Statistical physics

Exercises – Series 2

Exercice 1. — Long-distance connection

Show that for bond percolation on $\mathbb{Z}^d$, $\lim_{|x|\to\infty}\mathbb{P}_p(0\leftrightarrow x)=\mathbb{P}_p(0\leftrightarrow\infty)^2$.

Exercice 2. — Exponential decay with BK

Recover point 1 of Duminil-Copin–Tassion Theorem using the BK inequality.

Exercice 3. — Supercritical cluster in the plane

Show that for bond percolation on $\mathbb{Z}^2$, if $p>p_c$, there exists $c>0$ such that

$$\mathbb{P}_p(0\leftrightarrow\partial\Lambda_n, 0\not\leftrightarrow\infty)\le c^{-1}e^{-cn}.$$

Exercice 4. — Finite size criterion and susceptibility

1. Show that, if $\chi(p):=\mathbb{E}_p[|C_0|]<\infty$, there is exponential decay of $\mathbb{P}_p(0\leftrightarrow\partial\Lambda_n)$. Hint: show that there exists k such that $\sum_{x\in\partial\Lambda_k}\mathbb{P}_p(0\overset{\Lambda_k}{\leftrightarrow}x)<1$ and consider $\mathbb{P}_p(0\leftrightarrow\partial\Lambda_{Nk})$.
2. Show that for $p=p_c$, there is no exponential decay k. Hint: recall the condition in the first hint.

Exercice 5. — Correlation length

We want to show that $\mathbb{P}_p(0\leftrightarrow\partial\Lambda_n)^{1/n}$ converges to a limit in $(0,1)$ for $p\in(0,p_c)$.

1. For $p<p_c$, show that $\lim_{n\to\infty}\mathbb{P}_p(0\leftrightarrow(n,0,\dots,0))^{1/n}=\exp(-1/L(p))$ for some $L(p)>0$.
2. Show that $\mathbb{P}_p(0\leftrightarrow\partial\Lambda_n)^{1/n}$ has the same limit.
3. $L(p)$ is called correlation length. Why?

Exercice 6. — Triangular lattice

Check that the results of Sections I-IV of the lectures hold also for site percolation on the triangular lattice. What kind of duality do we use? Check in particular that $p_c=1/2$.