\sigma)) - exc(\Psi(\sigma)), des(\Psi(\sigma)), exc(\Psi(\sigma))).$$

As a straight corollary of Theorem 3.3, we obtain the equality

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}_2(\sigma)} y^{\widetilde{\mathrm{des}_2}(\sigma)} z^{\mathrm{inv}_2(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} y^{\mathrm{des}(\sigma)} z^{\mathrm{exc}(\sigma)} \tag{3.4}$$

which implies Equality (3.2).

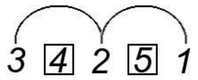


Figure 3.1: Linear graph of $\sigma = 34251 \in \mathfrak{S}_5$.

The rest of this chapter is organised as follows.

In Section 3.3, we introduce two graphical representations of a given permutation so as to highlight either the statistic $(\mathrm{maj} - \mathrm{exc}, \mathrm{des}, \mathrm{exc})$ or $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$. Practically speaking, the bijection Ψ of Theorem 3.3 will be defined by constructing one of the two graphical representations of $\Psi(\sigma)$ for a given permutation $\sigma \in \mathfrak{S}_n$.

We define Ψ in Section 3.4.

In Section 3.5, we prove that Ψ is bijective by constructing Ψ^{-1}.

In Section 3.6, we show how Theorem 3.3 also proves a quasisymmetric function generalization of (3.2).

In Section 3.7, we discuss the potential use of Theorem 3.3 to prove combinatorially Equality (3.3).

3.3 Graphical representations

3.3.1 Linear graph

Let $\sigma \in \mathfrak{S}_n$. The linear graph of σ is a graph whose vertices are (from left to right) the integers $\sigma(1), \sigma(2), \ldots, \sigma(n)$ aligned in a row, where every $\sigma(k)$ (for $k \in \mathrm{DES}_2(\sigma)$) is boxed, and where an arc of circle is drawn from $\sigma(i)$ to $\sigma(j)$ for every $(i, j) \in \mathrm{INV}_2(\sigma)$.

For example, the permutation $\sigma = 34251 \in \mathfrak{S}_5$ (such that $(\mathrm{maj}_2(\sigma), \widetilde{\mathrm{des}_2}(\sigma), \mathrm{inv}_2(\sigma)) = (6, 3, 2)$) has the linear graph depicted in Figure 3.1.

3.3.2 Planar graph

Let $\tau \in \mathfrak{S}_n$. The planar graph of τ is a graph whose vertices are the integers $1, 2, ..., n$, organized in ascending and descending slopes (the height of each vertex doesn't matter) such that the i-th vertex (from left to right) is the integer $\tau(i)$, and where every vertex $\tau(i)$ with $i \in \mathrm{EXC}(\tau)$ is encircled.

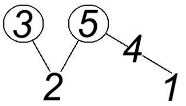


Figure 3.2: Planar graph of $\tau = 32541 \in \mathfrak{S}_5$.

For example, the permutation $\tau = 32541 \in \mathfrak{S}_5$ (such that $(\mathrm{maj}(\tau) - \mathrm{exc}(\tau), \mathrm{des}(\tau), \mathrm{exc}(\tau)) = (6, 3, 2)$) has the planar graph depicted in Figure 3.2.

3.4 Definition of the map Ψ of Theorem 3.3

Let $\sigma \in \mathfrak{S}_n$. We set $(r, s) = (\mathrm{des}_2(\sigma), \mathrm{inv}_2(\sigma))$, and

$$\mathrm{DES}_2(\sigma) = \left\{d_2^k(\sigma) : k \in [r]\right\},$$
$$\mathrm{INV}_2(\sigma) = \{(i_l(\sigma), j_l(\sigma)) : l \in [s]\}$$

with $d_2^k(\sigma) < d_2^{k+1}(\sigma)$ for all k and $i_l(\sigma) < i_{l+1}(\sigma)$ for all l.

We intend to define $\Psi(\sigma)$ by constructing its planar graph. To do so, we first construct (in Subsection 3.4.1) a graph $\mathcal{G}(\sigma)$ made of n circles or dots organized in ascending or descending slopes such that two consecutive vertices are necessarily in a same descending slope if the first vertex is a circle and the second vertex is a dot. Then, in Subsection 3.4.2, we label the vertices of this graph with the integers $1, 2, \ldots, n$ in such a way that, if y_i is the label of the i-th vertex $v_i(\sigma)$ (from left to right) of $\mathcal{G}(\sigma)$ for all $i \in [n]$, then :

1. $y_i < y_{i+1}$ if and only if v_i and v_{i+1} are in a same ascending slope;
2. $y_i > i$ if and only if v_i is a circle.

The permutation $\tau = \Psi(\sigma)$ will then be defined as $y_1 y_2 \ldots y_n$, *i.e.* the permutation whose planar graph is the labelled graph $\mathcal{G}(\sigma)$.

With precision, we will obtain

$$\tau\left(\mathrm{EXC}(\tau)\right) = \{j_k(\sigma) : k \in [s]\}$$

(in particular $\mathrm{exc}(\tau) = s = \mathrm{inv}_2(\sigma)$), and

$$\mathrm{DES}(\tau) = \begin{cases} \{d^k(\sigma) : k \in [1, r]\} \text{ if } \widetilde{\mathrm{des}_2}(\sigma) = r, \\ \{d^k(\sigma) : k \in [0, r]\} \text{ if } \widetilde{\mathrm{des}_2}(\sigma) = r + 1 \end{cases}$$

for integers $0 \leq d^0(\sigma) < d^1(\sigma) < \ldots < d^r(\sigma) \leq n$ (with $d^0(\sigma) = 0 \Leftrightarrow \widetilde{\mathrm{des}_2}(\sigma) = \mathrm{des}_2(\sigma)$) defined by

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

(with $d_2^0(\sigma) := 0$) where $(c_k(\sigma))_{k\in[0,r]}$ is a sequence defined in Subsection 3.4.1 such that $\sum_k c_k(\sigma) = \mathrm{inv}_2(\sigma) = \mathrm{exc}(\tau)$. Thus, we will obtain $\mathrm{des}(\tau) = \widetilde{\mathrm{des}_2}(\sigma)$ and $\mathrm{maj}(\tau) = \mathrm{maj}_2(\sigma) + \mathrm{exc}(\tau)$.

3.4.1 Construction of the unlabelled graph $\mathcal{G}(\sigma)$

We set $(d_2^0(\sigma), \sigma(d_2^0(\sigma))) = (0, n+1)$ and $\left(d_2^{r+1}(\sigma), \sigma(n+1)\right) = (n, 0)$.

For all $k \in [r]$, we define the top $t_k(\sigma)$ of the 2-descent $d_2^k(\sigma)$ as

$$t_k(\sigma) = \min\{d_2^l(\sigma) : 1 \leq l \leq k, d_2^l(\sigma) = d_2^k(\sigma) - (k-l)\}, \qquad (3.5)$$

in other words $t_k(\sigma)$ is the smallest 2-descent $d_2^l(\sigma)$ such that the 2-descents $d_2^l(\sigma), d_2^{l+1}(\sigma), \ldots, d_2^k(\sigma)$ are consecutive integers.

The following algorithm provides a sequence $(c_k^0(\sigma))_{k\in[0,r]}$ of nonnegative integers.

Algorithm 3.4. Let $I_r(\sigma) = \mathrm{INV}_2(\sigma)$. For k from $r = \mathrm{des}_2(\sigma)$ down to 0, we consider the set $S_k(\sigma)$ of sequences $(i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma))$ such that :

1. $(i_{k_p}(\sigma), j_{k_p}(\sigma)) \in I_k(\sigma)$ for all $p \in [m]$;
2. $t_k(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \ldots < i_{k_m}(\sigma)$;
3. $\sigma(i_{k_1}(\sigma)) < \sigma(i_{k_2}(\sigma)) < \ldots < \sigma(i_{k_m}(\sigma))$.

The *length* of such a sequence is defined as $l = \sum_{p=1}^m n_p$ where n_p is the length of the maximal sequence of 2-inversions from i_{k_p} to some integer $j > i_{k_p}$. If $I_k(\sigma) \neq \emptyset$, we consider the sequence $(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \ldots, i_{k_m^{max}}(\sigma)) \in S_k(\sigma)$ whose length $l^{max} = \sum_{p=1}^m n_p^{max}$ is maximal and whose sequence $(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \ldots, i_{k_m^{max}}(\sigma))$ also is maximal with respect to the lexicographic order (defined by $(a_1, a_2, \ldots, a_m) < (b_1, b_2, \ldots, b_m)$ if and only if $a_1 < b_1$ or if there exists $p \in [m-1]$ such that $a_i = b_i$ for all $i \leq p$ and $a_{p+1} < b_{p+1}$). Then,

– if $I_k(\sigma) \neq \emptyset$, we set $c_k^0(\sigma) = l^{max}$ and

$$I_{k-1}(\sigma) = I_k(\sigma) \backslash \left(\cup_{p=1}^m \{(i_{k_i^{max}}(\sigma), j_{k_i^{max}}(\sigma)) : i \in [n_p^{max}]\}\right);$$

- else we set $c^0_k(\sigma) = 0$ and $I_{k-1}(\sigma) = I_k(\sigma)$.

Example 3.5. Consider the permutation $\sigma = 549321867 \in \mathfrak{S}_9$, with $\mathrm{DES}_2(\sigma) = \{3, 7\}$ and $I_2(\sigma) = \mathrm{INV}_2(\sigma) = \{(1, 2), (2, 4), (3, 7), (4, 5), (5, 6), (7, 9)\}$. In Figure 3.3 are depicted the $\mathrm{des}_2(\sigma) + 1 = 3$ steps $k \in \{2, 1, 0\}$ (at each step, the 2-inversions of the maximal sequence are drawed in red then erased at the following step) :

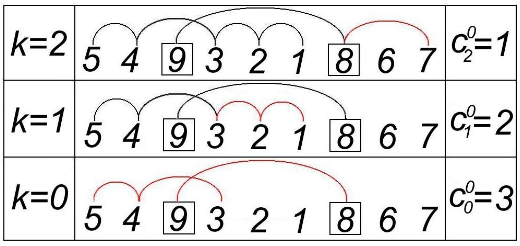


Figure 3.3: Computation of $(c^0_k(\sigma))_{k\in[0,\mathrm{des}_2(\sigma)]}$ for $\sigma = 549321867 \in \mathfrak{S}_9$.

- $k = 2$: there is only one legit sequence $(i_{k_1}(\sigma)) = (7)$, whose length is $l = n_1 = 1$. We set $c^0_2(\sigma) = 1$ and $I_1(\sigma) = I_2(\sigma)\backslash\{(7, 9)\}$.
- $k = 1$: there are three legit sequences $(i_{k_1}(\sigma)) = (3)$ (whose length is $l = n_1 = 1$) then $(i_{k_1}(\sigma)) = (4)$ (whose length is $l = n_1 = 2$) and $(i_{k_1}(\sigma)) = (5)$ (whose length is $l = n_1 = 1$). The maximal sequence is the second one, hence we set $c^0_1(\sigma) = 2$ and $I_0(\sigma) = I_1(\sigma)\backslash\{(4, 5), (5, 6)\}$.
- $k = 0$: there are three legit sequences $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (1, 3)$ (whose length is $l = n_1 + n_2 = 2 + 1 = 3$) then $(i_{k_1}(\sigma), i_{k_2}(\sigma)) = (2, 3)$ (whose length is $l = n_1 + n_2 = 1 + 1 = 2$) and $(i_{k_1}(\sigma)) = (3)$ (whose length is $l = n_1 = 1$). The maximal sequence is the first one, hence we set $c^0_0(\sigma) = 3$ and $I_{-1}(\sigma) = I_0(\sigma)\backslash\{(1, 2), (2, 4), (3, 7)\} = \emptyset$.

Lemma 3.6. *The sum* $\sum_k c^0_k(\sigma)$ *equals* $inv_2(\sigma)$ *(i.e.* $I_{-1}(\sigma) = \emptyset$*) and, for all* $k \in [0, r] = [0, des_2(\sigma)]$*, we have* $c^0_k(\sigma) \leq d^{k+1}_2(\sigma) - d^k_2(\sigma)$ *with equality only if* $c^0_{k+1}(\sigma) > 0$ *(where* $c^0_{r+1}(\sigma)$ *is defined as* 0*).*

Proof. With precision, we show by induction that, for all $k \in \{\mathrm{des}_2(\sigma), \ldots, 1, 0\}$, the set $I_{k-1}(\sigma)$ contains no 2-inversion (i, j) such that $d^k_2(\sigma) < i$. For $k = 0$, it will mean $I_{-1}(\sigma) = \emptyset$ (recall that $d^0_2(\sigma)$ has been defined as 0).

$\star$ If $k = \mathrm{des}_2(\sigma) = r$, the goal is to prove that $c^0_r(\sigma) < n - d^r_2(\sigma)$. Suppose there exists a sequence $(i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma))$ of length $c^0_r(\sigma) \geq n - d^r_2(\sigma)$

with $t_r(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \ldots < i_{k_m}(\sigma)$. In particular, there exist $c_r^0(\sigma) \geq n - d_2^r(\sigma)$ 2-inversions (i,j) such that $d_2^r(\sigma) < j$, which forces $c_r^0(\sigma)$ to equal $n - d_2^r(\sigma)$ and every $j > d_2^r(\sigma)$ to be the arrival of a 2-inversion (i,j) such that $t_r(\sigma) \leq i$. In particular, this is true for $j = d_2^r(\sigma) + 1$, which is absurd because $\sigma(i) \geq \sigma\left(d_2^r(\sigma)\right) > \sigma\left(d_2^r(\sigma)+1\right) + 1$ for all $i \in [t_r(\sigma), d_2^r(\sigma)]$. Therefore $c_r^0(\sigma) < n - d_2^r(\sigma)$. Also, it is easy to see that every $i > d_2^r(\sigma)$ that is the beginning of a 2-inversion (i,j) necessarily appears in the maximal sequence $\left(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \ldots, i_{k_m^{max}}(\sigma)\right)$ whose length defines $c_r^0(\sigma)$, hence $(i,j) \notin I_{r-1}(\sigma)$.

$\star$ Now, suppose that $c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)$ for some $k \in [\mathrm{des}_2(\sigma)]$ with equality only if $c_{k+1}^0(\sigma) > 0$, and that no 2-inversion (i,j) with $d_2^k(\sigma) < i$ belongs to $I_{k-1}(\sigma)$.

If $t_{k-1}(\sigma) = t_k(\sigma)$ (*i.e.*, if $d_2^{k-1}(\sigma) = d_2^k(\sigma) - 1$), since $I_{k-1}(\sigma)$ does not contain any 2-inversion (i,j) with $d_2^k(\sigma) < i$, then $c_{k-1}^0(\sigma) \leq 1 = d_2^k(\sigma) - d_2^{k-1}(\sigma)$. Moreover, if $c_{k-1}^0(\sigma) = 1$, then there exists a 2-inversion $(i,j) \in I_{k-1}(\sigma) \subset I_k(\sigma)$ such that $i \in [t_{k-1}(\sigma), d_2^k(\sigma)]$. Consequently (i) was a legit sequence for the computation of $c_k^0(\sigma)$ at the previous step (because $t_k(\sigma) = t_{k-1}(\sigma)$), which implies $c_k^0(\sigma)$ equals at least the length of (i). In particular $c_k^0(\sigma) > 0$.

Else, consider a sequence $(i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma))$ that fits the three conditions of Algorithm 3.4 at the step $k-1$. In particular $t_{k-1}(\sigma) \leq i_{k_1}(\sigma)$. Also $i_{k_m}(\sigma) \leq d_2^k(\sigma)$ by hypothesis. Since $\sigma(i_{k_p}(\sigma)) < \sigma(i_{k_{p+1}}(\sigma))$ for all p, and since $\sigma(t_{k-1}(\sigma)) > \sigma(t_{k-1}(\sigma)+1) > \ldots > \sigma\left(d_2^{k-1}(\sigma)\right) > \sigma\left(d_2^{k-1}(\sigma)+1\right)$, then only one element of the set $[t_{k-1}(\sigma), d_2^{k-1}(\sigma)+1]$ may equal $i_{k_p}(\sigma)$ for some $p \in [m]$. Thus, the length l of the sequence verifies $l \leq d_2^k(\sigma) - d_2^{k-1}(\sigma)$, with equality only if $i_{k_m}(\sigma) = d_2^k(\sigma)$ (which implies $c_k^0(\sigma) > 0$ as in the previous paragraph). In particular, this is true for $l = c_{k-1}^0(\sigma)$.

Finally, as for the $k = \mathrm{des}_2(\sigma)$ case, every $i \in [d_2^{k-1}(\sigma)+1, d_2^k(\sigma)]$ that is the beginning of a 2-inversion (i,j) necessarily appears in the maximal sequence $\left(i_{k_1^{max}}(\sigma), i_{k_2^{max}}(\sigma), \ldots, i_{k_m^{max}}(\sigma)\right)$ whose length defines $c_{k-1}^0(\sigma)$, hence $(i,j) \notin I_{k-2}(\sigma)$.

So the lemma is true by induction. □

Definition 3.7. We define a graph $\mathcal{G}^0(\sigma)$ made of circles and dots organised in ascending or descending slopes, by plotting :

- for all $k \in [0,r]$, an ascending slope of $c_k^0(\sigma)$ circles such that the first circle has abscissa $d_2^k(\sigma)+1$ and the last circle has abscissa $d_2^k(\sigma)+c_k^0(\sigma)$

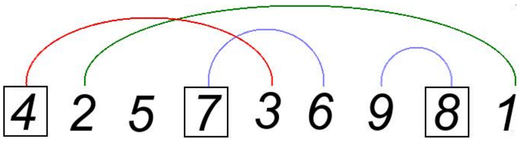


Figure 3.4: $(c_k^0(\sigma_0))_{k\in[0,3]} = (1, 1, 2, 0)$.

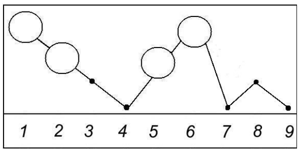


Figure 3.5: Graph $\mathcal{G}^0(\sigma_0)$.

(if $c_k^0(\sigma) = 0$, we plot nothing). All the abscissas are distinct because

$$c_k^0(\sigma) < d_2^{k+1}(\sigma) - d_2^k(\sigma) + c_{k+1}^0(\sigma)$$

for all $k \in [0, r]$ in view of Lemma 3.6;

- dots at the remaining $n - s = n - \mathrm{inv}_2(\sigma)$ abscissas from 1 to n, in ascending and descending slopes with respect to the descents and ascents of the word $w(\sigma)$ defined by

$$\omega(\sigma) = \sigma(u_1(\sigma))\sigma(u_2(\sigma)) \ldots \sigma(u_{n-s}(\sigma)) \tag{3.6}$$

where

$$\{u_1(\sigma) < u_2(\sigma) < \ldots < u_{n-s}(\sigma)\} := \mathfrak{S}_n \setminus \{i_1(\sigma) < i_2(\sigma) < \ldots < i_s(\sigma)\}.$$

Example 3.8. The permutation $\sigma_0 = 425736981 \in \mathfrak{S}_9$ (with $\mathrm{DES}_2(\sigma_0) = \{1, 4, 8\}$ and $\mathrm{INV}_2(\sigma_0) = \{(1, 5), (2, 9), (4, 6), (7, 8)\}$), which yields the sequence $(c_k^0(\sigma_0))_{k\in[0,3]} = (1, 1, 2, 0)$ (see Figure 3.4 where all the 2-inversions involved in the computation of a same $c_k^0(\sigma_0)$ are drawed in a same color) and the word $w(\sigma_0) = 53681$, provides the unlabelled graph $\mathcal{G}^0(\sigma_0)$ depicted in Figure 3.5.

Lemma 3.9. *For all $i \in [n]$, if the i-th vertex (from left to right) $v_i^0(\sigma)$ of $\mathcal{G}^0(\sigma)$ is a dot and if i is a descent of $\mathcal{G}^0(\sigma)$ (i.e., if $v_i^0(\sigma)$ and $v_{i+1}^0(\sigma)$ are two dots in a same descending slope) whereas $i \notin DES_2(\sigma)$, let k_i such that*

$$d_2^{k_i}(\sigma) + c_{k_i}^0(\sigma) < i < d_2^{k_i+1}(\sigma)$$

and let $p \in [n-s]$ such that $v_i^0(\sigma)$ is the p-th dot (from left to right) of $\mathcal{G}^0(\sigma)$. Then:

1. *$u_p(\sigma)$ is the greatest integer $u < d_2^{k_i+1}(\sigma)$ that is not the beginning of a 2-inversion of σ;*
2. *$d_2^{k_i+1}(\sigma)$ is the beginning of a 2-inversion of σ;*
3. *$i = d_2^{k_i+1}(\sigma) - 1$;*
4. *there exists $k \in [des_2(\sigma)]$ such that $t_k(\sigma) = d_2^{k_i+1}(\sigma)$ and $c_k^0(\sigma) > 0$.*

Proof. (1) implies (3) and (2) implies (4). The proofs of (1) and (2) come from $\sigma(u_p(\sigma)) > \sigma(u_{p+1}(\sigma))$ (since i is a descent of $\mathcal{G}^0(\sigma)$) and the fact that

$$\sigma(u_q(\sigma)) < \sigma(u_{q'}(\sigma))$$

for all (q, q') such that $d_2^{k_i}(\sigma) < u_q(\sigma) < u_{q'}(\sigma) \leq d_2^{k_i+1}(\sigma)$. □

Lemma 3.9 motivates the following definition.

Definition 3.10. For i from 1 to $n-1$, let $k_i \in [0, r]$ such that

$$d_2^{k_i}(\sigma) \ + \ c_{k_i}^0(\sigma) \ < \ i < \ d_2^{k_i+1}(\sigma).$$

If i fits the conditions of Lemma 3.9, then we define a sequence $(c_k^i(\sigma))_{k\in[0,r]}$ by

$$\begin{aligned} c_{k_i}^i(\sigma) &= c_{k_i}^{i-1}(\sigma) + 1, \\ c_{k_i+1}^i(\sigma) &= c_{k_i+1}^{i-1}(\sigma) - 1, \\ c_k^i(\sigma) &= c_k^{i-1}(\sigma) \text{ for all } k \notin \{k_i, k_i+1\}. \end{aligned}$$

Else, we define $(c_k^i(\sigma))_{k\in[0,r]}$ as $(c_k^{i-1}(\sigma))_{k\in[0,r]}$.

The final sequence $(c_k^n(\sigma))_{k\in[0,r]}$ is denoted by

$$(c_k(\sigma))_{k\in[0,r]}.$$

By construction, and from Lemma 3.6, the sequence $(c_k(\sigma))_{k\in[0,r]}$ has the same properties as $(c_k^0(\sigma))_{k\in[0,r]}$ detailed in Lemma 3.6.

Consequently, we may define an unlabelled graph

$$\mathcal{G}(\sigma)$$

by replacing $(c_k^0(\sigma))_{k\in[0,r]}$ with $(c_k(\sigma))_{k\in[0,r]}$ in Definition 3.7.

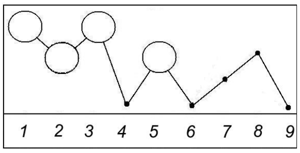


Figure 3.6: Graph $\mathcal{G}(\sigma_0)$.

Example 3.11. In the graph $\mathcal{G}^0(\sigma)$ depicted in Figure 3.5 where $\sigma_0 = 425736981 \in \mathfrak{S}_9$, we can see that the dot $v_3^0(\sigma_0)$ is a descent whereas $3 \notin \mathrm{DES}_2(\sigma_0)$, hence, from the sequence $(c_k^0(\sigma_0))_{k\in[0,3]} = (1,1,2,0)$, we compute $(c_k(\sigma_0))_{k\in[0,3]} = (1,2,1,0)$ and we obtain the graph $\mathcal{G}(\sigma_0)$ depicted in Figure 3.6.

Let $v_1(\sigma), v_2(\sigma), \ldots, v_n(\sigma)$ be the n vertices of $\mathcal{G}(\sigma)$ from left to right.

Lemma 3.12. *The descents of the unlabelled graph $\mathcal{G}(\sigma)$ (i.e., the integers $i \in [n-1]$ such that $v_i(\sigma)$ and $v_{i+1}(\sigma)$ are in a same descending slope) are the integers*

$$d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$$

for all $k \in [0,r]$.

Proof. By Definition 3.7, the descents of $\mathcal{G}^0(\sigma)$ are made of :

1. the integers $d_2^k(\sigma) + c_k^0(\sigma)$,
2. the integers $i \in [n-1]$ that fit the conditions of Lemma 3.9.

Definition 3.10 consists in potentially moving the upper circle of given ascending slopes of circles to the previous slope, so the integers $d_2^k(\sigma) + c_k(\sigma)$ are descents of the graph $\mathcal{G}(\sigma)$, and the possible other descents of $\mathcal{G}(\sigma)$ are of the kind $j \in [n-1]$ such that $i = j-1$ fits the conditions of Lemma 3.9. Now, if i fits the conditions of Lemma 3.9, by using the same notations as those of the lemma, we know that $i = d_2^{k_i+1}(\sigma) - 1$, so if $j = i+1$ is a descent of $\mathcal{G}(\sigma)$, since the vertices $v_{d_2^{k_i+1}(\sigma)}(\sigma), v_{d_2^{k_i+1}(\sigma)+1}(\sigma), \ldots, v_{d_2^{k_i+1}(\sigma)+c_{k_i+1}(\sigma)}(\sigma)$ form an ascending slope (whose last $c_{k_i+1}(\sigma)$ elements are circles), then $c_{k_i+1}(\sigma) = 0$ and $j = d_2^{k_i+1}(\sigma) + c_{k_i+1}(\sigma)$. $\square$

3.4.2 Labelling of the graph $\mathcal{G}(\sigma)$

Labelling of the circles

We intend to label the s circles of $\mathcal{G}(\sigma)$ with the integers

$$j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)$$

so that each circle is labelled by an integer that exceeds its abscissa.

Algorithm 3.13. We first label, from right to left, the circles of $\mathcal{G}(\sigma)$ with sets of integers that exceed the abscissas of the circles. Consider a maximal descending slope of circles from abscissa x to abscissa $x+a \geq x$, and let b and c be the nonnegative integers such that $v_{x+a}(\sigma), v_{x+a+1}(\sigma), \ldots, v_{x+a+b}(\sigma)$ form a maximal ascending slope of circles, and $v_{x+a+b}(\sigma), v_{x+a+b+1}(\sigma), \ldots, v_{x+a+b+c}(\sigma)$ form a maximal descending slope of circles (see Figure 3.7). By construction of $\mathcal{G}(\sigma)$ and in view of Lemma 3.12, the integers $x, x+1, \ldots, x+a-1$ are 2-descents of σ but the integers $x+a, x+a+1, \ldots, x+a+b-1$ are not. Let $k \in [0, r]$ such that $x+a-1 = d_2^k(\sigma)$. Because the $a+b+1$ vertices from abscissa x to abscissa $x+a+b$ are circles, it is necessary that there exist at least $a+b+1$ two inversions (i, j) such that $t_k(\sigma) \leq i$, which forces every of the corresponding j to equal at least $x+a+1$ because $t_k(\sigma)+1, t_k(\sigma)+2, \ldots, x+a-1$ are 2-descents. Consequently, for all $l \in \{0, 1, \ldots, b\}$, there exist at least $a+b+1-l$ elements of $\{j_1(\sigma), \ldots, j_s(\sigma)\}$ that exceed $x+a+l$. Likewise, the existence of the c circles $v_{x+a+b+1}(\sigma), \ldots, v_{x+a+b+c}(\sigma)$ implies the existence of c other (and distinct) $j_k(\sigma) \geq x+a+b$. As a result, for all $l \in \{0, 1, \ldots, b\}$, there exist at least $a+b+c+1-l$ elements of $\{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\}$ that exceed $x+a+l$. This proves that the following labelling by non-empty sets is well-defined :

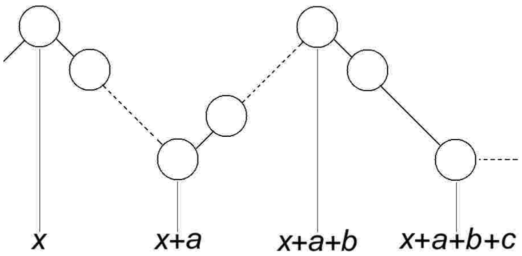


Figure 3.7: Sequence of circles.

- we label $v_{x+a}(\sigma)$ with the set $\{j_k(\sigma) : j_k(\sigma) > x + a\}$ (wich contains at least $a + b + c + 1$ elements) from which we delete the $a + b$ greatest values;
- for l from $a-1$ down to 1, we label $v_{x+l}(\sigma)$ with the set $\{j_k(\sigma) : j_k(\sigma) > x + l\}$ (which contains at least $a + b + c + 1$ elements) from which we delete the l greatest values and the minimal element of the label of $v_{x+l+1}(\sigma)$;
- for l from 1 to $b - 1$, we label $v_{x+a+l}(\sigma)$ with the set $\{j_k(\sigma) : j_k(\sigma) > x+a+l\}$ (which contains at least $a+b+c+1-l$ elements) from which we delete the $b-l$ greatest values and the minimal element of the label of $v_{x+a+l-1}(\sigma)$ if it appears in the resulted set (it is easy to see that in that case the number of integers $j_k(\sigma)$ that exceed $x + a + l$ is at least $a + b + c + 2 - l$, so that the label of $v_{x+l}(\sigma)$ is not empty);
- we label $v_{x+a+b}(\sigma)$ with the set $\{j_k(\sigma) : j_k(\sigma) > x + a + b\}$ (which contains at least $a+c+1$ elements) from which we remove the c smallest elements and the minimal element of the label of $v_{x+a+b-1}(\sigma)$ if it appears in the resulted set (like before, if it is the case, then there are necessarily at least $a + c + 2$ integers $j_k(\sigma)$ that exceed $x + a + b$).

Note that the circle $v_x(\sigma)$ is not labelled at this step but at the next one.

When every circle is labelled by a set, if an integer $j_k(\sigma)$ appears in only one label of a circle $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with $j_k(\sigma)$. Afterwards, we consider the greatest integer $m \in [s]$ such that $j_m(\sigma)$ is not the label of a circle, we consider the last (from left to right) circle whose label is still a set and contains $j_m(\sigma)$, we label this circle with $j_m(\sigma)$ and we repeat the process of this paragraph until every circle is labelled by an integer.

Example 3.14. For $\sigma_0 = 425736981$ (see Figure 3.4) whose graph $\mathcal{G}(\sigma_0)$ is depicted in Figure 3.6, we have $s = \text{inv}_2(\sigma) = 4$ and $\{j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)\} = \{5, 6, 8, 9\}$, which provides first the graph labelled by sets depicted in Figure 3.8. Afterwards, since the label of $v_2(\sigma_0)$ is the only set that contains 5, then we label $v_2(\sigma_0)$ with 5 (see Figure 3.9). Finally, the subsequence $L = (9, 6, 8)$ of $(j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)) = (5, 9, 6, 8)$ gives the order (from left to right) of apparition of the remaining integers $6, 8, 9$ (see Figure 3.10).

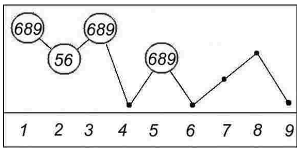


Figure 3.8: Incomplete $\mathcal{G}(\sigma_0)$.

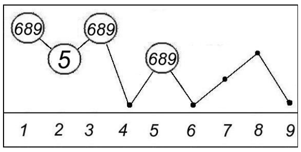


Figure 3.9: Incomplete $\mathcal{G}(\sigma_0)$.

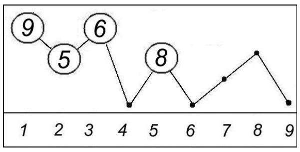


Figure 3.10: Incomplete $\mathcal{G}(\sigma_0)$.

Labelling of the dots

Let

$$\{p_1(\sigma) < p_2(\sigma) < \ldots < p_{n-s}(\sigma)\} = [n] \backslash \bigsqcup_{k=0}^{r}]d_2^k(\sigma), d^k(\sigma)].$$

We intend to label the dots $\{v_{p_i(\sigma)}(\sigma) : i \in [n-s]\}$ of $\mathcal{G}(\sigma)$ with the elements of

$$\{1 = e_1(\sigma) < e_2(\sigma) < \ldots < e_{n-s}(\sigma)\} = [n] \backslash \{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\}.$$

Lemma 3.15. *Let $k \in [n-s]$ and let $l \in [0, r]$ such that $d_2^l(\sigma)+1 \notin DES_2(\sigma)$ and such that $u_k(\sigma) \in [t_l(\sigma), d_2^{l+1}(\sigma)[$. Then:*

1. *If $u_k(\sigma) \in [t_l(\sigma), d_2^l(\sigma)]$, we have $u_k(\sigma) \geq p_k(\sigma)$. With precision, the nonnegative integer $u_k(\sigma) - p_k(\sigma)$ is the number of 2-inversions (i, j) of σ such that $i \in [t_l(\sigma), u_k(\sigma)[$ and such that (i, j) generates a circle whose abscissa exceeds i (i.e., such that (i, j) is part of a sequence of 2-inversions whose length defines $c_{l'}(\sigma)$ for some $l' \geq i$).*
2. *If $u_k(\sigma) = d_2^l(\sigma) + 1$,*
   - *(a) if there exists a 2-inversion (i, j) with $i \in [t_l(\sigma), d_2^l(\sigma)]$ that is part of the sequence of 2-inversions whose length defines $c_l(\sigma)$ (we know that there exists at most one such 2-inversion and that if it exists then i is the greatest integer not exceeding $d_2^l(\sigma)$ that is the beginning of a 2-inversion), then $u_k(\sigma) > p_k(\sigma)$, with precision the positive integer $u_k(\sigma) - p_k(\sigma)$ is the number of 2-inversions (i, j) of σ such that $i \in [t_l(\sigma), d_2^l(\sigma)]$ and such that (i, j) generates a circle whose abscissa exceeds i;*
   - *(b) otherwise $u_k(\sigma) \leq p_k(\sigma)$, with precision the nonnegative integer $p_k(\sigma) - u_k(\sigma)$ equals $c_l(\sigma)$.*

3. *If* $u_k(\sigma) \in]d_2^l(\sigma)+1, d_2^{l+1}(\sigma)[$*, we have* $u_k(\sigma) \leq p_k(\sigma)$*. With precision, the nonnegative integer* $p_k(\sigma) - u_k(\sigma)$ *is the number of 2-inversions* (i,j) *of* σ *such that* $i \in]u_k(\sigma), d_2^{l+1}(\sigma)]$ *and such that* (i,j) *generates a circle whose abscissa exceeds* $d_2^l(\sigma)$ *(i.e., such that* (i,j) *is part of the sequence of 2-inversions whose length defines* $c_l(\sigma)$*).*

Proof. Let n_1 (respectively n_2) be the number of 2-inversions (i,j) of σ such that $i \in [t_l(\sigma), d_2^l(\sigma)]$ and such that (i,j) generates a circle whose abscissa exceeds i (resp. the number of 2-inversions (i,j) of σ such that $i \in [d_2^l(\sigma)+1, d_2^{l+1}(\sigma)]$ and such that (i,j) generates a circle whose abscissa belongs to $[d_2^l(\sigma), d_2^l(\sigma)+c_l(\sigma)]$). In fact, the integer n_2 equals either $c_l(\sigma)-1$ (if there exists a 2-inversion (i,j) with $i \in [t_l(\sigma), d_2^l(\sigma)]$ that is part of the sequence of 2-inversions whose length defines $c_l(\sigma)$) or $c_l(\sigma)$.

If $n_2 = c_l(\sigma)$ (resp. $n_2 = c_l(\sigma)-1$), then the n_1 2-inversions (i,j) such that $i \in [t_l(\sigma), d_2^l(\sigma)]$ (resp. the $n_1 - 1$ first such 2-inversions) generate n_1 (respectively $n_1 - 1$) circles at the abscissas $d_2^l(\sigma) - n_1 + 1, \ldots, d_2^l(\sigma)$ (resp. $d_2^l(\sigma) - n_1 + 2, \ldots, d_2^l(\sigma)$), and the $n_2 = c_l(\sigma)$ 2-inversions (i,j) such that $i \in [d_2^l(\sigma)+1, d_2^{l+1}(\sigma)]$ (resp. the $n_2 + 1 = c_l(\sigma)$ 2-inversions of the sequence that defines $c_l(\sigma)$) generate $c_l(\sigma)$ circles at the abscissas $d_2^l(\sigma)+1, \ldots, d_2^l(\sigma)+c_l(\sigma)$. Now, every integer $u_k(\sigma) \in [t_l(\sigma), d_2^l(\sigma)+1]$ (resp. $[t_l(\sigma), d_2^l(\sigma)]$) generates a dot $v_{p_k(\sigma)}(\sigma)$ whose abscissa $p_k(\sigma)$ is the integer $u_k(\sigma)$ that has been "pushed to the left" by the insertion of the n_1 (resp. $n_1 - 1$) circles at the abscissas $d_2^l(\sigma) - n_1 + 1, \ldots, d_2^l(\sigma)$ (resp. $d_2^l(\sigma) - n_1 + 2, \ldots, d_2^l(\sigma)$), and it has been pushed to the left a number of times that equals the number of $(i,j) \in \mathrm{INV}_2(\sigma)$ such that $i \in [t_l(\sigma), u_k(\sigma)[$. Likewise, every integer $u_k(\sigma) \in]d_2^l(\sigma)+1, d_2^{l+1}(\sigma)[$ (resp. $[d_2^l(\sigma)+1, d_2^{l+1}(\sigma)[$) generates a dot $v_{p_k(\sigma)}(\sigma)$ whose abscissa $p_k(\sigma)$ is the integer $u_k(\sigma)$ that has been "pushed to the right" by the insertion of the $c_l(\sigma)$ circles at the abscissas $d_2^l(\sigma)+1, \ldots, d_2^l(\sigma)+c_l(\sigma)$, and it has been pushed to the right a number of times that equals the number of $(i,j) \in \mathrm{INV}_2(\sigma)$ such that $i \in]u_k(\sigma), d_2^{l+1}(\sigma)]$ and such that (i,j) is part of the sequence of 2-inversions which defines $c_l(\sigma)$. □

Algorithm 3.16. We first label, from left to right, the dots $v_{p_k(\sigma)}(\sigma)$ with sets of integers that do not exceed $\min(p_k(\sigma), u_k(\sigma))$. Not exceeding $u_k(\sigma)$ will be necessary to insure the map Ψ is surjective. Consider a maximal ascending slope of dots from abscissa x to abscissa $x+a \geq x$, and let b and c be the nonnegative integers such that $v_{x+a}(\sigma), \ldots, v_{x+a+b}(\sigma)$ form a maximal descending slope of dots (if $b = 0$ then $v_{x+a+1}(\sigma)$ is a circle), and such that $v_{x-c}(\sigma), \ldots, v_x(\sigma)$ form a maximal ascending slope of circles (if $c = 0$ then $v_{x-1}(\sigma)$ is a dot), see Figure 3.11. There exists k such that $x = p_k(\sigma)$ (hence $x + q = p_{k+q}(\sigma)$ for all $q \in [0, a + b]$). Suppose that every dot $v_{p_{k'}(\sigma)}(\sigma)$ with $k' < k$ (and with $k' = k$ if $c = 0$, *i.e.*, if $v_{x-1}(\sigma)$ is a dot) has already been labelled by a set of integers not exceeding $\min(u_{k'}(\sigma), p_{k'}(\sigma))$. We intend to label $v_x(\sigma), v_{x+1}(\sigma), \ldots, v_{x+a+b-1}(\sigma)$ with sets of appropriate integers.

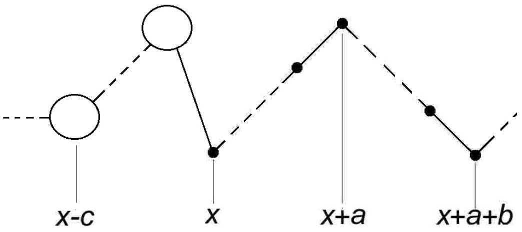


Figure 3.11: Sequence of dots.

By construction of $\mathcal{G}(\sigma)$, and in view of Lemma 3.12, the integers $x - c - 1, x + a, x + a + 1, \ldots, x + a + b - 1$ are 2-descents of σ, and the integers $x - c, x - c + 1, \ldots, x + a - 1$ are not 2-descents. Let l such that $x - c - 1 = d_2^l(\sigma)$. Following the proof of Lemma 3.15, we know that there exist exactly c or $c - 1$ elements of $[x - c, x + a - 1]$ that are the beginnings of arcs of circles, which implies that the other a or $a + 1$ elements are of the kind $u_{k'}(\sigma)$. With precision, the a greatest elements among these $u_{k'}(\sigma)$ are $u_k(\sigma), u_{k+1}(\sigma), \ldots, u_{k+a-1}(\sigma)$ and belong to $]d_2^l(\sigma)+1, d_2^{l+1}(\sigma)[=]x-c, x+a[$, and still following Lemma 3.15, each of these $u_{k'}(\sigma)$ verifies $u_{k'}(\sigma) \leq p_{k'}(\sigma)$. Now, either $u_{k'}(\sigma)$ is of the kind $e_i(\sigma)$ (*i.e.*, is not the arrival of an arc of circle), or it is the arrival of a sequence of 2-inversions whose beginning is of the kind $e_i(\sigma) < u_{k'}(\sigma) \leq p_{k'}(\sigma)$. Since $u_k(\sigma), \ldots, u_{k+a-1}(\sigma)$ are not the beginnings of arcs of circles, the a integers of the kind $e_i(\sigma)$ that they induce are necessarily distinct, hence, for all $q \in [0, a - 1]$, there exist at least $q + 1$ integers of the kind $e_i(\sigma)$ that do not exceed $\min(u_{k+q}(\sigma), p_{k+q}(\sigma)) = u_{k+q}(\sigma)$.

Afterwards, since the vertices $v_{x+a}(\sigma), \ldots, v_{x+a+b}(\sigma)$ are dots and $x + a, \ldots,$
$x + a + b - 1$ are 2-descents, there exist at most one element of $[x + a, x + a + b]$ that is the beginning of an arc of circle (otherwise $v_{x+a+b}(\sigma)$ would be a circle by construction since $x + a = d_2^{l+1}(\sigma) = t_{l+1}(\sigma)$ by hypothesis). Consequently, at least b elements of $[x+a, x+a+b]$ are of the kind $u_{k'}(\sigma)$

and the b first of them are more precisely $u_{k+a}(\sigma), u_{k+a+1}(\sigma), \ldots, u_{k+a+b-1}(\sigma)$. By definition of the $u_{k'}(\sigma)$, it implies that, for all $q \in [0, b-1]$, the integer $\sigma^{-1}(\sigma(u_{k+a+q}(\sigma)) - 1) < u_{k+a+q}(\sigma)$, and since the elements of $[x+a, x+a+b-1]$ are 2-descents, we have in fact $\sigma^{-1}(\sigma(u_{k+a+q}(\sigma)) - 1) < x+a$. Obviously the b integers $\sigma^{-1}(\sigma(u_{k+a+q}(\sigma)) - 1)$ are b distinct integers of the kind $e_i(\sigma)$, so there exist at least b integers of the kind $e_i(\sigma)$ that do not exceed $x+a-1 < x+a+q = p_{k+a+q}(\sigma) = \min(u_{k+a+q}(\sigma), p_{k+a+q}(\sigma))$ for all $q \in [0, b]$ according to Lemma 3.15. Moreover, either $x+a = d_2^{l+1}(\sigma)$ is of the kind $e_i(\sigma)$, or it is the arrival of a sequence of 2-inversions whose beginning is of the kind $e_i(\sigma) < x+a \leq x+a+q = p_{k+a+q}(\sigma) = \min(u_{k+a+q}(\sigma), p_{k+a+q}(\sigma))$ for all $q \in [0, b]$, so there are finally at least $b+1$ integers of the kind $e_i(\sigma)$ that do not exceed $\min(u_{k+a+q}(\sigma), p_{k+a+q}(\sigma))$ for all $q \in [0, b]$.

This proves that the following labelling by non-empty sets is well-defined :

- for q from 0 to b, we label $v_{x+a+q}(\sigma)$ with the set $\{e_i(\sigma) : e_i(\sigma) \leq \min(p_{k+a+q}(\sigma), u_{k+a+q}(\sigma)) = x+a+q\}$ (which contains at least $\max(a, b+1)$ elements) from which we delete the $b-q$ smallest integers and the q greatest integers;
- for q from 1 to a (or $a-1$ if $v_x(\sigma)$ has already been labelled by a set), we label $v_{x+a-q}(\sigma)$ with the set $\{e_i(\sigma) : e_i(\sigma) \leq \min(p_{k+a-q}(\sigma), u_{k+a-q}(\sigma)) = u_{k+a-q}(\sigma)\}$ (which contains at least $a-q+1$ elements) from which we delete the $a-q$ smallest integers, and eventually the maximal integer of the label of $v_{x+a-q+1}(\sigma)$ if it appears in the resulted set (in which case, there are in fact at least $a-q+2$ integers $e_i(\sigma) \leq u_{k+a-q}(\sigma)$ because the a integers of the kind $e_i(\sigma)$ induced by the a integers $u_k(\sigma), u_{k+1}(\sigma), \ldots, u_{k+a-1}(\sigma)$ and $x+a$ are distinct), so the final set is not empty.

When every dot is labelled by a set, if an integer $e_i(\sigma)$ appears in only one label of a dot, then we replace the label of this dot with $e_i(\sigma)$.

Finally, for k from 1 to $n-s$, let

$$w_1^k(\sigma) < w_2^k(\sigma) < \ldots < w_{q_k(\sigma)}^k(\sigma) \tag{3.7}$$

such that

$$\{p_{w_i^k(\sigma)}(\sigma) : i\} = \{p_i(\sigma) : \ e_k(\sigma) \text{ appears in the label of } p_i(\sigma)\},$$

and let $i(k) \in [q_k(\sigma)]$ such that

$$\sigma\left(u_{w_{i(k)}^k(\sigma)}(\sigma)\right) = \min\{\sigma\left(u_{w_i^k(\sigma)}(\sigma)\right) : i \in [q_k(\sigma)]\}.$$

Then, we replace the label of the dot $p_{w^k_{i(k)}(\sigma)}(\sigma)$ with the integer $e_k(\sigma)$ and we erase $e_k(\sigma)$ from any other label (and if an integer l appears in only one label of a dot $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with l).

Example 3.17. For $\sigma_0 = 425736981$ whose graph $\mathcal{G}(\sigma_0)$ has its circles labelled in Figure 3.10, the sequence $(u_1(\sigma_0), u_2(\sigma_0), u_3(\sigma_0), u_4(\sigma_0), u_5(\sigma_0)) = (3, 5, 6, 8, 9)$ provides first the graph labelled by sets depicted in Figure 3.12. The rest of the algorithm goes from $k = 1$ to $n - s = 9 - 4 = 5$.

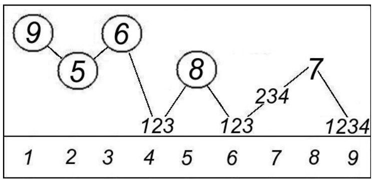


Figure 3.12: Incomplete $\mathcal{G}(\sigma_0)$.

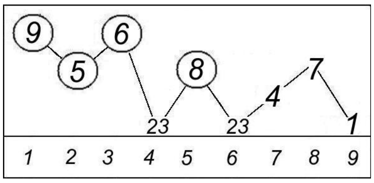


Figure 3.13: Incomplete $\mathcal{G}(\sigma_0)$.

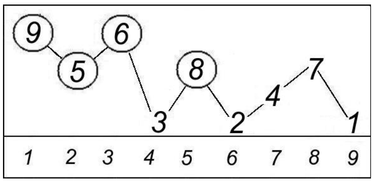


Figure 3.14: Complete graph $\mathcal{G}(\sigma_0)$.

- $k = 1:$ in Figure 3.12, the integer $e_1(\sigma_0) = 1$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$, $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ and $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$, so, from
$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0)), \sigma_0(u_5(\sigma_0)) = (5, 3, 1),$$
we label the dot $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$ with the integer $e_1(\sigma_0) = 1$ and we erase 1 from any other label, and since the integer 4 now only appears in the label of the dot $v_7(\sigma_0)$, then we label $v_7(\sigma_0)$ with 4 (see Figure 3.13).
- $k = 2$: in Figure 3.13, the integer $e_2(\sigma_0) = 2$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$ and $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ so, from
$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0))) = (5, 3),$$
we label the dot $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ with the integer $e_2(\sigma_0) = 2$ and we erase 2 from any other label, which provides the graph labelled by integers depicted in Figure 3.14.

- The three steps $k = 3, 4, 5$ change nothing because every dot of $\mathcal{G}(\sigma_0)$ is already labelled by an integer at the end of the previous step.

So the final version of the labelled graph $\mathcal{G}(\sigma_0)$ is the one depicted in Figure 3.14.

3.4.3 Definition of $\Psi(\sigma)$

By construction of the labelled graph $\mathcal{G}(\sigma)$, the word $y_1 y_2 \dots y_n$ (where the integer y_i is the label of the vertex $v_i(\sigma)$ for all i) obviously is a permutation of the set $[n]$, whose planar graph is $\mathcal{G}(\sigma)$.

We define $\Psi(\sigma) \in \mathfrak{S}_n$ as this permutation.

For the example $\sigma_0 = 425736981 \in \mathfrak{S}_9$ whose labelled graph $\mathcal{G}(\sigma_0)$ is depicted in Figure 3.14, we obtain $\Psi(\sigma_0) = 956382471 \in \mathfrak{S}_9$.

In general, by construction of $\tau = \Psi(\sigma) \in \mathfrak{S}_n$, we have

$$\tau\,(\mathrm{EXC}(\tau)) = \{j_k(\sigma) : k \in [\mathrm{inv}_2(\sigma)]\} \tag{3.8}$$

and

$$\mathrm{DES}(\tau) = \begin{cases} \{d^k(\sigma) : k \in [1, \mathrm{des}_2(\sigma)]\} & \text{if } c_0(\sigma) = 0 (\Leftrightarrow d^0(\sigma) = 0), \\ \{d^k(\sigma) : k \in [0, \mathrm{des}_2(\sigma)]\} & \text{otherwise.} \end{cases} \tag{3.9}$$

Equality (3.8) provides

$$\mathrm{exc}(\tau) = \mathrm{inv}_2(\sigma).$$

By $d^k(\sigma) = d_2^k(\sigma) + c_k(\sigma)$ for all k, Equality (3.9) provides

$$\mathrm{maj}(\tau) = \mathrm{maj}_2(\sigma) + \sum_{k \geq 0} c_k(\sigma),$$

and by definition of $(c_k(\sigma))_k$ and Lemma 3.6 we have

$$\sum_{k \geq 0} c_k(\sigma) = \sum_{k \geq 0} c_k^0(\sigma) = \mathrm{inv}_2(\sigma) = \mathrm{exc}(\tau)$$

hence

$$\mathrm{maj}(\tau) - \mathrm{exc}(\tau) = \mathrm{maj}_2(\sigma)$$

Finally, it is easy to see that $\widetilde{\mathrm{des}_2}(\sigma) = \mathrm{des}_2(\sigma)$ if and only if $c_0(\sigma) = 0$, so Equality (3.9) also provides

$$\mathrm{des}(\tau) = \widetilde{\mathrm{des}_2}(\sigma).$$

As a conclusion, we obtain

$$(\mathrm{maj}(\tau) - \mathrm{exc}(\tau), \mathrm{des}(\tau), \mathrm{exc}(\tau)) = (\mathrm{maj}_2(\sigma), \widetilde{\mathrm{des}_2}(\sigma), \mathrm{inv}_2(\sigma))$$

as required by Theorem 3.3.

3.5 Construction of Ψ^{-1}

To end the proof of Theorem 3.3, it remains to show that $\Psi : \mathfrak{S}_n \to \mathfrak{S}_n$ is surjective. Let $\tau \in \mathfrak{S}_n$. We introduce integers $r \geq 0$, $s = \mathrm{exc}(\tau)$, and

$$0 \leq d^{0,\tau} < d^{1,\tau} < \ldots < d^{r,\tau} < n$$

such that

$$\begin{aligned} \mathrm{DES}(\tau) &= \{d^{k,\tau} : k \in [0,r]\} \cap \mathbb{N}_{>0}, \\ d^{0,\tau} &= 0 \Leftrightarrow \tau(1) = 1. \end{aligned}$$

In particular $\mathrm{des}(\tau) = \begin{cases} r & \text{if } \tau(1) = 1, \\ r+1 & \text{otherwise.} \end{cases}$

For all $k \in [0,r]$, we define

$$\begin{aligned} c_k^\tau &= \mathrm{EXC}(\tau) \cap]d^{k-1,\tau}, d^{k,\tau}] \text{ (with } d^{-1,\tau} := 0), \\ d_2^{k,\tau} &= d^{k,\tau} - c_k^\tau. \end{aligned}$$

We have

$$0 = d_2^{0,\tau} < d_2^{1,\tau} < \ldots < d_2^{r,\tau} < n$$

and similarly as Formula 3.5, we define

$$t_k^\tau = \min\{d_2^{l,\tau} : 1 \leq l \leq k, d_2^{l,\tau} = d_2^{k,\tau} - (k-l)\} \tag{3.10}$$

for all $k \in [r]$.

We intend to construct a graph $\mathcal{H}(\tau)$ which is the linear graph of a permutation $\sigma \in \mathfrak{S}_n$ such that $\Psi(\sigma) = \tau$.

3.5.1 Skeleton of the graph $\mathcal{H}(\tau)$

We consider a graph $\mathcal{H}(\tau)$ whose vertices $v_1^\tau, v_2^\tau, \ldots, v_n^\tau$ (from left to right) are n dots, aligned in a row, among which we box the $d_2^{k,\tau}$-th vertex $v_{d_2^{k,\tau}}^\tau$ for all $k \in [r]$. Afterwards, we draw the end of an arc of circle above every vertex v_j^τ such that $j = \tau(i)$ for some $i \in \mathrm{EXC}(\tau)$. Also, if the first (from left to right) non-excedance point of τ is $i \in [n]$, in order to obtain $\Psi(\sigma_\tau)(i) = \tau(i)$ we draw beginnings of arcs of circles above the vertices $v_1^\tau, v_2^\tau, \ldots, v_{\tau(i)-1}^\tau$ (because the first dot from left to right of $\mathcal{G}(\sigma_\tau)$ will be labelled with an integer that does not exceed $u_1(\sigma_\tau)$).

For the example $\tau_0 = 956382471 \in \mathfrak{S}_9$ (whose planar graph is depicted in Figure 3.14), we have $r = \mathrm{des}(\tau_0) - 1 = 3$ and

$$\begin{aligned}
(c_k^{\tau_0})_{k\in[0,3]} &= (1,2,1,0),\\
(d_2^{k,\tau_0})_{k\in[0,3]} &= (1-1, 3-2, 5-1, 8-0) = (0,1,4,8),\\
\tau_0(\mathrm{EXC}(\tau_0)) &= \{5,6,8,9\},\\
\tau_0\left(\min\{i \in [n] : \tau_0(i) \le i\}\right) &= 3,
\end{aligned}$$

and we obtain the graph $\mathcal{H}(\tau_0)$ depicted in Figure 3.15.

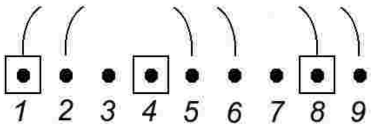


Figure 3.15: Incomplete graph $\mathcal{H}(\tau_0)$.

In general, by definition of $\Psi(\sigma)$ for all $\sigma \in \mathfrak{S}_n$, if $\Psi(\sigma) = \tau$, then $r = \mathrm{des}_2(\sigma)$ and $d_2^k(\sigma)$ (respectively $c_k(\sigma), d^k(\sigma), t_k(\sigma)$) equals $d_2^{k,\tau}$ (resp. $c_k^\tau, d^{k,\tau}, t_k^\tau$) for all $k \in [0,r]$ and $\{j_l(\sigma) : l \in [\mathrm{inv}_2(\sigma)]\} = \tau(\mathrm{EXC}(\tau))$. Consequently, the linear graph of σ necessarily have the same skeleton as that of $\mathcal{H}(\tau)$.

The following lemma is easy.

Lemma 3.18. *If $\tau = \Psi(\sigma)$ for some $\sigma \in \mathfrak{S}_n$, then :*

1. *If $j = \tau(l)$ with $l \in$ EXC(τ) such that $l \in]d_2^{k,\tau}, d^{k,\tau}]$, and if $(i,j) \in$ INV$_2(\sigma)$, then $t_k^\tau \le i \le t_{k'}^\tau$ where $t_{k'}^\tau = d_2^{k',\tau}$ is the smallest 2-descent of σ such that $t_k^\tau < t_{k'}^\tau$ (if there is no such 2-descent, then we define $t_{k'}^\tau$ as n).*

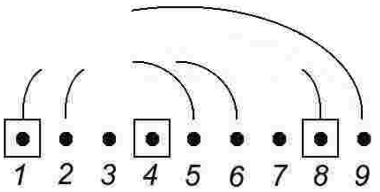


Figure 3.16: Incomplete graph $\mathcal{H}(\tau_0)$.

2. *A pair $(i, i+1)$ cannot be a 2-inversion of σ if $i \in DES_2(\sigma)$ ($\Leftrightarrow$ if the vertex v_i^τ of $\mathcal{H}(\tau)$ is boxed).*
3. *For all pair $(l, l') \in EXC(\tau)^2$, if the labels of the two circles $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $\mathcal{G}(\sigma)$, let i and i' such that $(i, l) \in INV_2(\sigma)$ and $(i', l') \in INV_2(\sigma)$, then $i < i' \Leftrightarrow l < l'$.*
4. *If $(i, j) \in INV_2(\sigma)$ such that $j \in]d_2^{k,\tau} + 1, d_2^{k+1,\tau}[$, then either $i = j - 1$, or $i \leq d_2^{k,\tau}$.*

Consequently, in order to construct the linear graph of a permutation $\sigma \in \mathfrak{S}_n$ such that $\tau = \Psi(\sigma)$ from $\mathcal{H}(\tau)$, it is necessary to extend the arcs of circles of $\mathcal{H}(\tau)$ to reflect the facts of Lemma 3.18. When it becomes known, at any step of this section, that a vertex is necessarily the beginning of an arc of circle, we draw the beginning of an arc of circle above it. When there is only one vertex v_i^τ that can be the beginning of an arc of circle, we complete the latter by making it start from v_i^τ.

Example 3.19. For $\tau_0 = 956382471 \in \mathfrak{S}_9$, the graph $\mathcal{H}(\tau_0)$ becomes as depicted in Figure 3.16. Note that the arc of circle ending at $v_6^{\tau_0}$ cannot begin at $v_5^{\tau_0}$ because otherwise, from the third point of Lemma 3.18, and since $(6, 8) = (\tau_0(l), \tau_0(l'))$ with $3 = l < l' = 5$, it would force the arc of circle ending at $v_8^{\tau_0}$ to begin at $v_{i'}^{\tau_0}$ with $6 \leq i'$, which is absurd because a permutation $\sigma \in \mathfrak{S}_9$ whose linear graph would be of the kind $\mathcal{H}(\tau_0)$ would have $c_2(\sigma) = 2 \neq 1 = c_2^{\tau_0}$. Also, still in view of the third point of Lemma 3.18, and since $\tau_0^{-1}(9) < \tau^{-1}(6)$, the arc of circle ending at $v_9^{\tau_0}$ must start before the arc of circle ending at $v_6^{\tau_0}$, hence the configuration of $\mathcal{H}(\tau_0)$ in Figure 3.16.

The following two facts are obvious.

Fact 3.20. If $\tau = \Psi(\sigma)$ for some $\sigma \in \mathfrak{S}_n$, then :

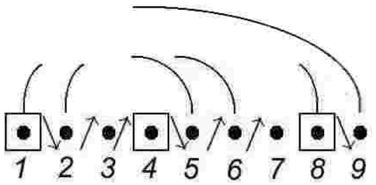


Figure 3.17: Incomplete graph $\mathcal{H}(\tau_0)$.

1. A vertex v_i^τ of $\mathcal{H}(\tau)$ is boxed if and only if $i \in \mathrm{DES}_2(\sigma)$. In that case, in particular i is a descent of σ.
2. If a pair $(i, i+1)$ is not a 2-descent of σ and if v_i^τ is not boxed, then i is an ascent of σ, *i.e.* $\sigma(i) < \sigma(i+1)$.

To reflect Facts 3.20, we draw an ascending arrow (respectively a descending arrow) between the vertices v_i^τ and v_{i+1}^τ of $\mathcal{H}(\tau)$ whenever it is known that $\sigma(i) < \sigma(i+1)$ (resp. $\sigma(i) > \sigma(i+1)$) for all $\sigma \in \mathfrak{S}_n$ such that $\Psi(\sigma) = \tau$.

For the example $\tau_0 = 956382471 \in \mathfrak{S}_9$, the graph $\mathcal{H}(\tau_0)$ becomes as depicted in Figure 3.17. Note that it is not known yet if there is an ascending or descending arrow between $v_7^{\tau_0}$ and $v_8^{\tau_0}$.

3.5.2 Completion and labelling of $\mathcal{H}(\tau)$

The following lemma is analogous to the third point of Lemma 3.18 for the dots instead of the circles and follows straightly from the definition of $\Psi(\sigma)$ for all $\sigma \in \mathfrak{S}_n$.

Lemma 3.21. *Let $\sigma \in \mathfrak{S}_n$ such that $\Psi(\sigma) = \tau$. For all pair $(l, l') \in ([n] \backslash \mathit{EXC}(\tau))^2$, if the labels of the two dots $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $\mathcal{G}(\sigma)$, let k and k' such that $l = p_k(\sigma)$ and $l' = p_{k'}(\sigma)$ (by hypothesis, we have $\tau(l') \leq p_k(\sigma)$), if we also suppose that $\tau(l') \leq u_k(\sigma)$, then $\tau(l) < \tau(l') \Leftrightarrow \sigma(u_k(\sigma)) < \sigma(u_{k'}(\sigma))$.*

Proof. This comes from Algorithm 3.16 in view of $\tau(l') \leq \min(u_k(\sigma), p_k(\sigma))$ and $\tau(l) \leq \min(u_k(\sigma), p_k(\sigma)) < \min(u_{k'}(\sigma), p_{k'}(\sigma))$. □

Now, the ascending and descending arrows between the vertices of $\mathcal{H}(\tau)$ introduced earlier, and Lemma 3.21, induce a partial order on the set $\{v_i^\tau : i \in [n]\}$:

Definition 3.22. We define a partial order $\succ$ on $\{v_i^\tau : i \in [n]\}$ by :

- $v_i^\tau \prec v_{i+1}^\tau$ (resp. $v_i^\tau \succ v_{i+1}^\tau$) if there exists an ascending (resp. descending) arrow between v_i^τ and v_{i+1}^τ;
- $v_i^\tau \succ v_j^\tau$ (with $i < j$) if there exists an arc of circle from v_i^τ to v_j^τ;
- if two vertices v_i^τ and v_j^τ are known to be respectively the k-th and k'-th vertices of $\mathcal{H}(\tau)$ that cannot be the beginning of a complete arc of circle, let l and l' be respectively the k-th and k'-th non-excedance point of τ (from left to right), if (l, l') fits the conditions of Lemma 3.21, then we set $v_i^\tau \prec v_j^\tau$ (resp. $v_i^\tau \succ v_j^\tau$) if $\tau(l) < \tau(l')$ (resp. $\tau(l) > \tau(l')$).

Example 3.23. For the example $\tau_0 = 956382471$, according to the first point of Definition 3.22, the arrows of Figure 3.17 provide

$$v_1^{\tau_0} \succ v_2^{\tau_0} \prec v_3^{\tau_0} \prec v_4^{\tau_0} \succ v_5^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0}$$

and

$$v_8^{\tau_0} \succ v_9^{\tau_0}.$$

Remark 3.24. As a consequence of Lemma 3.21, if v_u^τ and $v_{u'}^\tau$ (with $u < u'$) are two vertices that must not be the beginnings of arcs of circles (in order for $\mathcal{H}(\tau)$ to be the skeleton of the planar graph of a permutation σ such that $\Psi(\sigma) = \tau$), that correspond with two dots d and d' of the planar graph of τ whose abscissas are respectively l and l' (*i.e.*, if d and d' are respectively the p-th and p'-th dots of the planar graph of τ, then v_u^τ and $v_{u'}^\tau$ are respectively the p-th and p'-th vertices of $\mathcal{H}(\tau)$ not to be the beginnings of arcs of circles), and such that $\tau(l) < \tau(l')$ and $v_u^\tau \succ v_{u'}^\tau$, then it is necessary that $u < \tau(l')$.

Remark 3.24 eventually brings information on vertices of $\mathcal{H}(\tau)$ that won't be the beginnings of arcs of circles, and at any step of this section, we will immediatly complete the arcs of circles of the graph as soon as it is known that they must begin at a given vertex in view of Remark 3.24.

Definition 3.25. A vertex v_i^τ of $\mathcal{H}(\tau)$ is said to be *minimal* on a subset $S \subset [n]$ if $v_i^\tau \not\succ v_j^\tau$ for all $j \in S$.

Let

$$1 = e_1^\tau < e_2^\tau < \ldots < e_{n-s}^\tau$$

be the non-excedance values of τ (*i.e.*, the labels of the dots of the planar graph of τ).

Algorithm 3.26. Let $S = [n]$ and $l = 1$. While the vertices $\{v_i^\tau : i \in [n]\}$ have not all been labelled with the elements of $[n]$, apply the following algorithm.

1. If there exists a unique minimal vertex v_i^τ of τ on S, we label it with l, then we set $l := l + 1$ and $S := S \backslash \{v_i^\tau\}$. Afterwards,
   (a) If v_i^τ is the ending of an arc of circle starting from a vertex v_j^τ, then we label v_j^τ with the integer l and we set $l := l + 1$ and $S := S \backslash \{v_j^\tau\}$.
   (b) If v_i^τ is the arrival of an incomplete arc of circle (in particular $i = \tau(l)$ for some $l \in \mathrm{EXC}(\tau)$), we intend to complete the arc by making it start from a vertex v_j^τ for some integer $j \in [t_k^\tau, j[$ (where $l \in]d_2^{k,\tau}, d^{k,\tau}]$) in view of the first point of Lemma 3.18. We choose v_j^τ as the rightest minimal vertex on $[t_k^\tau, j[\cap S$ from which it may start in view of the third point of Lemma 3.18, and we label this vertex v_j^τ with the integer l. Then we set $l := l + 1$ and $S := S \backslash \{v_j^\tau\}$.

   Now, if there exists an arc of circle from v_j^τ (for some j) to v_i^τ, we apply above steps (a) and (b) to the vertex v_j^τ in place of v_i^τ.
2. Otherwise, let $k \geq 0$ be the number of vertices v_i^τ that have already been labelled and that are not the beginning of arcs of circles. Let
$$l_1 < l_2 < \ldots < l_q$$
   be the integers $l \in [n]$ such that $l \geq \tau(l) \geq e_{k+1}^\tau$ and such that we can exchange the labels of dots $\tau(l)$ and e_{k+1}^τ in the planar graph of τ without modifying the skeleton of the graph. It is easy to see that q is precisely the number of minimal vertices of τ on S. Let $l_{i_{k+1}} = \tau^{-1}(e_{k+1}^\tau)$ and let v_j^τ be the i_{k+1}-th minimal vertex (from left to right) on S. We label v_j^τ with l, then we set $l := l + 1$ and $S := S \backslash \{v_j^\tau\}$, and we apply steps 1.(a) and (b) to v_j^τ instead of v_i^τ.

By construction, the labelled graph $\mathcal{H}(\tau)$ is the linear graph of a permutation $\sigma \in \mathfrak{S}_n$ such that
$$\mathrm{DES}_2(\sigma) = \{d_2^{k,\tau} : k \in [r]\}$$
and
$$\{j_l(\sigma) : l \in [\mathrm{inv}_2(\sigma)]\} = \tau(\mathrm{EXC}(\tau)).$$

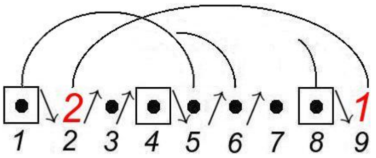


Figure 3.18: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

Example 3.27. Consider $\tau_0 = 956382471 \in \mathfrak{S}_9$ whose unlabelled and incomplete graph $\mathcal{H}(\tau_0)$ is depicted in Figure 3.17.

- As stated in Example 3.23, the minimal vertices of τ_0 on $S = [9]$ are $(v_2^{\tau_0}, v_5^{\tau_0}, v_9^{\tau_0})$. Following step 2 of Algorithm 3.26, $k = 0$ and the integers $l \in [9]$ such that $\tau_0(l) \geq e_{k+1}^{\tau_0} = 1$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 1 in the planar graph of τ_0 (see Figure 3.14) are $(l_1, l_2, l_3) = (4, 6, 9)$. By $\tau_0^{-1}(1) = 9 = l_3$, we label the third minimal vertex on [9], *i.e.* the vertex $v_9^{\tau_0}$, with the integer $l = 1$.

  Afterwards, following step 1.(b), since $v_9^{\tau_0}$ is the arrival of an incomplete arc of circle starting from a vertex $v_j^{\tau_0}$ with $1 = t_1^{\tau_0} \leq j$, and with $j < 5$ because that arc of circle must begin before the arc of circle ending at $v_6^{\tau_0}$ in view of Fact 3 of Lemma 3.18, we complete that arc of circle by making it start from the unique minimal vertex $v_j^{\tau_0}$ on $[1, 5[$, *i.e.* $j = 2$, and we label $v_2^{\tau_0}$ with the integer $l = 2$ (see Figure 3.18). Note that as from now we know that the arc of circle ending at $v_5^{\tau_0}$ necessarily begins at $v_1^{\tau_0}$, because otherwise $v_1^{\tau_0}$, being the beginning of an arc of circle, would be the beginning of the arc of circle ending at $v_6^{\tau_0}$, which is absurd in view of Fact 3 of Lemma 3.18 because $\tau_0^{-1}(9) < \tau^{-1}(6)$, so we complete that arc of circle by making it start from $v_1^{\tau_0}$, which has been depicted in Figure 3.18.

  We now have $S = [9]\backslash\{2, 9\}$ and $l = 3$.
- From Figure 3.18, the minimal vertices on $S = [9]\backslash\{2, 9\}$ are $(v_3^{\tau_0}, v_5^{\tau_0})$. Following step 2 of Algorithm 3.26, $k = 1$ and the integers $l \in [9]$ such that $l \geq \tau_0(l) \geq e_{k+1}^{\tau_0} = 2$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 2 in the planar graph of τ_0 (see Figure 3.14) are $(l_1, l_2) = (4, 6)$. By $\tau_0^{-1}(2) = 6 = l_2$, we label the second minimal vertex on S, *i.e.* the vertex $v_5^{\tau_0}$, with the integer $l = 3$.

  Afterwards, following step 1.(a), since $v_5^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_1^{\tau_0}$, we label $v_1^{\tau_0}$ with the integer $l = 4$ (see Figure 3.19).

  We now have $S = [9]\backslash\{1, 2, 5, 9\}$ and $l = 5$.

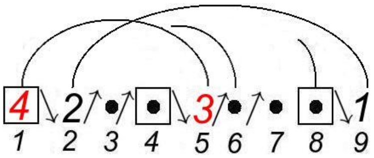


Figure 3.19: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

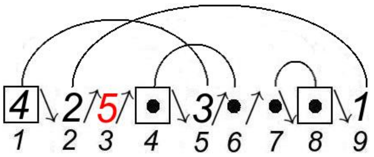


Figure 3.20: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

- From Figure 3.19, the minimal vertices on $S = \{3, 4, 6, 7, 8\}$ are $(v_3^{\tau_0}, v_6^{\tau_0})$. Following step 2 of Algorithm 3.26, $k = 2$ and the integers $l \in [9]$ such that $l \geq \tau_0(l) \geq e_{k+1}^{\tau_0} = 3$ and such that the labels of dots $\tau_0(l)$ can be exchanged with 3 in the planar graph of τ_0 (see Figure 3.14) are $(l_1, l_2) = (4, 7)$. By $\tau_0^{-1}(3) = 4 = l_1$, we label the first minimal vertex on S, *i.e.* the vertex $v_3^{\tau_0}$, with the integer $l = 5$ (see Figure 3.20). Note that as from now we know that the arc of circle ending at $v_6^{\tau_0}$ necessarily begins at $v_4^{\tau_0}$ since it it is the only vertex left it may start from. Consequently, the arc of circle ending at $v_8^{\tau_0}$ necessarily starts from $v_7^{\tau_0}$ (otherwise it would start from $v_6^{\tau_0}$, which is prevented by Definition 3.22 because we cannot have $v_8^{\tau_0} \prec v_6^{\tau_0} \prec v_7^{\tau_0} \prec v_8^{\tau_0}$). The two latter remarks are taken into account in Figure 3.20.
  We now have $S = \{4, 6, 7, 8\}$ and $l = 6$.
- From Figure 3.20, there is only one minimal vertex on $S = \{4, 6, 7, 8\}$, *i.e.* the vertex $v_6^{\tau_0}$. Following step 1 of Algorithm 3.26, we label $v_6^{\tau_0}$ with $l = 6$.
  Afterwards, following step 1.(a), since $v_6^{\tau_0}$ is the arrival of the arc of circle starting from the vertex $v_4^{\tau_0}$, we label $v_4^{\tau_0}$ with the integer $l = 7$ (see Figure 3.21).
  We now have $S = \{7, 8\}$ and $l = 8$.
- From Figure 3.21, there is only one minimal vertex on $S = \{7, 8\}$, *i.e.* the vertex $v_6^{\tau_0}$. Following step 1 of Algorithm 3.26, we label $v_6^{\tau_0}$ with $l = 8$.
  Afterwards, following step 1.(a), since $v_8^{\tau_0}$ is the arrival of the arc of

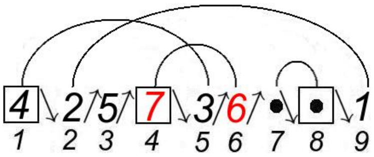


Figure 3.21: Beginning of the labelling of $\mathcal{H}(\tau_0)$.

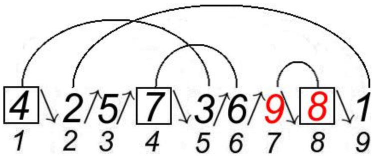


Figure 3.22: Labelled graph $\mathcal{H}(\tau_0)$.

circle starting from the vertex $v_7^{\tau_0}$, we label $v_7^{\tau_0}$ with the integer $l = 9$ (see Figure 3.22).

As a conclusion, the graph $\mathcal{H}(\tau_0)$ is the linear graph of the permutation $\sigma_0 = 425736981 \in \mathfrak{S}_9$, which is mapped to τ_0 by Ψ.

Proposition 3.28. *We have $\Psi(\sigma) = \tau$, hence Ψ is bijective.*

Proof. We first show that $\mathcal{G}(\sigma_\tau)$ and the planar graph of τ have the same skeleton. By construction of σ_τ, for all $k \in [0, \mathrm{des}_2(\sigma_\tau)] = [0, r]$,

$$d_2^k(\sigma_\tau) = d^{k,\tau} - c_k^\tau,$$
$$c_k^0(\sigma_\tau) - c_k^\tau \in \{-1, 0, 1\}.$$

With precision, we know that if $c_k^0(\sigma_\tau) = c_k^\tau - 1$ for some k (which easily implies $k < r$), then $c_{k+1}^0(\sigma_\tau) = c_{k+1}^\tau + 1$. For any such k, it is easy to see that necessarily $d_2^k(\sigma_\tau) < d_2^{k+1}(\sigma_\tau) - 1$ and that the $\big(d_2^{k+1,\tau}\big)$-th and $\big(d^{k+1,\tau} + 1\big)$-th vertices of the planar graph of τ are dots, which implies that the vertices $v^0_{d_2^{k+1}(\sigma_\tau)-1}$ and $v^0_{d_2^{k+1}(\sigma_\tau)}$ of $\mathcal{G}^0(\sigma_\tau)$ are two consecutive dots, say, the p-th and $(p+1)$-th dots of $\mathcal{G}^0(\sigma_\tau)$. Now, following step (2)(b) of Algorithm 3.26, the vertex $v^\tau_{d_2^{k+1,\tau}}$ of $\mathcal{H}(\tau)$ necessarily is the beginning of an arc of circle, and in order to respect Lemma 3.21 as requested by this step, it is necessary that $\sigma_\tau(u_p(\sigma_\tau)) > \sigma_\tau(u_{p+1}(\sigma_\tau))$, which, following Definition 3.7 of $\mathcal{G}^0(\sigma)$, implies that $d_2^{k+1,\tau} - 1 = d_2^{k+1}(\sigma_\tau) - 1$ is a descent of $\mathcal{G}^0(\sigma_\tau)$ whereas $d_2^{k+1}(\sigma_\tau) - 1 \notin \mathrm{DES}_2(\sigma_\tau)$ since we have $d_2^k(\sigma_\tau) < d_2^{k+1}(\sigma_\tau) - 1 < d_2^{k+1}(\sigma_\tau)$. So, following Lemma 3.9 and Definition 3.10 of

$\mathcal{G}(\sigma_\tau)$, we obtain $(c_k(\sigma_\tau), c_{k+1}(\sigma_\tau)) = (c_k^\tau, c_{k+1}^\tau)$. Consesquently $\mathcal{G}(\sigma_\tau)$ and the planar graph of τ have the same skeleton, *i.e.* $\mathrm{DES}(\Psi(\sigma_\tau)) = \mathrm{DES}(\tau)$ and $\mathrm{EXC}(\Psi(\sigma_\tau)) = \mathrm{EXC}(\tau)$.

Afterwards, the labels of the circles of $\mathcal{G}(\sigma_\tau)$ are the elements of

$$\{j_l(\sigma_\tau) : l \in [s]\} = \tau(\mathrm{EXC}(\tau)),$$

and following step (2)(b) of Algorithm 3.26, which respects the third point of Lemma 3.18, every pair $(l, l') \in \mathrm{EXC}(\tau)^2$ such that we can exchange the labels $\tau(l)$ and $\tau(l')$ in the planar graph of τ is such that

$$i < i' \Leftrightarrow l < l'$$

where $(i, \tau(l))$ and $(i, \tau(l'))$ are the two corresponding 2-inversions of σ_τ. Consequently, by definition of $\Psi(\sigma_\tau)$, the labels of the circles of $\mathcal{G}(\sigma_\tau)$ appear in the same order as in the planar graph of τ (*i.e.* $\Psi(\sigma_\tau)(i) = \tau(i)$ for all $i \in \mathrm{EXC}(\Psi(\sigma_\tau)) = \mathrm{EXC}(\tau)$).

As a consequence, the dots of $\mathcal{G}(\sigma_\tau)$ and the planar graph of τ are labelled with the elements

$$1 = e_1(\sigma_\tau) = e_1^\tau < e_2(\sigma_\tau) = e_2^\tau < \ldots < e_{n-s}(\sigma_\tau) = e_{n-s}^\tau.$$

To show that the above integers appear in the same order in $\mathcal{G}(\sigma_\tau)$ and the planar graph of τ, it suffices to prove that

$$\Psi(\sigma_\tau)^{-1}(e_i^\tau) < \Psi(\sigma_\tau)^{-1}(e_j^\tau) \Leftrightarrow \tau^{-1}(e_i^\tau) < \tau^{-1}(e_j^\tau) \tag{3.11}$$

for all pair (i, j) with $i < j$ and such that we can exchange the labels e_i^τ and e_j^τ in the planar graph of τ. Let $i \in [n-s-1]$ and suppose that the equivalence is true for all $i' \leq i-1 \in [0, n-s-2]$ and for all $j \in]i', n-s]$ where we set $\Psi(\sigma_\tau)(0) = \tau(0) = 0$ (so the equivalence is true for $i' = 0$ and for all $j \in [n-s]$). In particular $\Psi(\sigma_\tau)^{-1}(e_{i'}^\tau) = \tau^{-1}(e_{i'}^\tau)$ for all $i' < i$. Consider $j \in]i, n-s]$, and let $(k, k') \in [n-s]^2$ and $(l, l') \in ([n] \backslash \mathrm{EXC}(\tau))^2$ such that $(\Psi(\sigma_\tau)^{-1}(e_i^\tau), \Psi(\sigma_\tau)^{-1}(e_j^\tau)) = (p_k(\sigma_\tau), p_{k'}(\sigma_\tau))$ and $(\tau^{-1}(e_i^\tau), \tau^{-1}(e_j^\tau)) = (l, l')$. The equivalence of (3.11) is straightforward if the label of $v_{u_k(\sigma_\tau)}^\tau$ has been defined in the context (1)(b) of Algorithm 3.26. Now, since the equivalence of (3.11) is true for all $i' < i$, if there exists $l'' \in]\min(l, l'), \max(l, l')[$ such that $\tau(l'') \leq l''$ and $\tau(l'')$ is of the kind $e_{i'}^\tau$ with $i' < i$, then the equivalence of (3.11) is true by transitivity. Otherwise, for all $l'' \in]\min(l, l'), \max(l, l')[$, either $\tau(l'') > l''$, or $\tau(l'')$ is of the kind $e_{i''}^\tau$ with

$i'' > i$. In fact, since the labels e_i^τ and e_j^τ can be exchanged by hypothesis, then $]\min(l,l'),\max(l,l')[$ is not empty and $\tau(l'') > \max(e_i^\tau, e_j^\tau) = e_j^\tau$ for all $l'' \in]\min(l,l'),\max(l,l')[$. It implies that $u_k(\sigma_\tau)$ and $u_{k'}(\sigma_\tau)$ are both minimal vertices on S at the step of the computation of the label of $v^\tau_{u_k(\sigma_\tau)}$, thence we are in the context (1)(b) of Algorithm 3.26 (we are not in the context (2) because $v^\tau_{u_k(\sigma_\tau)}$ is not the beginning of an arc of circle by definition of $u_k(\sigma_\tau)$) and the equivalence of (3.11) is true in this situation as stated earlier.

As a conclusion, the planar graph of τ is in fact $\mathcal{G}(\sigma)$, *i.e.* $\tau = \Psi(\sigma)$. □

3.6 Extension

For all subset S of $[n-1]$, let $D(S)$ be the set of functions f from $[n]$ to $\mathbb{N}_{>0} = \{1,2,\ldots\}$ such that $f(i) \geq f(i+1)$ for all $i \in [n-1]$ and $f(i) > f(i+1)$ for all $i \in S$. The functions

$$F_{n,S} = \sum_{f \in D(S)} x_{f(1)} x_{f(2)} \cdots$$

for $S \subset [n-1]$ are named Gessel's fundamental quasisymmetric functions. They form a basis for the free $\mathbb{Z}$-module of homogeneous degree n quasisymmetric functions with coefficients in $\mathbb{Z}$ (see [SW12]).

Afterwards, let $[\bar{n}] = \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$, we totally order the alphabet $[\bar{n}] \sqcup [n]$ by

$$\bar{1} < \bar{2} < \ldots < \bar{n} < 1 < 2 < \ldots < n.$$

For a permutation $\tau = \tau(1)\tau(2)\ldots\tau(n) \in \mathfrak{S}_n$, let $\bar{\tau}$ be the word over $[\bar{n}] \sqcup [n]$ obtained from τ by replacing $\tau(i)$ with $\overline{\tau(i)}$ for all $i \in \mathrm{EXC}(\tau)$. For example, if $\tau = \Psi(\sigma_0) = 956382471$ (see Figure 3.18), we obtain $\bar{\tau} = \bar{9}\bar{5}\bar{6}3\bar{8}2471$. We define a *descent* in a word $\omega = \omega_1\omega_2\ldots\omega_n$ over any totally ordered alphabet to be any $i \in [n-1]$ such that $\omega_i > \omega_{i+1}$. Now, for all $\tau \in \mathfrak{S}_n$, let

$$\mathrm{DEX}(\tau) := \mathrm{DES}(\bar{\tau}).$$

For the example $\tau = \Psi(\sigma_0) = 956382471$, we obtain $\mathrm{DEX}(\tau) = \{1,4,8\} = \mathrm{DES}_2(\sigma_0)$. In general, for all $\sigma \in \mathfrak{S}_n$, since $\mathrm{EXC}(\Psi(\sigma)) = \bigsqcup_{k=0}^{r}]d_2^k(\sigma), d^k(\sigma)]$ in view of Lemma 3.12, it is straightforward that $\mathrm{DEX}(\Psi(\sigma)) = \mathrm{DES}_2(\sigma)$. Consequently, Theorem 3.3 proves bijectively the following quasisymmetric function generalization of (3.2) obtained by Shareshian and Wachs (see [[SW12], Eq. (4.8)]):

$$\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_2(\sigma)} F_{n,\mathrm{DES}_2(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} F_{n,\mathrm{DEX}(\sigma)}. \tag{3.12}$$

3.7 Open problem

In view of Formula (3.3) and Theorem 3.3, it is natural to look for a bijection $\mathfrak{S}_n \to \mathfrak{S}_n$ that maps $(\mathrm{maj}_2, \widetilde{\mathrm{des}_2}, \mathrm{inv}_2)$ to $(\mathrm{amaj}_2, \widetilde{\mathrm{asc}_2}, \mathrm{inv}_2)$.

Recall that for a permutation $\sigma \in \mathfrak{S}_n$, the equality $\widetilde{\mathrm{des}_2}(\sigma) = \mathrm{des}_2(\sigma)$ is equivalent to $\Psi(\sigma)(1) = 1$, which is similar to the equivalence $\widetilde{\mathrm{asc}_2}(\tau) = \mathrm{asc}_2(\tau) \Leftrightarrow \tau(1) = 1$ for all $\tau \in \mathfrak{S}_n$.

Note that if $\mathrm{DES}_2(\sigma) = \bigsqcup_{p=1}^{r} [i_p, j_p]$ with $j_p + 1 < i_{p+1}$ for all p, the permutation $\pi = \rho_1 \circ \rho_2 \circ \ldots \circ \rho_r \circ \sigma$, where ρ_p is the $(j_p - i_p + 2)$-cycle

$$\begin{pmatrix} i_p & i_p+1 & i_p+2 & \ldots & j_p & j_p+1 \\ \sigma(j_p+1) & \sigma(j_p) & \sigma(j_p-1) & \ldots & \sigma(i_p+1) & \sigma(i_p) \end{pmatrix}$$

for all p, is such that $\mathrm{DES}_2(\sigma) \subset \mathrm{ASC}_2(\pi)$ and $\mathrm{INV}_2(\sigma) = \mathrm{INV}_2(\pi)$. One can try to get rid of the eventual unwanted 2-ascents $i \in \mathrm{ASC}_2(\pi) \setminus \mathrm{DES}_2(\sigma)$ by composing π with adequate permutations.

See also [Lin13, LZ15] for other interpretations and problems related to the polynomials which appear in this chapter.

Appendix A

Dellac configurations and Genocchi numbers

A.1 Proof of the statistic preservation formula (1.6)

First notice that Formula formula (1.6) is true for $C = C_1(n)$, the unique Dellac configuration with $\binom{n}{2}$ falls (see Definition 1.14): indeed $\phi(C_1(n))$ is the involution $\sigma = 214365\ldots(2n+2)(2n+1) \in \mathcal{D}'_{n+1}$, consequently the two words

$$\phi(C_1(n))^e = 135\ldots(2n+1)$$
$$\phi(C_1(n))^o = 246\ldots(2n+2)$$

have no inversion, hence

$$\mathrm{st}(\phi(C_1(n))) = (n+1)^2 - (1+3+5+\ldots+(2n+1)) = 0.$$

Let $C \in \mathrm{DC}(n)$. From Lemma 1.39, there exists a finite sequence of switching transformations $(C^0, C^1, \ldots, C^m)$ from $C^0 = C_1(n)$ to $C^m = C$. For all $k \in \{0, 1, \ldots, m-1\}$, let $i_k \in [2n]$ such that $C^{k+1} = Sw^{i_k}(C^k)$. We can suppose that $C^{k+1} \neq C^k$, *i.e.*, that $\mathrm{fal}(C^{k+1}) = \mathrm{fal}(C^k) \pm 1$. Since Formula (1.6) is true for $C_1(n)$, it will be true for C by induction if we show that

$$\mathrm{st}(\phi(C^{k+1})) - \mathrm{st}(\phi(C^k)) = \mathrm{fal}(C^k) - \mathrm{fal}(C^{k+1})$$

for all k. We know that the number $\mathrm{fal}(C^k) - \mathrm{fal}(C^{k+1})$ equals ± 1. From Fact 1.19, we have $Sw^{i_k}(C^{k+1}) = C^k$. Then, provided that C^k is replaced by

$Sw^{i_k}(C^k) = C^{k+1}$, we can assume that the number $\mathrm{fal}(C^k) - \mathrm{fal}(C^{k+1})$ equals 1, which means the pair $(e_{i_k}, e_{i_{k+1}})$ is a fall of C^k. Consequently, to achieve the proof of Theorem 1.23, it suffices to prove the equality

$$\mathrm{st}(\phi(C^{k+1})) - \mathrm{st}(\phi(C^k)) = 1 \tag{A.1}$$

under the hypothesis $\mathrm{fal}(C^k) - \mathrm{fal}(C^{k+1}) = 1$. Let $\sigma_k = \phi(C^k)$ and $\sigma_{k+1} = \phi(C^{k+1})$. Since e_{i_k} and e_{i_k+1} are not in the same column of C^k, we have $\sigma_{k+1} = \sigma_k \circ (e_{i_k}, e_{i_k+1})$ in view of Proposition 1.38.

(a) If e_{i_k} and e_{i_k+1} have the same parity (which is always true except for $i_k = n$), then the two integers e_{i_k} and e_{i_k+1} appear in the same subset $\{1, 3, \dots, 2n+1\}$ or $\{2, 4, \dots, 2n+\ 2\}$. Consequently, we obtain the two equalities

$$\sum_{i=1}^{n+1} \sigma_{k+1}(2i) = \sum_{i=1}^{n+1} \sigma_k(2i)$$

and

$$(\mathrm{fal}(\sigma^e_{k+1}) - \mathrm{fal}(\sigma^e_k), \mathrm{fal}(\sigma^o_{k+1}) - \mathrm{fal}(\sigma^o_k)) = (-1, 0) \text{ or } (0, -1),$$

thence $\mathrm{st}(\sigma_{k+1}) = \mathrm{st}(\sigma_k) + 1$, which brings Equality (A.1).

(b) Else $i_k = n$ and $(e_{i_k}, e_{i_k+1}) = (2n+2, 1)$. From $\sigma_{k+1} = \sigma_k \circ (e_{i_k}, e_{i_k+1})$, we obtain

$$\begin{aligned} \sigma^e_{k+1} &= \sigma_k(2)\sigma_k(4)\dots\sigma_k(2n)\sigma_k(1), \\ \sigma^o_{k+1} &= \sigma_k(2n+2)\sigma_k(3)\sigma_k(5)\dots\sigma_k(2n+1). \end{aligned}$$

This provides the three following equations.

$$\sum_{i=1}^{n+1} \sigma_{k+1}(2i) = \left(\sum_{i=1}^{n+1} \sigma_k(2i)\right) - \sigma_k(2n+2) + \sigma_k(1), \tag{A.2}$$

$$\begin{aligned} \mathrm{fal}(\sigma^e_{k+1}) = \mathrm{fal}(\sigma^e_k) &- |\{2i < 2n+2 : \sigma_k(2i) > \sigma_k(2n+2)\}| \\ &+ |\{2i < 2n+2 : \end{aligned} \tag{A.3}$$

$$\begin{aligned} \mathrm{fal}(\sigma^o_{k+1}) = \mathrm{fal}(\sigma^o_k) &- |\{1 < 2i+1 : \sigma_k(2i+1) < \sigma_k(1)\}| \\ &+ |\{1 < 2i+1 : \sigma_k(2i+1) < \sigma_k(2n+2)\}|. \end{aligned} \tag{A.4}$$

We need the following lemma to make explicit Equalities (A.3) and (A.4).

Lemma A.1. *We have the equalities*

$$|\{2i < 2n+2 : \sigma_k(2i) > \sigma_k(2n+2)\}| = r_{C^k}(2n+2) + \left(1 + (-1)^{\sigma_k(2n+2)}\right)/2, \tag{A.5}$$

$$|\{2i < 2n+2 : \sigma_k(2i) > \sigma_k(1)\}| = r_{C^k}(1) - \left(1 - (-1)^{\sigma_k(1)}\right)/2, \tag{A.6}$$

$$|\{1 < 2i+1 : \sigma_k(2i+1) < \sigma_k(1)\}| = l_{C^k}(1) + \left(1 - (-1)^{\sigma_k(1)}\right)/2, \tag{A.7}$$

$$|\{1 < 2i+1 : \sigma_k(2i+1) < \sigma_k(2n+2)\}| = l_{C^k}(2n+2) - \left(1 + (-1)^{\sigma_k(2n+2)}\right)/2. \tag{A.8}$$

Proof. We only demonstrate Equalities (A.5) and (A.6), because the proof of (A.7) is analogous to that of (A.5) and the proof of (A.8) is analogous to that of (A.6).

- Proof of (A.5): if the dot $e_{i_k} = 2n+2$ appears in the j_k-th column of C^k, and if the dot $e_{i-1} = 2i$ (with $1 \leq i-1 \leq n = i_k$) appears in the j_{i-1}-th column of C^k, then $\sigma_k(2n+2) \in \{2j_k, 2j_k+1\}$ and $\sigma_k(2i) \in \{2j_{i-1}, 2j_{i-1}+1\}$. Consequently, the two following assertions are equivalent:
  - $\sigma_k(2i) > \sigma_k(2n+2)$;
  - either $j_{i-1} > j_k$, or $j_{i-1} = j_k$ and $\sigma_k(2n+2) = 2j_{i-1}$ (which forces $\sigma_k(2i)$ to be $2j_{i-1}+1$).

  As a result,

  $$|\{2i < 2n+2 : \sigma_k(2i) > \sigma_k(2n+2)\}| = r_{C^k}(2n+2) + \delta_{\sigma_k(2n+2)}$$

  where $\delta_{\sigma_k(2n+2)} = 1$ if $\sigma_k(2n+2)$ is even, and $\delta_{\sigma_k(2n+2)} = 0$ if $\sigma_k(2n+2)$ is odd, *i.e.*, where $\delta_{\sigma_k(2n+2)} = \left(1 + (-1)^{\sigma_k(2n+2)}\right)/2$.
- Proof of (A.6): with the same reasoning as for (A.5), we find the equality

  $$|\{2i < 2n+2 : \sigma_k(2i) > \sigma_k(1)\}| = r_{C^k}(1) - 1 + \left(1 + (-1)^{\sigma_k(1)}\right)/2$$

  (with $r_{C^k}(1) - 1$ instead of $r_{C^k}(1)$ because there is a fall from $1 = e_{i_{k+1}}$ to $2n+2 = e_{i_k}$, whereas $2n+2$ is not counted in the number $|\{2i < 2n+2 : \sigma_k(2i) > \sigma_k(1)\}|$). Since $-1 + \left(1 + (-1)^{\sigma_k(1)}\right)/2 = -\left(1 - (-1)^{\sigma_k(1)}\right)/2$, we obtain (A.6).

□

In view of Lemma A.1, Equalities (A.3) and (A.4) become

$$\mathrm{fal}(\sigma^e_{k+1}) - \mathrm{fal}(\sigma^e_k) = r_{C^k}(1) - r_{C^k}(2n+2) - 1 + \left((-1)^{\sigma_k(1)} - (-1)^{\sigma_k(2n+2)}\right)/2, \tag{A.9}$$

$$\mathrm{fal}(\sigma^o_{k+1}) - \mathrm{fal}(\sigma^o_k) = l_{C^k}(2n+2) - l_{C^k}(1) - 1 + \left((-1)^{\sigma_k(1)} - (-1)^{\sigma_k(2n+2)}\right)/2. \tag{A.10}$$

Now, from Lemma 1.35, we know that

$$\sigma_k(1) = y_{n+1+l_{C^k}(1) - r_{C^k}(1)},$$
$$\sigma_k(2n+2) = y_{n+l_{C^k}(2n+2) - r_{C^k}(2n+2)}.$$

From $y_i = i + 1 - (-1)^i$ for all i, we deduce the two following formulas.

$$\sigma_k(1) = n + 2 + (-1)^n + l_{C^k}(1) - r_{C^k}(1) + (-1)^{n+1}\left(1 - (-1)^{l_{C^k}(1) - r_{C^k}(1)}\right), \tag{A.11}$$

$$\begin{aligned}\sigma_k(2n+2) = n + 1 - (-1)^n + l_{C^k}(2n+2) - r_{C^k}(2n+2) \\ + (-1)^n\left(1 - (-1)^{l_{C^k}(2n+2) - r_{C^k}(2n+2)}\right).\end{aligned} \tag{A.12}$$

By substituting Equalities (A.11) and (A.12) in Equalities (A.2), (A.9) and (A.10), we obtain the three new equalities

$$\begin{aligned}\sum_{i=1}^{n+1} \sigma_{k+1}(2i) - \sum_{i=1}^{n+1} \sigma_k(2i) = 1 + l_{C^k}(1) - l_{C^k}(2n+2) + r_{C^k}(2n+2) - r_{C^k}(1) \\ + (-1)^{n+l_{C^k}(1) - r_{C^k(1)}} + (-1)^{n+l_{C^k}(2n+2) - r_{C^k}(2n+2)},\end{aligned} \tag{A.13}$$

$$\begin{aligned}\mathrm{fal}(\sigma^e_{k+1}) - \mathrm{fal}(\sigma^e_k) = r_{C^k}(1) - r_{C^k}(2n+2) - 1 \\ - \left((-1)^{n+l_{C^k}(1) - r_{C^k}(1)} + (-1)^{n+l_{C^k}(2n+2) - r_{C^k}(2n+2}\right)/2,\end{aligned} \tag{A.14}$$

$$\begin{aligned}\mathrm{fal}(\sigma^o_{k+1}) - \mathrm{fal}(\sigma^o_k) = l_{C^k}(2n+2) - l_{C^k}(1) - 1 \\ - \left((-1)^{n+l_{C^k}(1) - r_{C^k}(1)} + (-1)^{n+l_{C^k}(2n+2) - r_{C^k}(2n+2}\right)/2.\end{aligned} \tag{A.15}$$

Finally, we obtain Equality (A.1) by summing Equalities (A.13), (A.14) and (A.15). This proves Theorem 1.23.

A.2 Proof of the bijectivity of ψ : DC(n) $\rightarrow$ DH(n)

Definition A.2. Let $S = (\gamma, \xi) \in \mathrm{DH}(n)$ with $\gamma = (p_0, p_1, \ldots, p_{2n})$ and $\xi = (\xi_1, \ldots, \xi_n)$. We define $\psi(S)$ as a tableau T of width n and height $2n$, in which we insert the $2n$ dots $e_1, e_2, \ldots, e_{2n}$ according to the two following (analogous and independant) algorithms.

1. **Insertion of the n odd dots $e_{n+1}, e_{n+2}, \ldots, e_{2n}$.** Let $\mathcal{I}_0^o$ be the sequence $(1, 2, \ldots, n)$. For $i = 1$ to n, consider $j_i \in [n]$ such that the i-th down step s_i^d of γ is one of the two steps (p_{2j_i-2}, p_{2j_i-1}) or (p_{2j_i-1}, p_{2j_i}). If the sequence $\mathcal{I}_{i-1}^o$ is defined, we denote by $H(i)$ the hypothesis "$\mathcal{I}_{i-1}^o$ has length $n+1-i$ such that for all $j \in \{i, i+1, \ldots, n\}$, the $(j-i+1)$-th element of $\mathcal{I}_{i-1}^o$ is inferior to $n+j$". If the hypothesis $H(i+1)$ is true, then we iterate the algorithm to $i+1$. At the beginning, the sequence $\mathcal{I}_0^o$ is defined and $H(1)$ is obviously true so we can initiate the algorithm.
   (a) If s_i^d is a down step in the context 1. or 2. of Definition 1.45, let $(n_1, n_2) = \xi_i$. In particular, since $n_2 \leq k = j_i - i$ (see Remark 1.44) and $j_i \leq n$, we have $1 + n_2 \leq n - i + 1$ so, from Hypothesis $H(i)$, we can consider the $(1+n_2)$-th element of $\mathcal{I}_{i-1}^o$, say, the integer q. We insert the odd dot e_{n+q} in the j_i-th column of T. From Hypothesis $H(i)$, the $(j_i - i + 1)$-th element of $\mathcal{I}_{i-1}^o$ is inferior to $n + j_i$, and $1 + n_2 \leq 1 + k = j_i - i + 1$. Consequently, the dot e_{n+q} is between the lines $y = x$ and $y = x + n$. Afterwards, we define $\mathcal{I}_i^o$ as the sequence $\mathcal{I}_{i-1}^o$ from which we have removed q (by abusing the notation, we write $\mathcal{I}_i^o := \mathcal{I}_{i-1}^o \backslash \{q\}$). Thus, the sequence $\mathcal{I}_i^o$ has length $n + 1 - (i+1)$. Also, if $j \in \{i+1, i+2, \ldots, n\}$, then following Hypothesis $H(i)$, the $(j-i)$-th element of $\mathcal{I}_{i-1}^o$ is inferior to $n + j - 1$, so the $(j - (i+1) + 1)$-th element of $\mathcal{I}_i^o$ is inferior to $n + j - 1 < n + j$. Therefore, Hypothesis $H(i+1)$ is true and we can iterate the algorithm to $i+1$.
   (b) If s_i^d and s_{i+1}^d are two consecutive down steps in the context 3. of Definition 1.45, let $(n_1, n_2) = \xi_{i+1}$. In particular $n_1 \leq n_2 \leq k-1 = j_i - i - 1 \leq n - i - 1$, so $1 + n_1 < 2 + n_2 \leq j_i - i + 1$. Consequently, following Hypothesis $H(i)$, we can consider the $(1+n_1)$-th element of $\mathcal{I}_{i-1}^o$, say, the integer q_1, and the $(2+n_2)$-th element of $\mathcal{I}_{i-1}^o$, say, the integer $q_2 > q_1$. We insert the two odd dots e_{n+q_1} and e_{n+q_2}

in the j-th column of T. With precision, by the same argument as for (a), those two dots are located between the lines $y = x$ and $y = x + n$. Afterwards, we set $\mathcal{I}^o_{i+1} := \mathcal{I}^o_{i-1} \setminus \{q_1, q_2\}$. Thus, the sequence $\mathcal{I}^o_{i+1}$ has length $n-(i+2)+1$, and if $j \in \{i+2, i+3, \dots, n\}$ then, by Hypothesis $H(i)$, the $(j-i-1)$-th element of $\mathcal{I}^o_{i-1}$ is inferior to $n+j-2$, so the $(j-(i+2)+1)$-th element of $\mathcal{I}^o_{i+1}$ is inferior to $n+j-2 < n+j$. Therefore, Hypothesis $H(i+2)$ is true and we can iterate the algorithm to $i+2$.

2. **Insertion of the n even dots $e_1, e_2, \dots, e_n$.** Let $\mathcal{I}^e_0 = (n, n-1, \dots, 1)$. For $i = 1$ to n, consider $j_i \in [n]$ such that the $(n+1-i)$-th up step s^u_{n+1-i} of γ is one of the two steps (p_{2j_i-2}, p_{2j_i-1}) or (p_{2j_i-1}, p_{2j_i}). If the sequence $\mathcal{I}^e_{i-1}$ is defined, we denote by $H'(i)$ the hypothesis "$\mathcal{I}^e_{i-1}$ has length $n+1-i$ such that for all $j \in [n-i+1]$, the $(n-i+2-j)$-th element of $\mathcal{I}^o_{i-1}$ is greater than j". If Hypothesis $H'(i+1)$ is true, we iterate the algorithm to $i+1$. In particular, the set $\mathcal{I}^e_0$ is defined and $H'(1)$ is true so we can initiate the algorithm.

   (a) If s^u_{n+1-i} is an up step in the the context 1. or 2. of Definition 1.45, then let $i_0 \in [n]$ such that $\{(p_{2j_i-2}, p_{2j_i-1}), (p_{2j_i-1}, p_{2j_i})\} = \{s^u_{n+1-i}, s^d_{i_0}\}$. Let $(n_1, n_2) = \xi_{i_0}$. From Remark 1.44, we have $1+n_1 \leq 1+k = n-i+2-j_i \leq n-i+1$ so, following Hypothesis $H'(i)$, we can consider the $(1+n_1)$-th element of $\mathcal{I}^e_{i-1}$, say, the integer p. We insert the even dot e_p in the j_i-th column of T. By Hypothesis $H'(i)$, the $(n-i+2-j_i)$-th element of $\mathcal{I}^e_{i-1}$ is greater than j_i, and $1+n_1 \leq 1+k = n-i-j_i+2$ so the dot e_p is located between the lines $y = x$ and $y = x+n$. Afterwards, we set $\mathcal{I}^e_i := \mathcal{I}^e_{i-1} \setminus \{p\}$. The sequence $\mathcal{I}^e_i$ has length $n+1-(i+1)$. Also, if $j \in \{1, 2, \dots, n+1-(i+1)\}$, then, by Hypothesis $H'(i)$, the $(n-i-j)$-th element of $\mathcal{I}^e_{i-1}$ is greater than $j+1$, so the $(n-(i+1)+1-j)$-th element of $\mathcal{I}^e_i$ is greater than $j+1 > j$. Therefore, Hypothesis $H'(i+1)$ is true and we can iterate the algorithm to $i+1$.

   (b) If $s^u_{n+1-(i+1)}$ and s^u_{n+1-i} are two consecutive up steps (p_{2j_i-2}, p_{2j_i-1}) and (p_{2j_i-1}, p_{2j_i}) from level $2k-2$ towards level $2k$ in γ, let $j_0 > j_i$ such that the two steps (p_{2j_0-2}, p_{2j_0-1}) and (p_{2j_0-1}, p_{2j_0}) are the next two consecutive down steps $s^d_{i_0}$ and $s^d_{i_0+1}$ from level $2k$ towards level $2k-2$ (see Figure 1.9). Let $(n_1, n_2) = \xi_{i_0}$. Being in the context 3. of Definition 1.45, we have $n_2 \leq n_1 \leq k-1 = n-i-j_0 \leq$

$n-i-1$, hence $1+n_2 < 2+n_1 \leq n-i+1$. Consequently, by Hypothesis $H'(i)$, we can consider the $(1+n_2)$-th element of $\mathcal{I}^e_{i-1}$, say, the integer p_1, and the $(2+n_1)$-th element of $\mathcal{I}^e_{i-1}$, say, the integer $p_2 < p_1$. We insert the two even dots e_{p_2} and e_{p_1} in the j_i-th column of T. With precision, for the same argument as for (a), those two dots are between the lines $y = x$ and $y = x+n$. Afterwards, we set $\mathcal{I}^e_{i+1} := \mathcal{I}^e_{i-1} \backslash \{p_2, p_1\}$. The sequence $\mathcal{I}^e_{i+1}$ has length $n-(i+2)+1$. Also, if $j \in \{1, 2, \ldots, n+1-(i+2)\}$, then by Hypothesis $H'(i)$, the $(n-i-j)$-th element of $\mathcal{I}^e_{i-1}$ is greater than $j+2$, so the $(n-(i+2)+2-j)$-th element of $\mathcal{I}^e_{i+1}$ is greater than $j+2 > j$. Therefore, Hypothesis $H'(i+2)$ is true and we can iterate the algorithm to $i+2$.

By construction, it is clear that $\tilde{\psi}(S) = T$ is a Dellac configuration.

Remark A.3. Let $S = (\gamma, \xi) \in \mathrm{DH}(n)$ and $C = \tilde{\psi}(S) \in \mathrm{DC}(n)$. For all $i \in [n]$, the i-th up step s^u_i (resp. down step s^d_i) of γ gives birth to the even dot $e_{p_C(i)}$ (resp. to the odd dot $e_{n+q_C(i)}$) (see Definition 1.11).

Example A.4. If $S \in \mathrm{DH}(6)$ is the Dellac history $\psi(C)$ of Example 1.54, we obtain $\tilde{\psi}(S) = C$.

Following Remark A.3, it is easy to prove the following lemma by induction on $i \in [n]$.

Lemma A.5. *Let $S \in DH(n)$. We consider the two sequences $(\mathcal{I}^o_i)_i$ and $(\mathcal{I}^e_i)_i$ defined in the computation of $C = \tilde{\psi}(S)$ (see Definition A.2). Then for all $i \in [n]$, the integer $q_C(i)$ is the $(1 + r^o_C(e_{n+q_C(i)}))$-th element of the sequence $\mathcal{I}^o_{i-1}$, and the integer $p_C(n+1-i)$ is the $(1 + l^e_C(e_{p_C(n+1-i)}))$-th element of the sequence $\mathcal{I}^e_{i-1}$.*

Proposition A.6. *The maps $\psi : DC(n) \to DH(n)$ and $\tilde{\psi} : DH(n) \to DC(n)$ are inverse maps.*

Proof. From Remarks 1.55 and A.3, it is easy to see that $\psi \circ \tilde{\psi} = Id_{\mathrm{DH}(n)}$. The equality $\tilde{\psi} \circ \psi = Id_{\mathrm{DC}(n)}$ is less straightforward. Let $C \in \mathrm{DC}(n)$ and $S = (\gamma, \xi) = \psi(C) \in \mathrm{DH}(n)$. We are going to show, by induction on $i \in [n]$, that $q_{\tilde{\psi}(S)}(i) = q_C(i)$ and $p_{\tilde{\psi}(S)}(i) = p_C(i)$ for all i, hence $\tilde{\psi}(S) = C$. The two proofs of $q_{\tilde{\psi}(S)}(i) = q_C(i)$ and $p_{\tilde{\psi}(S)}(i) = p_C(i)$ respectively being independant and analogous, we only prove $q_{\tilde{\psi}(S)}(i) = q_C(i)$ for all i. Let $i = 1$. In the context $1(a)$ of Definition A.2, from Remark 1.55, the first odd dot to be inserted is $e_{n+q_{\tilde{\psi}(S)}(1)}$. Therefore, by definition, the integer $q_{\tilde{\psi}(S)}(1)$ is the $(1+n_2)$-th element of $\mathcal{I}_0^o$ (*i.e.*, we obtain $q_{\tilde{\psi}(S)}(1) = 1 + n_2$ where $(n_1, n_2) = \xi_1$). In this situation, since $S = \psi(C)$, we know that $n_2 = r_C^o(e_{n+q_C(1)})$. Consequently, from Lemma A.5, we obtain $q_{\tilde{\psi}(S)}(1) = 1 + r_C^o(e_{n+q_C(1)}) = q_C(1)$. The proof in the context $1(b)$ is analogous. Now let $i \in \{2, 3, \ldots, n\}$. Suppose that $q_{\tilde{\psi}(S)}(k) = q_C(k)$ for all $k < i$. In the context $1(a)$ of Definition A.2, from Remark 1.55, the i-th odd dot to be inserted is $e_{n+q_{\tilde{\psi}(S)}(i)}$. Therefore, by definition, if $\xi_i = (n_1, n_2)$, then $q_{\tilde{\psi}(S)}$ is the $(1+n_2)$-th element of $\mathcal{I}_{i-1}^e = \mathcal{J}_{i-1}^e$. Since $S = \psi(C)$, we know that $n_2 = r_C^o(e_{n+q_C(i)})$ so, from Lemma A.5, we obtain $q_{\tilde{\psi}(S)}(i) = q_C(i)$. The proof in the context $1(b)$ is analogous. □

This proves Theorem 1.48.

Appendix B

Irreducible k-shapes and surjective pistols

B.1 Proof of Lemma 2.14

Lemma 2.14. *Let s be a partial k-shape obtained by adding rectangles to the empty partition, i.e., such that s is the result of a sequence of sums*

$$\begin{aligned} s_0 &= s_0 \oplus^k_{t(j_0)} \lceil (j_0+1)/2 \rceil^{z_0}, \\ s_2 &= s_1 \oplus^k_{t(j_1)} \lceil (j_1+1)/2 \rceil^{z_1}, \\ &\vdots \\ s &= s_m \oplus^k_{t(j_m)} \lceil (j_m+1)/2 \rceil^{z_m} \end{aligned}$$

where s_0 is the empty partition. Let $j \in [2k-4]$ such that every column of s is at least $\lceil (j+2)/2 \rceil$ cells high, and let $z \in \{0, 1, \ldots, k-1-\lceil j/2 \rceil\}$. We consider two consecutive columns (from left to right) of s, which we denote by C_1 and C_2, with the same height and the same label but not the same level, and such that C_1 has been lifted by the Rule 2 of Definition 2.12 (note that it cannot be by the Rule 1). If C_2 has been lifted at the same level as C_1 in $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$, then it is not by the Rule 2 of Definition 2.12.

Proof. Let R_1 (resp. R_2) be the row in which C_1 (resp. C_2) is rooted, and let R be the row beneath R_1. Let l be the length of R. Since C_1 and C_2 have the same height and the same label, and since C_1 has been lifted by the Rule 2 of Definition 2.12, then it is necessary that the length of R_2 equals l

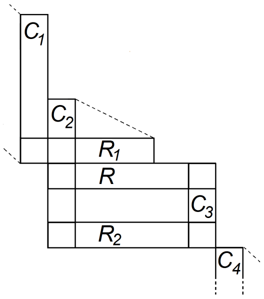


Figure B.1: Partial k-shape s.

as well. Consequently, the partial k-shape s is like depicted in Figure B.1. We also consider the last column C_3 to be rooted in R_2, and the column C_4 that follows C_3.

Now, suppose that, in $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$, the column C_2 has been lifted at the same level as C_1 by the Rule 2 of Definition 2.12. To do so, it is necessary that the row R gains cells from s to $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$, *i.e.*, that the column C_4 is lifted at the same level as C_3. By hypothesis, it means that C_4 must be lifted down to at least $\lceil (j+2)/2 \rceil$ cells from s to $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$. Obviously, every column have been lifted up to at most $\lceil (j+1)/2 \rceil$ cells from s to $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$, so it is necessary that $\lceil (j+1)/2 \rceil = \lceil (j+2)/2 \rceil$, *i.e.*, that there exists $p \in [k-2]$ such that $j = 2p$. This means that the z columns that we glue on the right of s in order to compute $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$ are $\lceil (j+1)/2 \rceil = p+1$ cells high and labelled by $t(j) = 0$, which cannot be because, according to the Rule 1 of Definition 2.12, only the p top cells of those z columns may lift the columns of s by the Rule 2, *i.e.*, the columns of s are lifted up to at most p cells in this context, implying that C_4 cannot be lifted at the same level as C_3. □

B.2 Proof of Lemma 2.15

Lemma 2.15. *Let s be a partial k-shape in the context of Lemma 2.14, and let $j \geq 1$ such that the height of every column of s is at least $\lceil (j+2)/2 \rceil$, and such that the quantity of integers $i \in [k-2]$ in which s is not saturated is at most $\lceil j/2 \rceil$. If s is not saturated in $i_0 \in [k-2]$, then there exists a unique integer $z \in [k-1-\lceil j/2 \rceil]$ such that the partial k-shape $s \oplus^k_{t(j)} \lceil (j+1)/2 \rceil^z$*

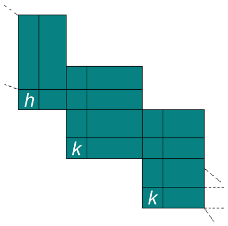


Figure B.2: Partial k-shape s.

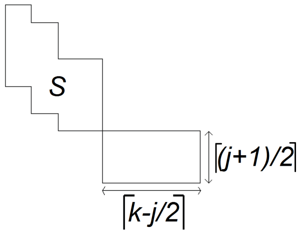


Figure B.3: Partial k-shape $s^{k-\lceil j/2\rceil}$.

is saturated in i_0*.*

Proof. It is easy to see that, in the partial k-shape s, the columns of height i_0+1 and label 0 are organized in $m \geq 1$ groups of columns rooted in a same row, such that the $m-1$ first groups from right to left are made of saturated columns, *i.e.*, such that the columns of these groups are rooted in rows whose greatest hook length is k, and such that the m-th group is made of non-saturated columns, *i.e.*, such that the columns $C_1, C_2, \ldots, C_q$ of this group (from left to right) are rooted in a row whose greatest hook length is $h < k$ (see Figure B.2).

Now, for all $p \in [k-\lceil j/2\rceil]$, let s^p be the partial k-shape $s \oplus^k_{t(j)} \lceil (j+1)/2\rceil^p$. For all $p \geq 2$, the partial k-shape s^p is obtained by gluing a column of height $\lceil (j+1)/2\rceil$ and label $t(j)$ right next to the last column of s^{p-1}, then by applying the three rules of Definition 2.12 so as to obtain a partial k-shape. We now focus on $s^{k-\lceil j/2\rceil}$. Let C be a column of s. Since the height of C is at least $\lceil (j+2)/2\rceil$, if the bottom cell c of C is located in the same row as one of the cells of the rectangle $\lceil (j+1)/2\rceil^{k-\lceil j/2\rceil}$ during the computation of $s^{k-\lceil j/2\rceil}$, then the hook length of c will be at least $\lceil (j+2)/2\rceil + k - \lceil j/2\rceil \geq k+1$. Consequently, by the Rule 2 of Definition 2.12, the column C is lifted as long as its bottom cell is in the same row as one of the cells of the rectangle $\lceil (j+1)/2\rceil^{k-\lceil j/2\rceil}$, and since this holds for every column C of s, then the partial k-shape $s^{k-\lceil j/2\rceil}$ is obtained by drawing the rectangle $\lceil (j+1)/2\rceil^{k-\lceil j/2\rceil}$ in the bottom right-hand corner of s (see Figure B.3). In particular, the columns $C_1, C_2, \ldots, C_q$ must have been been lifted $\lceil (j+1)/2\rceil$ times from s to $s^{k-\lceil j/2\rceil}$. The idea is to prove there exists a unique $p_0 \in [k-1-\lceil j/2\rceil]$ such that C_1 has been lifted by the Rule 2 of Definition 2.12 in s^{p_0+1}, implying the hook length h of its bottom cell c_1 equals k in s^{p_0}, which means s^{p_0} is saturated in i_0 according to Remark 2.13. Note that the columns $C_1, C_2, \ldots, C_q$

cannot be lifted by the Rule 3 of Definition 2.12 because their label is 0. If $m \geq 2$, the columns $C_1, C_2, \ldots, C_q$ cannot be lifted by the Rule 1, hence they are lifted by the Rule 2, so the existence and unicity of the integer p_0 is obvious. If $m = 1$, suppose C_1 is never lifted by the Rule 2, *i.e.*, that it is lifted $\lceil (j+1)/2 \rceil$ times by the Rule 1 between s and $s^{k-\lceil j/2 \rceil}$. Each of these $\lceil (j+1)/2 \rceil$ times, the first column labelled by 0 (from right to left) prompting the chain reaction of liftings by the Rule 1 (which leads to the lifting of C_1), cannot be saturated because from Remark 2.13 they would still be saturated when glued to the columns of different height or label that are lifted, meaning their bottom cell would be hooked lengthed by k and that the lifting would be due to the Rule 2, which is not the case by hypothesis. This non-saturation also implies that each time, the height $i+1$ of this first column prompting the liftings must be different from the other ones (because the columns of height $i+1$ and label 0 are also organized like depicted in Figure B.2). As a conclusion, it is necessary that in addition to i_0, there would exist $\lceil (j+1)/2 \rceil \geq \lceil j/2 \rceil$ different integers $i < i_0$ such that s is not saturated in i, which is absurd by hypothesis. □

B.3 Proof of Lemma 2.22

Lemma 2.22. *For all $\lambda \in IS_k$ and for all $j \in [2k-4]$, we have*

$$z_j(\lambda) \in \{0, 1, \ldots, k-1-\lceil j/2 \rceil\}.$$

Proof. By definition of an irreducible k-shape, we automatically have $z_{2i}(\lambda) = x_i(\lambda) < k-i$ for all $i \in [k-2]$. The proof of $z_{2i-1}(\lambda) = y_i(\lambda) < k-i$ is less straightforward. Suppose that $y_i(\lambda) \geq k-i$. Let C_0 be the first column (from left to right) of $\bigsqcup_{j=1}^{k-i} H_j^k(\lambda) \cap V_i^k(\lambda)$, let R_0 be the row in which C_0 is rooted, and let R_1 be the row beneath R_0. We denote by $l \in [y_i(\lambda)]$ the number of consecutive columns (from left to right) of $\bigsqcup_{j=1}^{k-i} H_j^k(\lambda) \cap V_i^k(\lambda)$ whose bottom cells are located in R_0, and l' the length of R_1 (see Figure B.4).

The hook length $h = i + l' + 1$ of the cell glued to the left of R_1 exceeds k because it is not in $\partial^k(\lambda)$, so $l' \geq k-i$. Now suppose that $l \geq k-i$: consequently, the hook length $h_0 \leq k-1$ of the bottom cell of C_0 is such that $h_0 \geq i + l - 1 \geq k-1$ hence $h_0 = k-1$. As a result, the first

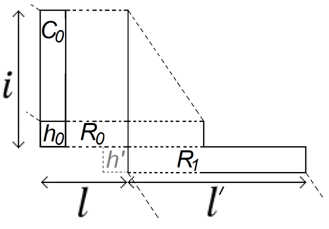


Figure B.4: $\lambda \in \mathrm{IS}_k$.

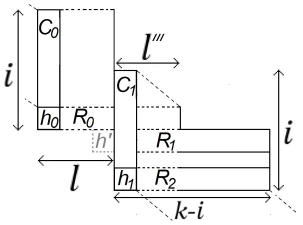


Figure B.5: $\lambda \in \mathrm{IS}_k$.

l columns of $\bigsqcup_{j=1}^{k-i} H_j^k(\lambda) \cap V_i^k(\lambda)$ are in fact located in $H_{k-i}^k(\lambda) \cap V_i^k(\lambda)$, therefore $|H_{k-i}^k(\lambda) \cap V_i^k(\lambda)| \geq l \geq k-i$, which cannot be because λ is irreducible. So $l \leq k-i-1 < y_i(\lambda)$. Consequently, there exists a column of $\bigsqcup_{j=1}^{k-i} H_j^k(\lambda) \cap V_i^k(\lambda)$ which intersects R_1. Consider the first column C_1 that does so, then its bottom cell c_1 is located in a row R_2 (whose length is denoted by $l'' \geq l'$) and c_1 is hook lengthed by the integer $h_1 = i + l'' - 1 < k$, thence $k-i \leq l' \leq l'' \leq k-i$, which implies $l' = l'' = k-i$. Now, let $l''' \in \{0, 1, \ldots, k-i\}$ be the number of columns of $\bigsqcup_{j=1}^{k-i} H_j^k(\lambda) \cap V_i^k(\lambda)$ intersecting R_1 but whose top cells are not located in R_1 (see Figure B.5). Since $h_0 = i - 1 + l + l''' \leq k-1$, we obtain $l + l''' \leq k-i$. With precision, it is necessary that $l + l''' \leq k-i-1$: otherwise, $l + l''' = k-i$ would imply $h_0 = k-1$ hence the first $l + l''' = k-i$ columns of $\bigsqcup_{j=1}^{k-i} H_j^k(\lambda) \cap V_i^k(\lambda)$ would be located in $H_{k-i}^k(\lambda) \cap V_i^k(\lambda)$, which cannot be because λ is irreducible. So $l + l''' < k-i \leq y_i(\lambda)$. It means there exists a column C_2 of $\bigsqcup_{j=1}^{k-i} H_j^k(\lambda) \cap V_i^k(\lambda)$ whose top box is located in R_1 (we may consider that it is the first from left to right), which forces its bottom cell to be located in a row R_3 of length $k-i$ (see Figure B.6) because the length of R_2 is $k-i$ and $rs(\lambda)$ is a partition (and the hook length h_2 of the bottom cell of C_2 being at most $k-1$, the length of R_3 is at most $k-i$ hence equals $k-i$ and $h_2 = k-1$).

But then the bottom cells of the $k-i$ columns intersecting R_1 are located in a row of length $k-i$, therefore the bottom cells of those $k-i$ columns are elements of the set $H_{k-i}^k(\lambda) \cap V_i^k(\lambda)$, which cannot be because λ is irreducible. As a conclusion, it is necessary that $y_i(\lambda) < k-i$. □

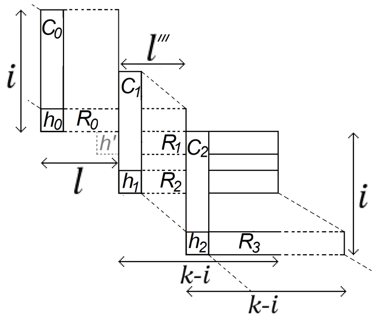


Figure B.6: $\lambda \in \mathrm{IS}_k$.

B.4 Proof of Lemma 2.24

Lemma 2.24. *We have* $s^1(\lambda) = \partial^k(\lambda)$ *for all* $\lambda \in IS_k$.

Proof. Let n be the number of columns of $\partial^k(\lambda)$, which is obviously the same as for $s^1(\lambda)$. For all $q \in [n]$, we define $\partial^k(\lambda)_q$ (respectively $s^1(\lambda)_q$) as the skew partition obtained by considering the q first columns (from right to left) of $\partial^k(\lambda)$ (resp. $s^1(\lambda)$). The idea is to prove that $\partial^k(\lambda)_q = s^1(\lambda)_q$ for all $q \in [n]$ by induction (the statement being obvious for $q = 1$). In particular, for $q = n$, we will obtain $\partial^k(\lambda) = s^1(\lambda)$. Suppose that $\partial^k(\lambda)_q = s^1(\lambda)_q$ for some $q \geq 1$. From right to left, the $(q+1)$-th column C (whose bottom cell is denoted by c) of $\partial^k(\lambda)_{q+1}$ is glued to the left of $\partial^k(\lambda)_q$, at the unique level such that the hook length h of c doesn't exceed k, and the hook length x of the cell beneath c exceeds k (see Figure B.7). Since the hook length of every cell of $s^1(\lambda)$ doesn't exceed k, the $(q+1)$-th column C' (whose bottom cell is denoted by c' with the hook length h') of $s^1(\lambda)_{q+1}$ is necessarily positioned above or at the same level as C (see Figure B.8). Note that in Figures B.7 and B.8, the q-th columns C_q and C'_q of respectively $\partial^k(\lambda)_{q+1}$ and $s^1(f)_{q+1}$ are at the same level because $\partial^k(\lambda)_q = s^1(f)_q$ by hypothesis.

Now, suppose that the columns C and C' are not at the same level, *i.e.*, that C' is at a higher level. In particular, the hook length x' of the cell beneath c' (see Figure B.8) is such that $x' \leq h \leq k$. Also, the bottom cell c'_q (whose hook length is denoted by h'_q) of the q-th column C'_q of $s^1(\lambda)_{q+1}$ is a corner.

1. If C' has been lifted by the Rule 2 of Definition 2.12, then since $x' \leq k$, then it is necessary that C' (as a column of a labelled skew partition) is labelled by 1 and that $x' = k$. Consequently, since the bottom cell c of C, hook lengthed by $h \leq k$, is at the same level or beneath the cell

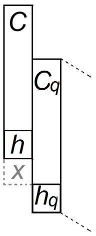


Figure B.7: $\partial^k(\lambda)_{q+1}$.

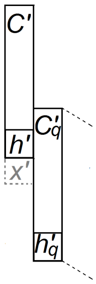


Figure B.8: $s^1(f)_{q+1}$.

hook lengthed by $x' = k$, then $h = k$ and those two cells are in fact at the same level. In particular, this implies that C' must be labelled by 0, which is absurd.

2. If C' has been lifted by the Rule 1 of Definition 2.12, then in particular C'_q is a column labelled by 0. Consequently, the hook length h'_q of c'_q, which is the same as the hook length h_q of c_q because $s^1(\lambda)_q = \partial^k(\lambda)_q$ by hypothesis, equals the integer k. Since $rs^k(\lambda)$ and $cs^k(\lambda)$ are partitions, this implies $h > h_q = k$, which is absurd.
3. Therefore, the column C' has necessarily been lifted by the Rule 3 of Definition 2.12. It implies that :
   (a) C' and C'_q have the same height and the same label 1 (and $h'_q = k-1$);
   (b) C' is located one cell higher than C'_q.

   In particular, from (b), since C' is supposed to be located at a higher level than C, then the bottom cell c of C is glued to the left of the bottom cell c_q of C_q. Since $h > h_q = h'_q = k-1$, we obtain $h = k$, which is in contradiction with C' being labelled by 1.

So C and C' are located at the same level, thence $\partial^k(\lambda)_{q+1} = s^1(\lambda)_{q+1}$. □

B.5 Illustration of Definition 2.16

Let $k = 6$ and $f = (4, 2, 6, 8, 6, 10, 8, 10, 10, 10) \in \mathrm{SP}_5$. The partial 6-shape $s^{2k-3}(f) = s^9(f)$ is defined as the empty partition.

1. $j = 8 :$ $f(8) = 10$ and $f(10) = 10$, so we are in the second case of Definition 2.16 and $s^8(f)$ is defined as $s^9(f) \oplus_0^6 5^z$ where $z = f(8)/2 - \lceil 8/2 \rceil = 1$, *i.e.* $s^8(f)$ is simply one column of height 5 and label 0 (see Figure B.9).

Figure B.9: $s^8(f) = s^7(f)$.

2. $j = 7 :$ $f(7) = 8$ and $f(8) > 8$ but $7 \neq \min\{j' : f(j') = 8\}$ so we are still in the second case, and since $f(7)/2 - \lceil 7/2 \rceil = 0$, the 6-shape $s^7(f)$ is still $s^8(f)$ (see Figure B.9).
3. $j = 6 :$ $f(6) = 10$ and $f(10) = 10$ so we are in the second case and $s^6(f) = s^7(f) \oplus_0^6 4^z$ where $z = f(6)/2 - \lceil 6/2 \rceil = 2$, see Figure B.10.

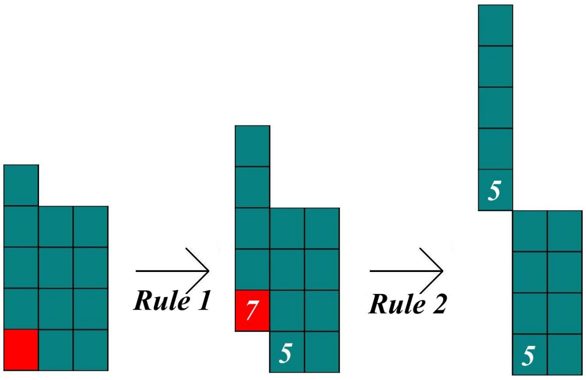


Figure B.10: $s^6(f) = s^5(f)$.

4. $j = 5 :$ as for the step $j = 7$, we obtain $s^5(f) = s^6(f)$.
5. $j = 4 :$ $f(4) = 8$, $f(8) > 8$, 4 is the smallest integer mapped to 8 by f, and $s^5(f)$ is not saturated in $i = 8/2 = 4 :$ indeed, the only

column C of $s^5(f)$ whose height is $i+1=5$ and whose label is 0 has its bottom cell hook lengthed by 5 instead of 6. So we are in the first case of Definition 2.16, and we look for the unique integer $z \in \{1,2,3\}$ such that $s^5(f) \oplus_0^6 3^z$ saturates C. We find $z=2$ and $s^4(f)$ is defined as $s^5(f) \oplus_0^6 3^2$, see Figure B.11.

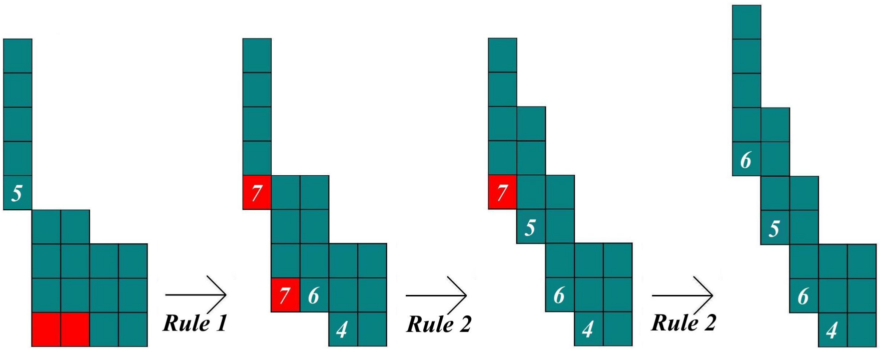


Figure B.11: $s^4(f)$.

6. $j=3:\ f(3)=6$, $f(6)>6$, 3 is the smallest integer mapped to 6 by f, and $s^4(f)$ is not saturated in $i=6/2=3:$ from left to right, the columns of $s^4(f)$ whose heights are $i+1=4$ and whose labels are 0, are the second and first columns C_2 and C_3 of $s^4(f)$, and C_2 is not saturated (but C_3 is). So we are in the first case and we define $s^3(f)$ as $s^4(f) \oplus_1^6 2^z$ (with $z \in \{1,2,3\}$) so that C_2 becomes saturated, in this case $z=3$ (see Figure B.12).
7. $j=2:\ f(2)=2$ so we are in the second case and $s^2(f)=s^3(f)$ (see Figure B.12).
8. $j=1:\ f(1)=4$, $f(4)>4$, 1 is the smallest integer mapped to 4 by f, and $s^2(f)$ is not saturated in $i=f(1)/2=2:$ from left to right, the columns of height $i+1=3$ and label 0 in s are the fourth column C_4 and the fifth column C_5, and C_4 is not saturated (but C_5 is). So we are in the first case, and we define $s^1(f)$ as $s^2(f) \oplus_1^6 1^z$ (with $z \in \{1,2,3,4\}$) so that C_4 becomes saturated, in this case $z=3$ (see Figure B.13).

Finally, the irreducible 6-shape $\varphi(f) \in \mathrm{IS}_6$ is the partial 6-shape depicted on Figure B.13 (on the right).

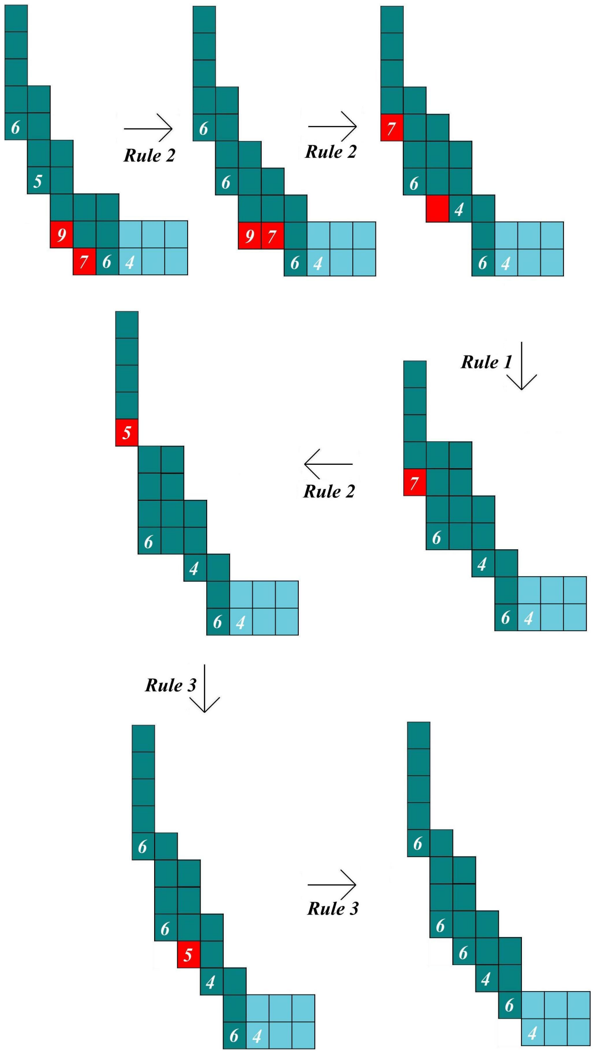


Figure B.12: $s^3(f) = s^2(f)$.

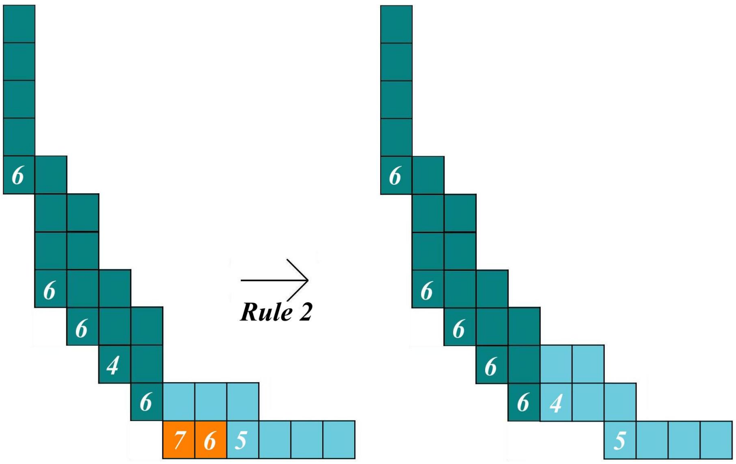


Figure B.13: $s^1(f)$.

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Index

Bijective combinatorics of permutations and Genocchi numbers

Abstract: This work is set in the context of enumerative combinatorics and constructs several statistic-preserving bijections between known or new combinatorial models of sequences of integers or polynomials, espacially the sequence of Genocchi numbers (and their extensions, the Gandhi polynomials) which appear in numerous mathematical theories and whose combinatorial properties are consequently intensively studied, and two sequences of q-Eulerian polynomials associated with the four fundamental statistics on permutations studied by MacMahon, and with analog statistics.

First of all, we define normalized Dumont permutations, a combinatorial model of the q-extended normalized median Genocchi numbers $\bar{c}_n(q)$ introduced by Han and Zeng, and we build a bijection between the latter model and the set of Dellac configurations, which have been proved by Feigin to generate $\bar{c}_n(q)$ by using the geometry of quiver Grassmannians. Then, in order to answer a question raised by the theory of continued fractions of Flajolet, we define a new combinatorial model of $\bar{c}_n(q)$, the set of Dellac histories, and we relate them with the previous combinatorial models through a second statistic-preserving bijection.

Afterwards, we study the set of irreducible k-shapes defined by Hivert and Mallet in the topic of k-Schur functions, which have been conjectured to generate the Gandhi polynomials with respect to the statistic of free k-sites. We construct a statistic-preserving bijection between the irreducible k-shapes and the surjective pistols of height $k-1$ (well-known combinatorial interpretation of the Gandhi polynomials with respect to the fixed points statistic) mapping the free k-sites to the fixed points, thence proving the conjecture.

Finally, we prove a new combinatorial identity between two eulerian polynomials defined on the set of permutations thanks to Eulerian and Mahonian statistics, by constructing a bijection on the permutations, which maps a finite sequence of statistics on another.

Keywords: Genocchi numbers; Gandhi polynomials; Dellac congigurations; Dellac histories; Irreducible k-shapes; Surjective pistols; q-Eulerian polynomials.

Combinatoire bijective des permutations et nombres de Genocchi

Résumé : Cette thèse a pour contexte la combinatoire énumérative et décrit la construction de plusieurs bijections entre modèles combinatoires connus ou nouveaux de suites d'entiers et polynômes, plus particulièrement celle des nombres de Genocchi (et de leurs extensions, les polynômes de Gandhi) qui interviennent dans diverses branches des mathématiques et dont les propriétés combinatoires sont de ce fait activement étudiées, et celles de polynômes q-eulériens associés aux quatre statistiques fondamentales de MacMahon sur les permutations ainsi qu'à des statistiques analogues.
On commence par définir les permutations de Dumont normalisées, un modèle combinatoire des nombres de Genocchi médians normalisés q-étendus, notés $\bar{c}_n(q)$ et définis par Han et Zeng, puis l'on construit une première bijection entre ce modèle et l'ensemble des configurations de Dellac, autre interprétation combinatoire de $\bar{c}_n(q)$ mise en évidence par Feigin dans le contexte de la géométrie des grassmanniennes de carquois. En s'appuyant sur la théorie des fractions continues de Flajolet, on en construit finalement un troisième modèle combinatoire à travers les histoires de Dellac, que l'on relie aux premiers modèles sus-cités au moyen d'une seconde bijection.
On s'intéresse ensuite à la classe combinatoire des k-formes irréductibles définies par Hivert et Mallet dans l'étude des k-fonctions de Schur, et qui faisaient l'objet d'une conjecture supposant que les polynômes de Gandhi sont générés par les k-formes irréductibles selon la statistique des k-sites libres. On construit une bijection entre les k-formes irréductibles et les pistolets surjectifs de hauteur $k-1$ (connus pour générer les polynômes de Gandhi selon la statistique des points fixes) envoyant les k-sites libres des premières sur les points fixes des seconds, démontrant de ce fait la conjecture.
Enfin, on établit une nouvelle identité combinatoire entre deux polynômes q-eulériens définis par des statistiques eulériennes et mahoniennes sur l'ensemble des permutations d'un ensemble fini, au moyen d'une dernière bijection sur les permutations, qui envoie une suite finie de statistiques sur une autre.

Mots clés : Nombres de Genocchi ; Polynômes de Gandhi ; Configurations de Dellac ; Histoires de Dellac ; k-formes irréductibles ; Pistolets surjectifs ; Polynômes q-eulériens.

Bijective combinatorics of permutations and Genocchi numbers

Keywords : Genocchi numbers ; Gandhi polynomials ; Dellac congigurations ; Dellac histories ; Irreducible k-shapes ; Surjective pistols ; q-Eulerian polynomials.

Image en couverture : Diagramme d'échange des configurations de Dellac de taille 3.

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