Statistical physics

Exercises – Series 4

Exercice 1. — Free energy

Give an example of $\Lambda_n \uparrow \mathbb{Z}^d$ and $(\omega_n)_{n\in\mathbb{N}}$ a sequence of boundary conditions such that $f^{\omega_n}_{\Lambda_n}(\beta,h)$ does not converge to $f(\beta,h)$.

Exercice 2. — Continuity of the magnetization

Show that there is uniqueness of the infinite volume Ising measure at (β,h) iff $h \mapsto \langle\sigma_0\rangle^+_{\beta,h}$ is continuous.

Exercice 3. — Low-temperature representation

Show that for β large enough, there exist $K,K'>0$ such that

$$\mu^+_{\Lambda_n,\beta,0}(\max\{|\gamma|,\gamma\,\text{contour in}\,\Lambda_n\} \in [K\log n, K'\log n]) \underset{n\to\infty}{\longrightarrow} 1,$$

where $\Lambda_n = [-n,n]^2$.

Exercice 4. — High-temperature representation

1. Let $A \subset \Lambda \Subset \mathbb{Z}^d$, $\beta \geq 0$. Find a high-temperature representation for $\langle\sigma_A\rangle^{\varnothing}_{\Lambda;\beta,0}$ and $\langle\sigma_A\rangle^+_{\Lambda;\beta,0}$.
2. If $d=1$, compute $\langle\sigma_{-1}\sigma_1\rangle^+_{[-n,n];\beta,0}$.
3. Show that, if β small enough (in the uniqueness regime), $\langle\sigma_0\sigma_x\rangle_{\beta,0}$ decreases exponentially fast in $\|x\|$.

Exercice 5. — Ising with periodic boundary condition

Fix the dimension d. Let $\mathbb{T}_N = \mathbb{Z}^d/(NZ^d)$. Recall that the set of vertices of $\mathbb{T}_N$ can be identified with $\Lambda_N = \{0,\dots,N-1\}^d$ and the set of edges defined by : $x=(x_1,\dots,x_d)$ and $y=(y_1,\dots,y_d)$ are connected by an edge if there exists $i \in \{1,\dots,d\}$ such that $y_i = x_i \pm 1 \mod N$ and $y_j = x_j$ for $j \neq i$. Let us denote $\mathcal{E}(\mathbb{T}_N)$ the set of these edges.

The Ising model on $\mathbb{T}_N$ is defined by the Hamiltonian

$$H^{\text{per}}_{\Lambda_N;\beta,h}(\sigma) = -\beta \sum_{\{x,y\}\in\mathcal{E}(\mathbb{T}_N)} \sigma_x\sigma_y - h\sum_{x\in\Lambda_N}\sigma_x.$$

We denote by $\langle\cdot\rangle^{\text{per}}_{\Lambda_N;\beta,h}$ and $Z^{\text{per}}_{\Lambda_N;\beta,h}$ respectively the associated Gibbs measure and partition function.

1. Show that if $h \geq 0$, for all $A \subset \Lambda_N$, $\langle\sigma_A\rangle^{\text{per}}_{\Lambda_N;\beta,h} \geq 0$.

2. Show that if f is a local non-decreasing function with support in Λ_{N-1},

$$\langle f\rangle^{-}_{\Lambda_{N-1};\beta,h} \leq \langle f\rangle^{\mathrm{per}}_{\Lambda_N;\beta,h} \leq \langle f\rangle^{+}_{\Lambda_{N-1};\beta,h}.$$

3. Let us define

$$f^{\mathrm{per}}_{\Lambda_N}(\beta,h) := \frac{1}{|\Lambda_N|}\log Z^{\mathrm{per}}_{\Lambda_N;\beta,h}.$$

   (a) Show that f^{per} is convex.

   (b) Show that $f^{\mathrm{per}}(\beta,\cdot)$ is even for all $\beta \geq 0$.

   (c) Show that $f^{\mathrm{per}}_{\Lambda_N}(\beta,h) \underset{N\to\infty}{\longrightarrow} f(\beta,h)$ for all $\beta \geq 0, h \in \mathbb{R}$.

Exercice 6. — Percus's transformation

Fix $\Lambda \Subset \mathbb{Z}^d$, $\beta \geq 0$, $h \in \mathbb{R}$. For any $\sigma,\eta \in \Omega_\Lambda$, we define $q_x = \frac{1}{2}(\sigma_x - \eta_x)$ and $t_x = \frac{1}{2}(\sigma_x + \eta_x)$ for $x \in \Lambda$. Let $q = (q_x)_{x\in\Lambda}$, $t = (t_x)_{x\in\Lambda}$.

1. Show that $\{(q,t) :\ (\sigma,\eta) \in \Omega_\Lambda \times \Omega_\Lambda\} = \{(q,t) \in \{-1,0,1\}^\Lambda \times \{-1,0,1\}^\Lambda :\ \forall x \in \Lambda,\ q_x = 0 \text{ iff } t_x \neq 0\}$.
2. Let σ,η be two independent copies of Ising configurations (i.e. random variables with law $\mu^{\varnothing}_{\Lambda;\beta,h}$. Denote their joint distribution by $\langle\cdot\rangle$. Show that for all $x,y \in \Lambda$, $\langle\sigma_x\rangle^{\varnothing}_{\Lambda;\beta,h} = \langle t_x\rangle$, $\langle\sigma_x\sigma_y\rangle^{\varnothing}_{\Lambda;\beta,h} - \langle\sigma_x\rangle^{\varnothing}_{\Lambda;\beta,h}\langle\sigma_y\rangle^{\varnothing}_{\Lambda;\beta,h} = 2\langle q_x q_y\rangle$.
3. Show that for all $(\sigma,\eta) \in \Omega_\Lambda \times \Omega_\Lambda$,

$$H^{\varnothing}_{\Lambda;\beta,h}(\sigma) + H^{\varnothing}_{\Lambda;\beta,h}(\eta) = H^{\varnothing}_{\Lambda;2\beta,0}(q) + H^{\varnothing}_{\Lambda;2\beta,2h}(t).$$

   Are q and t independent under $\langle\cdot\rangle$?

4. Show that for any functions $f,g : \{-1,0,1\}^\Lambda \to \mathbb{R}$,

$$\langle f(q)g(t)\rangle = \sum_{S\subset\Lambda} \frac{Z^{\varnothing}_{S;2\beta,0} Z^{\varnothing}_{S^c;2\beta,2h}}{(Z^{\varnothing}_{\Lambda;\beta,h})^2} \langle f^{(S)}\rangle^{\varnothing}_{S;2\beta} \langle g^{(S^c)}\rangle^{\varnothing}_{S^c;2\beta,2h},$$

   where for $S \subset \Lambda$ $f^{(S)} : \Omega_S \to \mathbb{R}$ is defined by $f^{(S)}(\sigma) = f(\tilde\sigma)$, where $\tilde\sigma_x = \sigma_x$ if $x \in S$, $\tilde\sigma_x = 0$ else.

5. For $A \subset \Lambda$, let $q_A = \prod_{x\in A} q_x$. Show that for any $A \subset \Lambda$, $\langle q_A\rangle \geq 0$.