Statistical physics

Exercises – Series 3

Exercice 1. — Box-crossing property

Let $\lambda > 1$, $n \in \mathbb{N}$, $R_n^\lambda = [0, \lceil \lambda n \rceil] \times [0,n]$. Consider critical bond percolation on $\mathbb{Z}^2$ and let C_n^λ be the event that there is a left-right open path in R_n^λ. Show that there are positive constants c_1, c_2, c_3, c_4 such that $c_1 e^{-c_2\lambda} \leq \mathbb{P}_{1/2}(C_n^\lambda) \leq c_3 e^{-c_4\lambda}$.

Exercice 2. — Close to criticality

Consider bond percolation on $\mathbb{Z}^2$.

1. Show that if A is an event depending on a finite number k of sites, for $p \leq q$ we have $\mathbb{P}_q(A) \leq (q/p)^k \mathbb{P}_p(A)$.
2. Show that there exists a finite constant C such that $\theta(1/2+\varepsilon) \leq C\varepsilon^{\alpha/2}$.

Exercice 3. — A proof of the FKG inequalities

We aim to give a proof of the FKG inequality through coupling arguments. Let $\Lambda \Subset \mathbb{Z}^d$ and μ a measure on $\Omega_\Lambda = \{-1,1\}^\Lambda$ such that $\mu(\omega) > 0$ for all $\omega \in \Omega_\Lambda$.

Let us define a Markov chain on Ω_Λ as follows. Assume the system is in state $\omega(n) \in \Omega_\Lambda$ after step n. To choose its next state, pick $x \in \Lambda$ uniformly at random and sample independently $U \sim \mathcal{U}([0,1])$. Then define $\omega(n+1)$ as $\omega(n+1)_y = \omega(n)_y$ if $x \neq y$, and $\omega(n+1)_x = 1$ if $U \leq \mu(\sigma_x = 1 | \sigma_{|\Lambda\setminus\{x\}} = \omega(n)_{|\Lambda\setminus\{x\}})$, $\omega(n+1)_x = -1$ else.

1. Show that this Markov chain is irreducible, aperiodic and reversible w.r.t. μ.
2. Let μ' another probability measures satisfying the same condition as μ. Assume that additionally we have for all $\omega, \omega' \in \Omega_\Lambda$ such that $\omega \leq \omega'$, for all $x \in \Lambda$,
$$\mu(\sigma_x = 1 | \sigma_y = \omega_y \ \forall y \neq x) \leq \mu'(\sigma_x = 1 | \sigma_y = \omega'_y \ \forall y \neq x).$$
Show that there exists a monotone coupling between the dynamics associated with μ and μ', *i.e.* a Markov chain $(\sigma(n), \sigma'(n))_n$ such that $(\sigma(n))_n$ (resp. $(\sigma'(n))_n$) follows the dynamics associated with μ (resp. μ') and $\sigma(n) \leq \sigma'(n)$ a.s. as soon as $\sigma(0) \leq \sigma'(0)$.
N.B. : The dynamics is said to be *attractive*.
3. Deduce that for f non-decreasing on Ω_Λ,
$$\mu(f) \leq \mu'(f).$$
N.B. : μ' is said to stochastically dominate μ, and we write $\mu \preccurlyeq \mu'$.

4. Let $\# \in \{-1,1\}^{\mathbb{Z}^d} \cup \{\varnothing\}$ a boundary condition. Using the previous result, show the FKG inequality for the Ising measure on Λ with boundary condition $\#$ and parameters $\beta \geq 0$, $h \in \mathbb{R}$.

Exercice 4. — GKS inequalities

Show the following properties.

1. For $\Lambda \Subset \mathbb{Z}^d$, $A \subset \Lambda$ and $J = (J_S)_{S \subset \Lambda}, J' = (J'_S)_{S \subset \Lambda}$ two collections of non-negative real numbers such that $J \leq J'$ coordinate-wise,

$$\langle \sigma_A \rangle_{\Lambda;J} \leq \langle \sigma_A \rangle_{\Lambda;J'},$$

   where we recall that $\langle \cdot \rangle_{\Lambda;J}$ is the probability measure which gives a weight proportional to $e^{-\sum_{S \subset \Lambda} J_S \sigma_S}$ to $\sigma \in \Omega_\Lambda$.

2. For $n \in \mathbb{N}$, $\Lambda_n(d) = [-n,n]^d \cap \mathbb{Z}^d$, $\langle \sigma_0 \rangle^+_{\Lambda_n(d);\beta,0}$ is non-decreasing in the dimension d.