Geometry of Geodesics in Integrable Models of Planar Last Passage Percolation

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Planar Last Passage Percolation

- IID space-time noise.
- The weight of a directed path, moving forward in time, is obtained by integrating the noise along the path.
- Maximizing the weight over all paths between two points gives the last passage time, optimizing path is called a geodesic.
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- Canonical models are believed to share universal features, but rigorous progress mostly for a few special integrable/exactly solvable models.

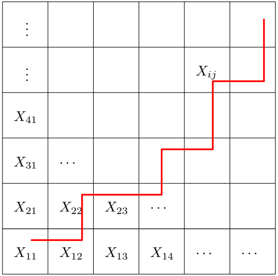
A special example we will focus on

- Exponential LPP: The underlying noise space made of i.i.d. Exponential Random Variables on $\mathbb{Z}^2$.

Exponential LPP on $\mathbb{Z}^2$

- Put i.i.d. weights $X_v \sim \text{Exp}(1)$ on each vertex of $\mathbb{Z}^2$.
- The last passage time from u to v.

$$T_{u,v} = \max_{\pi:u\to v} \sum_{w\in\pi} X_w.$$

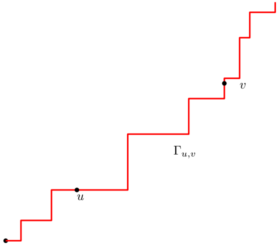


$X_{ij} \sim$ i.i.d. Exponential Variables.

Almost surely, for each u, v, there exists a unique geodesic $\Gamma_{u,v}$ between u and v.

Semi-infinite and bi-infinite geodesics

- An up-right path γ indexed by $\mathbb{N}$ (resp. $\mathbb{Z}$) is called a semi-infinite (resp. bi-infinite) geodesic if its restriction between any two points $u, v \in \gamma$ is the geodesic between u and v.
- Example: vertical and horizontal lines, a sub-sequential limit of $\Gamma_{\mathbf{0},\mathbf{n}}$ etc..



Questions we shall consider in this talk

How do geodesics look like?

- How does is the transversal fluctuation of a finite geodesic scale, i.e., how far away is the the geodesic $\Gamma_{u,v}$ from the straight line joining u and v?
- Do semi-infinite geodesics have direction?
- Do bi-infinte geodesics exist (except the vertical and horizontal lines)?

Questions we shall consider in this talk

Do geodesics coalesce?

- Consider geodesics from two fixed points to a far away point, do they typically coalesce before reaching the endpoint?
- If so, what is the typical scale at which they coalesce?
- Same question for semi-infinite geodesics going off in the same direction started at different points.

Questions we shall consider in this talk

Geometry of disjoint geodesics

- Can there be two disjoint geodesics close to each other?
- What is the typical separation for disjoint geodesics going between two parallel lines?

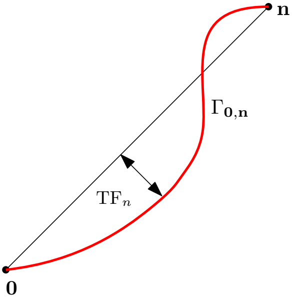
Scaling of time and space

- The answers to all the above questions depend on the proper scaling of time (the diagonal direction) and space (the anti-diagonal direction).
- The correct scaling can be deduced by considering the transversal fluctuation problem.
- We shall come back to the scaling question after we give a heuristic for the transversal fluctuation problem.

Transversal Fluctuation of Geodesics

The "$\chi = 2\xi - 1$ argument"

- Consider the geodesic Γ_n between $\mathbf{0}$ and $\mathbf{n}$.
- The transversal fluctuation of Γ_n, denoted, TF_n, is the smallest number such that Γ_n is contained in the strip $\{|x-y| \le \mathrm{TF}_n\}$.
- It is natural to predict that $\mathrm{TF}_n \sim n^{\xi}$ for some $\xi \in (0,1)$.

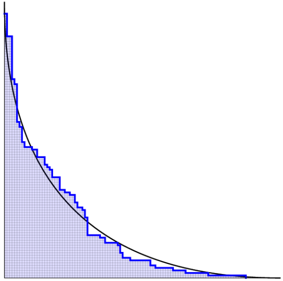


If n^{χ} is the order of the fluctuation of the passage time between two points at distance n, and n^{ξ} is the transversal fluctuation of the geodesic joining the two points, then

$$\chi = 2\xi - 1.$$

The "$\chi = 2\xi - 1$ argument"

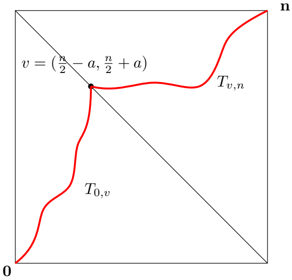
- Sub-additivity implies that $\lim_{n\to\infty} \frac{1}{n} T_{\mathbf{0},(nx,ny)} = g(x,y)$ a.s..
- The limit shape, $\{g(x,y) : x+y=2\}$ is expected to be curved with a maxima at $(1,1)$.
- This implies that if a path deviates too far from the straight line joining $\mathbf{0}$ and $\mathbf{n}$ it is penalized in expectation.



The deviation of the path should be at the scale where the penalty in the mean is of the same order as the fluctuations.

The "$\chi = 2\xi - 1$ argument"

- Suppose the geodesic passes through $v = (\frac{n}{2} + a, \frac{n}{2} - a)$.
- The geodesic weight is $T_{\mathbf{0},v} + T_{v,\mathbf{n}}$.
- This has expected weight



$$\frac{n}{2}g(1 - \frac{a}{n}, 1 + \frac{a}{n}) + \frac{n}{2}g(1 - \frac{a}{n}, 1 + \frac{a}{n}) \approx ng(1,1) + ag'(1,1) - \Theta(\frac{a^2}{n}).$$

- Applying the previous heuristic with $a \approx n^{\xi}$ we get

$$\frac{(n^{\xi})^2}{n} \approx n^{\chi} \Rightarrow \chi = 2\xi - 1.$$

Making it rigorous for exponential LPP

Curvature of Limit Shape and Fluctuations of T_n

- $\frac{T_{\mathbf{0},(nx,ny)}}{n} \to (\sqrt{x}+\sqrt{y})^2$. Rost (1981)
- Fluctuation exponent $\chi = 1/3$: $\frac{T_{\mathbf{0},\mathbf{n}}-4n}{2^{4/3}n^{1/3}} \Rightarrow F_{\mathrm{GUE}}$. Johansson (1999)
- Similar result available uniform in directions bounded away from axial directions.
- Moderate deviation bounds for $T_{\mathbf{0},\mathbf{n}}$. Ledoux-Rider(2010)
- Uniformly in directions.

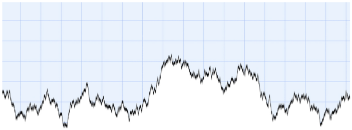
$\xi = 2/3$

- Based on similar inputs it was shown $\xi = \frac{2}{3}$. Johansson (2000).
- Showed $\mathrm{TF}_n = n^{2/3+o(1)}$ w.h.p.: not quantitatively optimal.
- It was done for Poissonian and Geometric LPP (two other exactly solvable models), but essentially same proof works for Exponential LPP.
- Exponent for exponential LPP obtained also via a queuing correspondence. Balász, Cator, Seppäläinen (2006)
- Similar results are obtained in different and more general settings before and after. Newman (1996), Wüthrich (1998), Chatterjee(2011)
- Optimal quantitative results for exponential LPP later in the talk.

An interlude on KPZ universality and Universal Scaling limits

Planar growth models in the KPZ class

The KPZ equation is a stochastic PDE predicted to model random interface growth in a universal way with slope dependent growth speed, subject to two forces: a surface tension whose effect is smoothening, and a local random force whose effect is to roughen the surface.



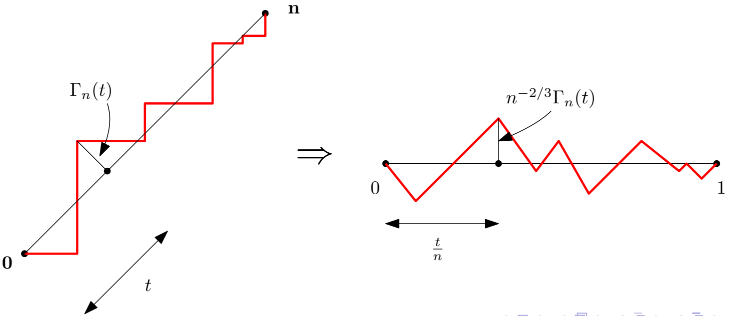
- The theory of KPZ universality predicts that these models share a triple $(1, 1/3, 2/3)$ of exponents.
- Planar LPP is a canonical model believed to exhibit KPZ universal behaviour.

Exactly Solvable Models

- While planar first and last passage percolation models are believed to exhibit KPZ scaling for general class of weights, it has rigorously been verified only for a handful of exactly solvable models.
- There are some remarkable bijections which allow exact computation for the distribution function of last passage times in exactly solvable LPP.
- For exponential LPP, last passage time has the same distribution as the largest eigenvalue of a random matrix ensemble with an explicit eigenvalue density.
- Other examples: Poissonian LPP on $\mathbb{R}^2$, Geometric LPP on $\mathbb{Z}^2$, semi-discrete Brownian LPP.
- In all these models, it is predicted that scaling time direction by n and space direction by $n^{2/3}$ gives rise to universal scaling limits.

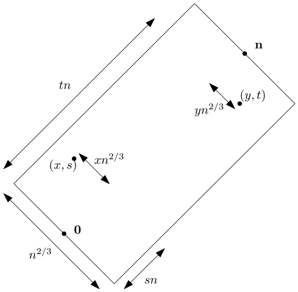
Conjectural Limit for the Geodesic

- For $t \in [0, 2n]$, let $\Gamma_n(t) = x(t) - y(t)$ where $(x(t), y(t))$ is the unique point at which Γ_n intersects the line $x + y = t$.
- Set $\pi_n(s) = n^{-2/3}\Gamma_n(2ns)$ for $s \in [0, 1]$.
- It is believed that π_n weakly converges to a $C[0, 1]$ valued stochastic process π.



Space time scaling and the conjectural limit

- Scale the time direction by n and the spatial direction by $n^{2/3}$, i.e., for $s, x \in \mathbb{R}$ the point $(sn + x(2n)^{2/3}, sn - x(2n)^{2/3})$ is mapped to (x, s).
- For $(x, s), (y, t) \in \mathbb{R}^2$ with $s < t$, define the four parameter random field $\mathcal{W}_n(x, s; y, t)$ by considering the last passage time from (x, s) to (y, t) (in the scaled co-ordinates) centered by $4(t - s)n$ and scaled by $2^{4/3} n^{1/3}$ (well defined for n sufficiently large).



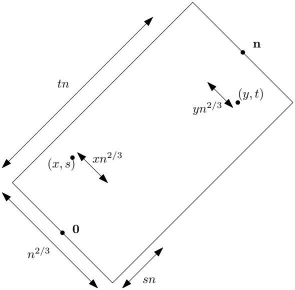
Space time scaling and the conjectural limit

- It is expected that as $n \to \infty$

$$\mathcal{W}_n(x,s;y,t) \Rightarrow \mathcal{W}(x,s;y,t),$$

  where $\mathcal{W}$ is a universal random object.

- Both these limits are recently shown to exist starting with the exactly solvable model of Brownian LPP.
  Dauvergne-Ortmann-Virag (2018)



Robustness of our methods

- In this talk, we shall only talk about geodesics in exponential LPP, but our methods are largely not specific to the exponential case.
- For the most part, we only use curvature of limit shape, Tracy-Widom convergence and uniform moderate deviation estimates.
- These are available for all known exactly solvable models of planar LPP.
- Hence variants of many of our results are expected to hold for other models and in the limit.

Results

Quantitative Results for Transversal Fluctuations

Theorem (B., Sidoravicius, Sly (2014))

For all x and n sufficiently large, we have for some $c > 0$

$$\mathbb{P}\left(TF_n \geq xn^{2/3}\right) \leq e^{-cx^3}.$$

- One point estimate is obtained by tightening Johansson's calculation presented before, and the rest is a chaining argument.
- Matching lower bound is available. Hammond-Sarkar (2020)

Quantitative Results for Transversal Fluctuations

Theorem (B., Bhatia (2020+))

For $\delta > 0$ small, and n sufficiently large, we have for some $c, c' > 0$

$$e^{-c'\delta^{-3/2}} \leq \mathbb{P}\left(TF_n \leq \delta n^{2/3}\right) \leq e^{-c\delta^{-3/2}}.$$

- The upper bound is a calculation of the probability of the large deviation event that the probability of the best path constrained in the small ball is competitive with the global best path.
- The lower bound is a geometric construction of a favourable event on which there is a good path in the small ball and all paths exiting the small ball are uncompetitive.

Semi-infinite Geodesics

- We only describe the picture for semi-infinite geodesics in the direction $(1,1)$, similar results hold in all fixed non-axial directions.

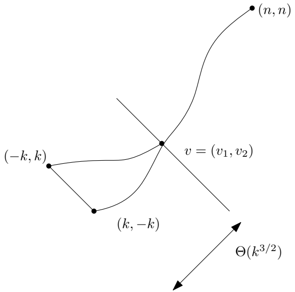
Almost surely the following hold:

- Starting from any $x \in \mathbb{Z}^2$, there exists a unique semi-infinite geodesic Γ_x in the direction $(1,1)$.
- Every sequence of finite geodesics from x to y_n where y_n has asymptotic direction $(1,1)$ converges to Γ_x.
- For $x \neq x'$, Γ_x and $\Gamma_{x'}$ coalesce.

Ferrari-Pimentel (2005), Coupier (20111)

Coalescence of Semi-infinite Geodesics

- Consider the semi-infinite geodesics from $(k,-k)$ and $(-k,k)$ in the direction $(1,1)$.
- $C(k)$ be such that the first point of intersection of these two geodesics lie on the line $x+y=C(k)$.



Theorem (B., Sarkar, Sly (2019))

There exists $C_1, C_2 > 0$ such that

$$C_1 R^{-2/3} \leq \mathbb{P}(C(k) \geq Rk^{3/2}) \leq C_2 R^{-2/3}.$$

- Lower bound was independently proved before. Pimentel (2016)

Coalescence of Finite Geodesics

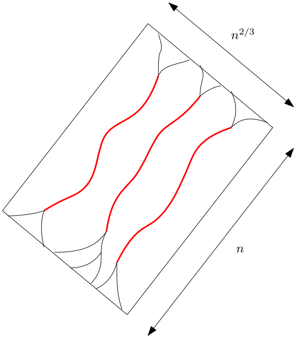
- Consider the same question as before but now for geodesics to (n, n).
- There exists $c > 0$ such that for $n \gg Rk$, $\mathbb{P}(C(k) \geq Rk^{3/2}) \leq R^{-c}$.

  B., Sarkar, Sly (2019)
- For $n \gg Rk$, $\mathbb{P}(C(k) \geq Rk^{3/2}) \leq R^{-2/3}$. Zhang (2020)
- Parallel results using joint distribution of Busemann increments.

  Balász, Busani, Seppäläinen(2020)

Disjoint Geodesics across a parallelogram

- Consider the parallelogram $\{0 \leq x + y \leq 2n, |x - y| \leq n^{2/3}\}$.
- Let N_n denote the maximum number of disjoint geodesics between the two sides of length $n^{2/3}$.
- Since any attractive region is likely to be used by every nearby geodesic, one expects most geodesics to merge with finitely many "highways".



Disjoint geodesics and nonexistence of bigeodesics

- One can use the one point estimates and the BK inequality to make this rigorous.
- N_n is uniformly tight with stretched exponential tails.

B., Hoffman, Sly (2018)

B., Ganguly, Hammond, Hegde (2020)

- This result goes into the proof of the optimal coalescence estimates.
- Also used to settle the bigeodesic existence problem.

Theorem (B., Hoffman, Sly (2018))

Almost surely the only bigeodesics in exponential LPP are lines parallel to the co-ordinate axes.

Key technical inputs

Integrable Inputs

- Curvature of the limit shape.
- Tracy-Widom convergence for point-to-point passage times.
- Uniform moderate deviation estimates:
  - $\mathbb{P}(T_{\mathbf{0},(m,n)} - (\sqrt{m}+\sqrt{n})^2 \geq xn^{1/3}) \leq Ce^{-cx^{3/2}}$.
  - $\mathbb{P}(T_{\mathbf{0},(m,n)} - (\sqrt{m}+\sqrt{n})^2 \leq -xn^{1/3}) \leq Ce^{-cx^{3}}$.

Tools from Percolation

- Correlation inequalities.
- Chaining argument.
- Geometric construction of favourable events at various scales.

Summary

- Exponential LPP is an exactly solvable model of last passage percolation where the geometry of geodesics is well understood.
- The methods include limited and streamlined inputs from integrable probability (curvature of limit shape together with one point moderate deviation estimates) together with percolation techniques.
- Expected to apply to all known models of exactly solvable planar LPP and also in the limit in some cases.
- Finer results than what we discussed today are known including the behaviour of geodesic trees, local geometry of the geodesics etc.
- Other techniques include stationary LPP, Busemann functions, Brownian Gibbs property etc.

Thank You

Questions?