Statistical physics

Exercises – Series 1

Exercice 1. — Connectivity constants

1. Show that the connectivity constant of $\mathbb{Z}^2$ satisfies $\mu \in (2,3)$.

2. Show that for the "ladder graph" $\mathbb{Z} \times \{0,1\}$, $\mu = \frac{1+\sqrt{5}}{2}$. You might consider B_n the number of paths of length n directed to the right.

Exercice 2. — Site percolation

On a graph $G = (V,E)$, consider $\omega \in \{0,1\}^V$ with law $\mathrm{Ber}(p)^{\otimes V}$. We say that x is open if $\omega_x = 1$ and that $x \leftrightarrow y$ if there exists a sequence $x_0 = x, x_1, \ldots, x_n = y$ of open sites such that $(x_i, x_{i+1}) \in E$ for all $i = 0, \ldots n-1$.

1. Show that $p_c < 1$ for site percolation in $\mathbb{Z}^d$.

2. Show that any bond percolation is equivalent to a site percolation on a modified lattice. Is the converse property true? Draw the modified lattice for $\mathbb{Z}^2$. What do you notice?

Exercice 3. — Percolation on a tree

Consider bond percolation on the d–regular tree $\mathbb{T}^d$ (in which all vertices have degree d; *e.g.* $\mathbb{T}^2 = \mathbb{Z}$). Analyse $p \mapsto \theta_{\mathbb{T}^d}(p)$: show the existence of and compute $p_c(\mathbb{T}^d)$. How many infinite clusters are there in the supercritical regime? Find α such that $\frac{\theta_{\mathbb{T}^d}(p)}{(p-p_c)^\alpha}$ converges as $p \to p_c$.

Exercice 4. — Measurability

Consider bond percolation on $\mathbb{Z}^d$.

1. Show that, for $k \in \mathbb{N} \cup \{\infty\}$, the event

$$C_\infty^k = \{\text{there exist exactly } k \text{ infinite clusters}\}$$

is measurable.

2. Show the fact admitted in class: for all measurable event A, there exists a sequence A_n of events depending on a finite number of bonds such that $\mathbb{P}_p(A \triangle A_n) \underset{n\to\infty}{\longrightarrow} 0$.

Exercice 5. — More Harris

1. What happens to the Harris inequality for A non-decreasing, B non-increasing?
2. Let μ a probability measure on $\mathbb{R}$ and $f,g:\mathbb{R}\to\mathbb{R}$ two non-decreasing functions in $L^2(\mu)$. Show that $\mu(fg)\geq\mu(f)\mu(g)$. What is the fundamental difference between $\mathbb{R}$ and Ω that makes the inequality so simple here?
3. Let Λ a countable set, $\mathbb{P}$ a product probability measure on $\{0,1\}^\Lambda$ and A a non-decreasing event. Does $\mathbb{P}(\cdot|A)$ satisfy the Harris inequality?

Exercice 6. — Witnesses

Consider bond percolation on $\mathbb{Z}^d$. Fix A an event (not necessarily non-decreasing) and $\omega\in A$. For $I\subset\mathbb{E}^d$ we say that I is a witness of A in ω if $\forall\omega'\in\Omega$, $\omega'_I=\omega_I\Rightarrow\omega'\in A$.

1. Give examples of witnesses when $A=\{|\mathcal{C}(0)|\geq 5\}$, $A=\{|\mathcal{C}(0)|=3\}$, $A=\{|\mathcal{C}(0)|\leq 4\}$.
2. If A is non-decreasing (resp. non-increasing) and $\omega\in A$, show that there exists I a witness of A in ω s.t. $\omega_I\equiv 1$ (resp. $\omega_I\equiv 0$).

Exercice 7. — A bound on the one-arm exponent

Consider bond percolation on $\mathbb{Z}^2$ with parameter $1/2$ and the rectangle $R_n=[-n,n]\times[0,2n-1]$.

1. Let A_n be the event that there exists an open path connecting the left and right boundaries of R_n remaining inside R_n. Show that $\mathbb{P}(A_n)=1/2$.
2. Deduce that, if $B_n=\{0\leftrightarrow\partial\Lambda_n\}$, we have

$$\mathbb{P}(B_n\circ B_n)\geq\frac{1}{4n}.$$

What can you say about $\mathbb{P}(0\leftrightarrow\partial\Lambda_n)$?