

Deux problèmes d'interface dans des systèmes aléatoires avec contraintes cinétiques

Two interface problems in stochastic systems with kinetic constraints

Oriane Blondel



5 December 2022

Systèmes de particules avec contraintes cinétiques



Le verre est-il plus proche de l'eau
(liquide) ou des glaçons ?

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Beaucoup d'approches proposées : MCT, spin glasses, RFOT, Coulomb frustrated models (arguments thermodynamiques).

Un autre angle d'attaque [Fredrickson-Andersen '84] : étudier des modèles

- ▶ avec un mécanisme de facilitation,
- ▶ sans interactions thermodynamiques.

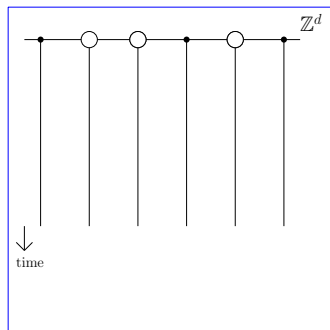
KCM (Kinetically Constrained Models)

Modèle de Fredrickson-Andersen

Processus aléatoire Markovien $(\eta_t)_{t \geq 0}$, où $\eta_t \in \{0, 1\}^{\mathbb{Z}^d}$.

- $\eta_t(x) = 1 \leftrightarrow$ particule en x
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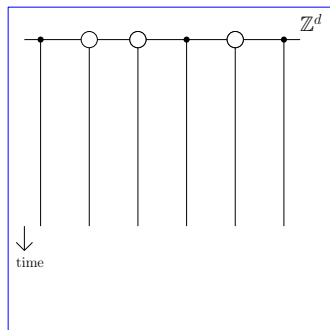


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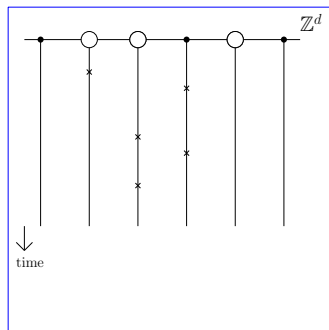
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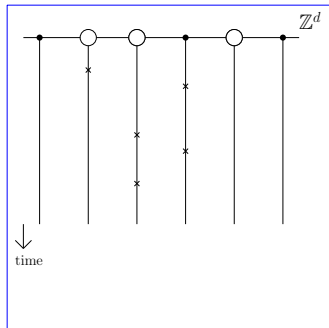
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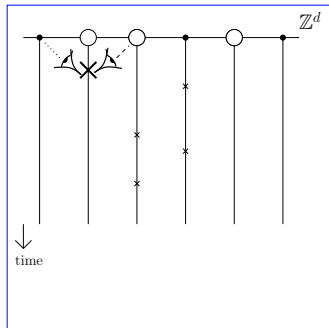
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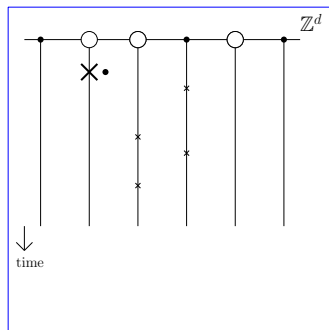
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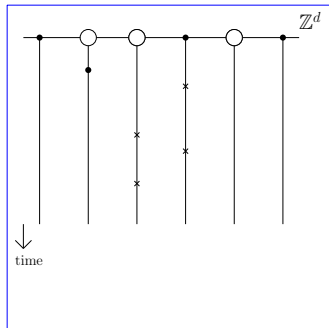
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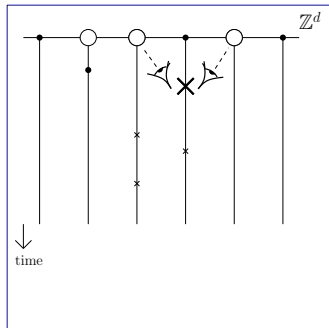
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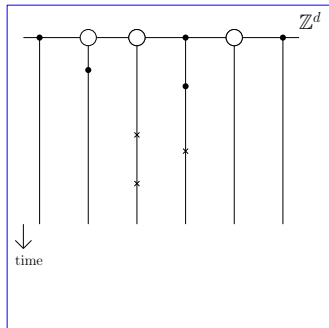
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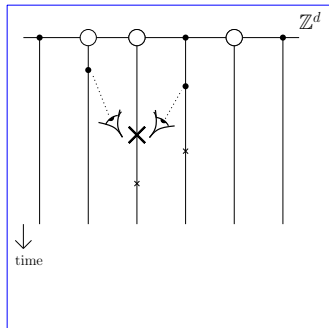
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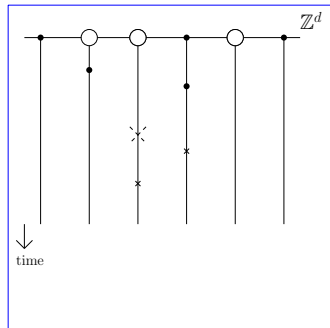
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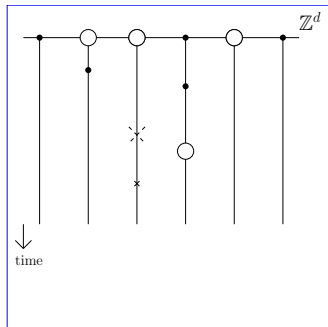
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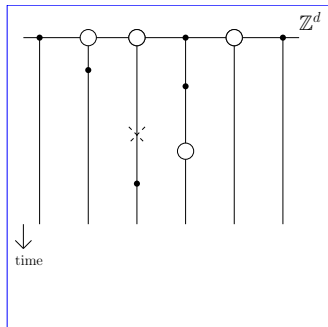
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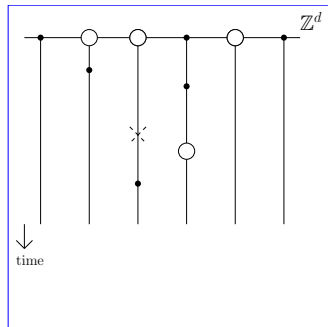
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Autres modèles : Est (voisin de droite vide), FA- j f (j voisins vides), etc.
Modèles conservatifs.

Les KCM sont des systèmes :

- ▶ (réversibles par rapport à $\mu = \text{Ber}(p)^{\otimes \mathbb{Z}^d}$),
- ▶ avec des transitions interdites,
- ▶ non attractifs/monotones.

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1. Systèmes de particules avec contraintes cinétiques/*Kinetically constrained models*
2. Limites hydrodynamiques/*Hydrodynamic limits*
3. Marches aléatoires en environnement aléatoire (dynamique)/*Random Walks in (Dynamic) Random Environment*

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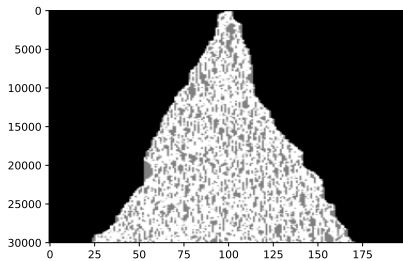
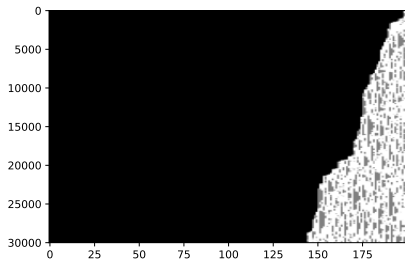
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→ Processus d'exclusion facilité/*Facilitated Exclusion Process*
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Front progression in kinetically constrained models

Propagation of zeros



Front progression, $d = 1$

Initial state: η_0 s.t. $\eta_0(x) = 1 \quad \forall x < 0, \eta_0(0) = 0.$

Front: X_t left-most zero of η_t . N.B.: X_t is not subadditive a priori.

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Theorem (B., Ganguly-Lubetsky-Martinelli, B.-Deshayes-Toninelli)

If $p < \bar{p}$ (with $\bar{p} = 1$ for East, $\bar{p} \approx 0.24$ for FA-1f), there exist $v < 0, \sigma \geq 0$ such that

$$\begin{aligned} \frac{X_t}{t} &\xrightarrow[t \rightarrow \infty]{} v \quad \text{a.s.} \\ \frac{X_t - vt}{\sqrt{t}} &\xRightarrow[t \rightarrow \infty]{} \mathcal{N}(0, \sigma^2). \end{aligned}$$

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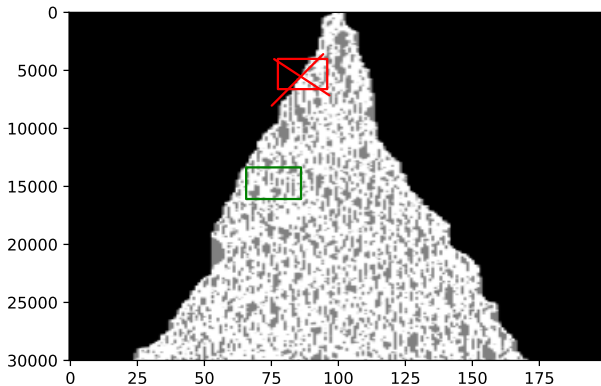
For the East model, with $\eta_0 \sim \text{Ber}(p)^{\otimes \mathbb{N}^*}$ on \mathbb{N}^* , the LLN is also a consequence of [B.-Hilário-Teixeira], where it comes with concentration bounds: for any $K \geq 0$,

$$\mathbb{P} \left(\left| \frac{X_t}{t} - v \right| \geq \varepsilon \right) \leq C(\varepsilon) t^{-K}.$$

Main ingredient: non-equilibrium relaxation

If there are initially many zeros in the system, and $p < \bar{p}$,

η_t close to μ_p for large t .



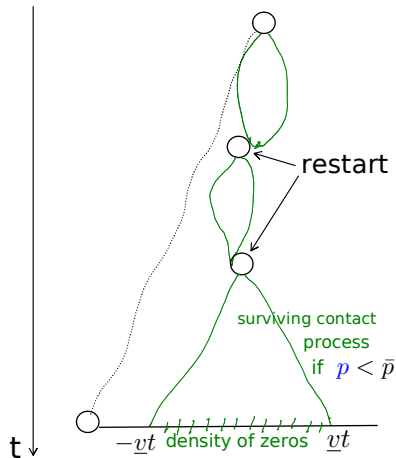
Step 1: generating enough zeros

FA-1f: comparison with a (threshold) contact process.

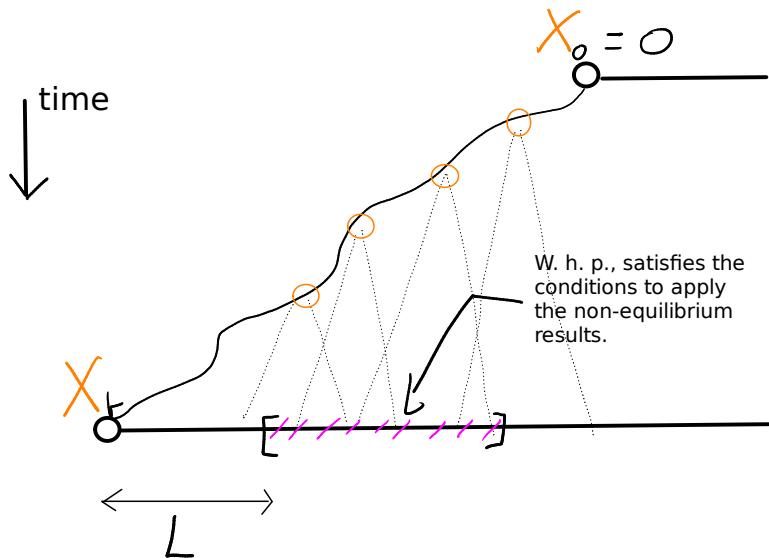
	Contact ξ_t	FA-1f η_t
$\circ \rightarrow \bullet$	p	$p c_x(\eta)$
$\bullet \rightarrow \circ$	$q c_x(\eta)$	$q c_x(\eta)$

Basic coupling:

$$\xi_0 \geq \eta_0 \Rightarrow \xi_t \geq \eta_t.$$



Step 1: generating enough zeros



Step 2: invariant measure behind the front

Theorem (ibid.)

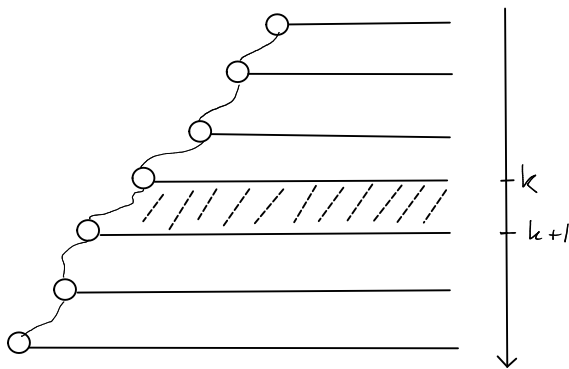
Consider the process seen from the front $\tilde{\eta}_t(\cdot) := \eta_t(X_t + \cdot)$. If $p < \bar{p}$, $\tilde{\eta}_t$ admits a unique invariant measure π and converges rapidly towards it.

Proof: Construction of a coupling between the processes seen from the front started from two different initial configurations. Based on Step 1 and non-equilibrium relaxation results.

Step 3: general CLT for mixing sequences

$X_{k+1} - X_k$ depends on

- $\tilde{\eta}_k$
- a slice of randomness
(|| for $\neq k$)



By the previous result, $\tilde{\eta}_k$ and $\tilde{\eta}_{k'}$ are not very correlated for $|k - k'|$ large enough.

\implies The increments $(X_{k+1} - X_k)_k$ form a *mixing sequence*.

A consequence: cutoff phenomenon

Theorem (GLM , Ertul)

If $p < \bar{p}$ on $[0, L]$ with b.c. 0 on $L + 1$, 1 on -1 , there is cutoff at time $\frac{L}{|v|}$ with window $O(\sqrt{L})$.

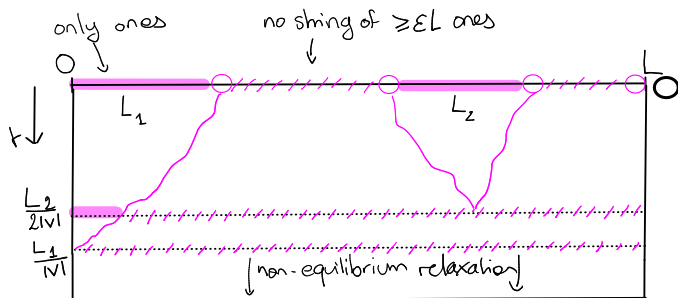
In other words, if $d(t) := \sup_{\eta} \|\mathcal{L}(\eta_t) - \mu_p\|_{TV}$, for all $\epsilon > 0$ there exists $\alpha(\epsilon) \geq 0$ s.t.

$$d\left(\frac{L}{|v|} - \alpha(\epsilon)\sqrt{L}\right) \xrightarrow{L \rightarrow \infty} 1,$$

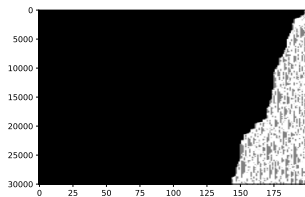
$$d\left(\frac{L}{|v|} + \alpha(\epsilon)\sqrt{L}\right) \xrightarrow{L \rightarrow \infty} 0.$$

Strategy of proof

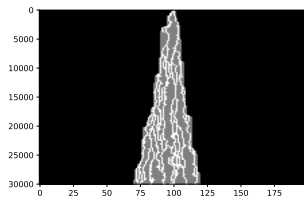
FA-1f [Ertul]:



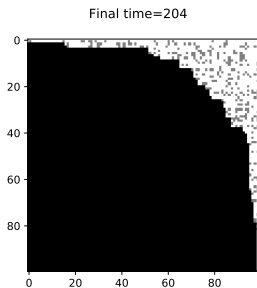
Open questions



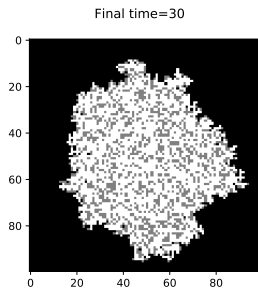
East, $d = 1$ ✓



FA-1f, $d = 1$. $p > \bar{p}$?



North-East ??
 $p = 0 \leftrightarrow$ LPP



FA-1f, $d = 2$??
 $p = 0 \leftrightarrow$ FPP

Stefan problem for the facilitated exclusion process

Facilitated exclusion process

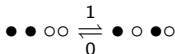
On the torus $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ or \mathbb{Z} , configuration $\eta \in \{0, 1\}^{\mathbb{T}_N}$ or $\{0, 1\}^{\mathbb{Z}}$.

Evolution: $\bullet\bullet\circ \rightarrow \bullet\circ\bullet$ at rate 1
 $\circ\bullet\bullet \rightarrow \bullet\circ\bullet$ at rate 1

Introduced to study active/inactive phase transitions in the presence of a conserved quantity [Rossi–Pastor-Satorras–Vespignani PRL'00].

KCM ?

- ▶ Kinetic constraint
- ▶ Not reversible



Active/inactive phase transition

Proposition

For fixed k, N , the process on \mathbb{T}_N with k particles satisfies:

1. If $k \leq \frac{N}{2}$, every invariant measure is concentrated on frozen configurations (absorbing states).
2. If $k > \frac{N}{2}$, the invariant measure $\pi_{k,N}$ is reversible and uniform on \mathcal{E} .

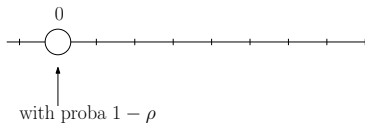
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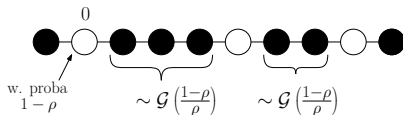
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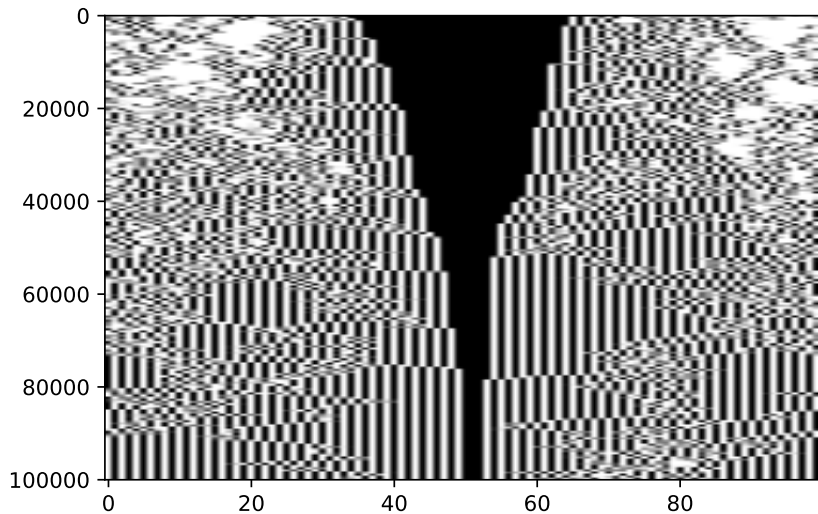
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Lemma (B.-Erignoux-Simon '21)

Let $\bar{\mu}$ be a translation invariant, stationary measure for the FEP on \mathbb{Z} . Then there exists $\lambda \in [0, 1]$ and a probability measure β on $[\frac{1}{2}, 1]$ such that

$$\bar{\mu} = \lambda \mu_{\text{frozen}} + \int_{1/2}^1 \pi_\rho d\beta(\rho).$$

Facilitated exclusion process



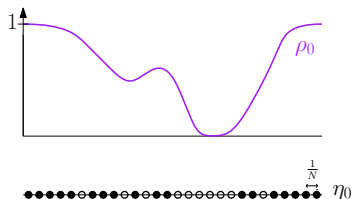
Hydrodynamic limit

See \mathbb{T}_N as a discrete approximation of $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ ($N \rightarrow \infty$).

$\rho_0 : \mathbb{T} \rightarrow [0, 1]$ *initial profile*.

Initial configuration

$$\{0, 1\}^{\mathbb{T}_N} \ni \eta_0 \sim \prod_{x \in \mathbb{T}_N} \text{Ber}(\rho_0(x/N)).$$



Hydrodynamic limit

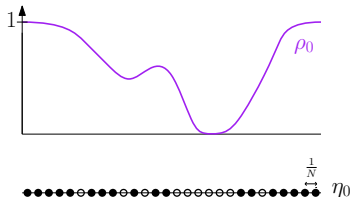
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$(\eta_t)_{t \geq 0}$ FEP starting from η_0 .



Is η_t associated with a profile? $\mathbb{P}(\eta_t(x) = 1) \approx \rho_t(x/N)$?
Evolution equation determining ρ_t ?

Hydrodynamic limit

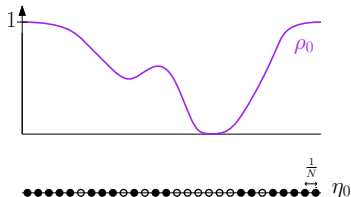
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Hydrodynamic limit for the FEP

Theorem (B.-Erignoux-Sasada-Simon, B.-Erignoux-Simon)

The hydrodynamic limit of the FEP with initial profile ρ_0 is given by the Stefan problem

$$\partial_t \rho = \Delta \left(\frac{2\rho - 1}{\rho} 1_{\rho > 1/2} \right) \quad (S)$$

with initial condition ρ_0 .

More precisely, for all smooth G , $\epsilon > 0$,

$$\mathbb{P} \left(\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G(x/N) \eta_{tN^2}(x) - \int_{\mathbb{T}} G(u) \rho_t(u) du \right| > \epsilon \right) \xrightarrow[N \rightarrow \infty]{} 0,$$

where ρ_t is the (unique) weak solution to (S) with initial condition ρ_0 .

Movie!

by Clément Erignoux.

Ingredients of proof

$$\begin{aligned}\eta_{tN^2}(x) - \eta_0(x) &= \int_0^t N^2 [1_{\circ\bullet\bullet} + 1_{\bullet\bullet\circ} - 1_{\bullet\bullet\circ} - 1_{\circ\bullet\bullet}] (\eta_s N^2) ds + \text{martingale} \\ &= \int_0^t N^2 [\tau_{x+1} h + \tau_{x-1} h - 2\tau_x h] (\eta_s N^2) ds + \text{martingale} \\ &= \int_0^t \Delta_N h(\eta_s N^2) ds + \text{martingale},\end{aligned}$$

where $h(\eta) = \eta(0)\eta(1) + \eta(0)\eta(-1) - \eta(-1)\eta(0)\eta(1)$ (gradient model!).

For $\rho > 1/2$, $\pi_\rho(h) = \frac{2\rho-1}{\rho}$; for η frozen, $h(\eta) = 0$. Taking the expectation and $N \rightarrow \infty$, we expect

$$\rho_t - \rho_0 = \int_0^t \Delta \left(\frac{2\rho_s - 1}{\rho_s} 1_{\rho_s > 1/2} \right) ds.$$

Main argument: for any translation invariant stationary measure ν ,

$$\nu \left(\left| \frac{1}{2\ell + 1} \sum_{|y| \leq \ell} \tau_y h(\eta) - \frac{2\rho^\ell - 1}{\rho^\ell} 1_{\rho^\ell > 1/2} \right| \right) \xrightarrow{\ell \rightarrow \infty} 0.$$

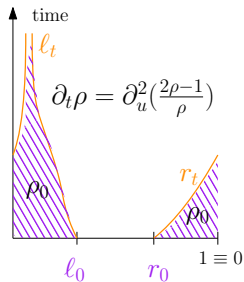
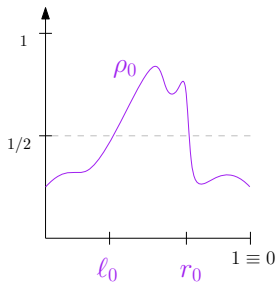
Active/inactive phases

Proposition

For $\rho_0 \in C^2(\mathbb{T})$ with $|\rho_0^{-1}(1/2)| < \infty$ and non-vanishing derivatives on this set, the weak solution to the Stefan problem is also a strong solution: there exist finitely many open intervals $I_1(t), \dots, I_k(t)$ s.t.

$$\begin{cases} \partial_t \rho = \Delta \left(\frac{2\rho-1}{\rho} \right) & \text{and } \rho > 1/2 & \text{on } I_j(t) \\ \rho = \rho_0 & \text{and } \rho_0 < 1/2 & \text{on } (I_j(t)^c)^\circ \end{cases}$$

with appropriate conditions at the boundaries $\partial I_j(t)$.



Microscopic phase separation

Theorem (B.-Erignoux-Simon)

For such ρ_0 , with high probability, the FEP system on \mathbb{T}_N at time tN^2 can be split into finitely many active/inactive phases, which coincide in the limit with the $I_j(t)$'s.

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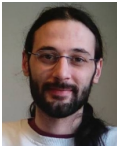
For ρ_0 as before, at time $N^{-1/4}$, the FEP system on \mathbb{T}_N can be split into finitely many active/inactive phases, which coincide in the limit with the $I_j(0)$'s.

Proof by coupling arguments.

Open questions

Higher dimension:

- ▶ Critical density? densities?
- ▶ Invariant measures?
- ▶ Hydrodynamic limit? Interface?



Merci !

