The Wigner semicircle law for tensors

Răzvan Gurău

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whenever it exists.

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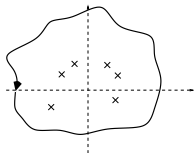Finite dimension – (λ, v) eigenpair $Tv = \lambda v \Rightarrow v \in \mathrm{Ker}(\lambda - T)$

The singular locus of the resolvent is the *spectrum of T*

Spectral theorem:

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Single trace invariants $\qquad \dfrac{1}{N}\text{Tr}\big[f(T)\big] = \displaystyle\int_\gamma \frac{dw}{2\pi i} \, f(w) \, \underbrace{\dfrac{1}{N}\text{Tr}\left[\dfrac{1}{w - T}\right]}_{\omega(w;T) \text{ function}}$
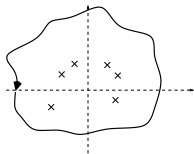
WHY IS THE RESOLVENT USEFUL?

Spectral theorem:

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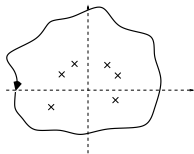γ simple curve encircling the spectrum.

Single trace invariants

$$\frac{1}{N}\mathrm{Tr}\big[f(T)\big] = \int_\gamma \frac{dw}{2\pi\imath} \, f(w) \, \underbrace{\frac{1}{N}\mathrm{Tr}\left[\frac{1}{w-T}\right]}_{\omega(w;T) \text{ function}}$$

Finite dimension – $\omega(w;T)$ has poles of order 1 at the eigenvalues of T with residues the multiplicities.

The Wigner semicircle law

Take T a real symmetric random matrix, distributed on GOE

$$d\nu(T) = \Big(\prod_{a \leq b} dT_{ab} \Big) \ \exp \Big\{ -\frac{N}{4} \sum_{a,b} T_{ab}^2 \Big\}$$

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$$\omega(w) = \lim_{N \to \infty} \int d\nu(T) \, \omega(w; T) = \frac{w}{2} \left(1 - \sqrt{1 - \frac{4}{w^2}} \right) \equiv w^{-1} \underbrace{T_2(w^{-2})}_{T_2 = 1 + u(T_2)^2} \; .$$

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Spectral representation: cut at $[-2, 2]$ and

$$\omega(w) = \int_{-2}^{2} d\lambda\, \frac{\rho(\lambda)}{w - \lambda}\ , \qquad \rho(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2}$$
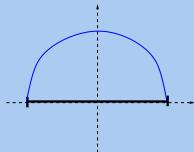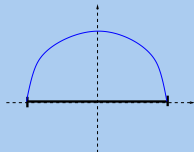
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$$\lim_{N \to \infty} \int d\nu(T) \, \frac{1}{N} \text{Tr}[f(T)] = \int d\lambda \, f(\lambda) \, \rho(\lambda)$$

For $\phi_a \in \mathbb{R}^N$ Gaussian field:

$$\mathcal{Z}(w; T) = \int [d\phi] \ \exp\left\{ -\underbrace{\frac{1}{2}\phi_a\left(\delta_{ab} - \frac{T_{ab}}{w}\right)\phi_b}_{S(\phi)} \right\} = \frac{1}{\sqrt{\det\left(1 - \frac{T}{w}\right)}}$$

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The resolvent is the two point function of the Gaussian 2−spin model

$$\omega(w; T) = \frac{w^{-1}}{\mathcal{Z}(w; T)} \int [d\phi] \, \frac{\phi^2}{N} \, e^{-S(\phi)} \qquad \omega(w; T) = \frac{1}{w} - \frac{2}{N}\frac{d}{dw}\ln \mathcal{Z}(w; T)$$

Tensors and Eigenvalues

T real symmetric tensor of order p has N^p real components

$$T_{a_1 \ldots a_p} = T_{a_{\sigma(1)} \ldots a_{\sigma(p)}} \ , \qquad T_{b_1 \ldots b_p} = \sum_{a_1 \ldots a_p} O_{a_1 b_1} \ldots O_{a_p b_p} T_{a_1 \ldots a_p}$$

$$\left(Tx^p = \sum_{a_1 \ldots a_p} T_{a_1 \ldots a_p} x_{a_1} \ldots x_{a_p}, \ Tx^{p-1} = \sum_{a_2 \ldots a_p} T_{a_1 \ldots a_p} x_{a_2} \ldots x_{a_p} \right)$$

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(λ, x) eigenpair of T iff $T x^{p-1} = \lambda x, \; x^2 = 1$

Real symmetric tensors have:

- at least two real eigenvalues (a continuous function on a compact set with no boundary attains its extrema which are critical points)
- at most $[(p-1)^N - 1]/(p-2)$ complex eigenvalues
- typically exponentially many real eigenvalues

Invariants in correspondence with p valent graphs $\mathcal{B}$:

$$\text{Tr}_{\mathcal{B}}(T) = \sum_{\{a\}} \prod_{v \text{ vertices}} T_{a_1^v \dots a_p^v} \prod_{(v,w) \text{ edges}} \delta_{a_i^v a_i^w}$$

Many graphs with n vertices...

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TRACE INVARIANTS

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$p = 2$ balanced invariant $I_n = \text{Tr}(T^n)$

$p \geq 3$ more complicated

$$I_2(T) = 3 \sum_{a_1, a_2, b} T_{a_1 a_1 b} T_{a_2 a_2 b} + 2 \sum_{a,b,c} T_{abc} T_{abc}$$

Partition function of the p–spin model:

$$\mathcal{Z}(w, T) = \int [d\phi] e^{-S(\phi)} , \qquad S(\phi) = \frac{1}{2}\phi^2 - \frac{1}{p\,w} \sum_{\{a\}} \underbrace{T_{a_1 \ldots a_p} \phi_{a_1} \ldots \phi_{a_p}}_{T\phi^p}$$

THE p–SPIN MODEL, $p \geq 3$

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Directional Borel Leroy sum of its perturbative series $w = |w| e^{i\,\psi}$:

$$\mathcal{Z}_\pm(w; T) = \int_{e^{i\,\theta_\pm}\,\mathbb{R}^N} [d\phi]\ e^{-S(\phi)} , \qquad \theta_\pm = \frac{1}{p}\left(\psi \mp \frac{\pi}{2}\right)$$

Cuts along $\mathbb{R}$, dominated by the largest and smallest real eigenvalues of T

$$\mathcal{Z}_+(y; T) - \mathcal{Z}_-(y; T) \sim \begin{cases} e^{-\frac{p-2}{2p}\left(\frac{y}{\lambda_{\max}}\right)^{\frac{2}{p-2}}} + \ldots , & y > 0 \\ e^{-\frac{p-2}{2p}\left(\frac{y}{\lambda_{\min}}\right)^{\frac{2}{p-2}}} + \ldots , & y < 0 \end{cases} .$$

The tensor resolvent – two point function of the p–spin model

$$\omega(w; T) = \frac{w^{-1}}{\mathcal{Z}(w; T)} \int [d\phi] \, \frac{\phi^2}{N} e^{-S(\phi)} \, , \quad \omega(w; T) = \frac{1}{w} - \frac{p}{N} \frac{d}{dw} \ln \mathcal{Z}(w; T)$$
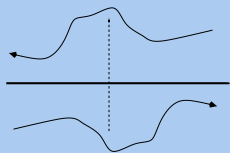
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Spectral theorem for balanced invariants:

$$I(T) = \sum a_n \frac{I_n(T)}{N} \, , \quad h_I(w) = \sum_n a_n w^n$$

$$I(T) = \int_\gamma \frac{dw}{2\pi \imath} \, h_I(w) \, \omega(w; T)$$

The generalized Wigner law

Take T a real symmetric random tensor, distributed on GOE

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Spectral representation: cut at $[-p^{p/2}/(p-1)^{(p-1)/2}, p^{p/2}/(p-1)^{(p-1)/2}]$ and

$$\omega(w) = \int_{cut} d\lambda\, \frac{\rho(\lambda)}{w - \lambda}, \qquad \rho(\lambda) = |\lambda| P_p(\lambda^2)$$

$$P_p(x) = \sum_{k=1}^{p-1} \Lambda_{k,p}\, x^{\frac{k-p}{p}}\, {}_{p-1}F_{p-2}\left(\left\{ 1 - \frac{1+j}{p-1} + \frac{k}{p} \right\}_{j=1}^{p-1}, \left\{ 1 + \frac{k-j}{p} \right\}_{\substack{j=1 \\ j \neq k}}^{p-1}; \frac{(p-1)^{p-1}}{p^p} x \right),$$

$$\Lambda_{k,p} = \frac{1}{(p-1)^{3/2}} \sqrt{\frac{p}{2\pi}} \left(\frac{(p-1)^{p-1}}{p^p} \right)^{\frac{k}{p}} \frac{\prod_{j=1,\ldots p-1}^{j \neq k} \Gamma\left(\frac{j-k}{p} \right)}{\prod_{j=1}^{p-1} \Gamma\left(\frac{j+1}{p-1} - \frac{k}{p} \right)} .$$

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$$\lim_{N \to \infty} \int d\nu(T)\, I(T) = \int d\lambda\, h_I(\lambda)\, \rho(\lambda)$$

The $p = 3$ case

$$\omega(w) = \frac{\imath}{3^{1/2}} \left[\left(\sqrt{1 - \frac{3^3/2^2}{w^2}} - \imath \frac{3^{3/2}/2}{w} \right)^{1/3} - \left(\sqrt{1 - \frac{3^3/2^2}{w^2}} + \imath \frac{3^{3/2}/2}{w} \right)^{1/3} \right],$$

$$\rho(\lambda) = \frac{1}{2\pi|\lambda|^{1/3}} \left(\frac{3^3}{2^2} \right)^{1/6} \left[\left(1 + \sqrt{1 - \frac{\lambda^2}{3^3/2^2}} \right)^{1/3} - \left(1 - \sqrt{1 - \frac{\lambda^2}{3^3/2^2}} \right)^{1/3} \right],$$

$$\int d\nu(T)\, \omega(w; T) = \frac{1}{w} - \frac{p}{N}\frac{d}{dw}\int d\nu(T)\, \ln \mathcal{Z}(w; T)$$

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quenched $\simeq$ annealed

$$\int d\nu(T)\, \ln \mathcal{Z}(w; T) = \ln \int d\nu(T)\, \mathcal{Z}(w; T)$$

How it works

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Annealed is

$$\int [dT][d\phi]\, \exp\left\{ -\frac{N^{p-1}}{2p}\sum T^2_{a_1\ldots a_p} - \frac{1}{2}\phi^2 + \frac{1}{p\,w}T\phi^p \right\}$$

integrate out T

$$\omega(w) = \frac{1}{w} - \frac{p}{N} \frac{d}{dw} \ln \int d\nu(T) \mathcal{Z}(w; T)$$

$$\int d\nu(T) \mathcal{Z}(w; T) \sim \int d\rho \, e^{Nf(\rho)} \qquad f(\rho) = \ln \rho - \frac{1}{2}\rho^2 + \frac{1}{2pw}\rho^{2p}$$

Resolvent is $\omega(w) = w^{-1}\rho_0^2$ with ρ_0 dominant saddle point.

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Saddle point equations

$$f'(\rho) = \frac{1}{\rho}\left(1 - \rho^2 + \frac{1}{w^2}\rho^{2p}\right) \;\Rightarrow\; \rho_0^2 = T_p(w^{-2})$$

Signal + Gaussian noise:

$$A_{a_1 \ldots a_p} = \frac{b}{N^{\frac{p}{2}-1}} \, v_{a_1} \ldots v_{a_p} + T_{a_1 \ldots a_p} \, ,$$

with v a fixed vector in $\mathbb{R}^N$ with $v^2 = 1$ and T GOE.

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For what value of b is the spike detectable?

THE SPIKED MODEL AND THE DETECTION THRESHOLD

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For what value of b is the spike detectable?

$$b_t = \frac{(p-1)^{p/2}}{(p-2)^{(p-2)/2}}$$

at b_t the largest non removable singularity of the resolvent jumps from $p^{p/2}/(p-1)^{(p-1)/2}$ to $p^{p/2}$.

$$\omega(w) = \frac{1}{w} - \frac{p}{N}\frac{d}{dw}\ln\int d\theta d\rho \, e^{Nf(\theta,\rho)}$$

$$f(\theta,\rho) = \ln(\sin\theta) + \ln\rho - \frac{1}{2}\rho^2 + \frac{b}{wp}\rho^p\cos^p\theta + \frac{1}{2pw^2}\rho^{2p} \; .$$

Resolvent is $\omega(w) = \frac{1}{w}\rho_\star^2$ with $\star$ the dominant saddle point

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Saddle point equations:

$$\partial_\theta f = \frac{\cos\theta}{\sin\theta} - \frac{b}{w}\rho^p\cos^{p-1}\theta\sin\theta \; , \;\; \partial_\rho f = \frac{1}{\rho}\left(1 - \rho^2 + \frac{b}{w}\rho^p\cos^p\theta + \frac{1}{w^2}\rho^{2p}\right) \; ,$$

Two solutions $\theta_0 = \pi/2$, $\rho_0^2 = T_p(w^{-2})$ and (θ_1, ρ_1^2) with $\theta_1 \neq \pi/2$.

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At b_t the saddle point (θ_1, ρ_1^2) becomes dominant at the largest non removable singularity of $\omega(w)$.

PROBLEM 1: SPIKED MELONS?

Quenched is

$$\int [dT] \, \exp\left\{ -\frac{N^{p-1}}{2p} \sum T^2_{a_1 \ldots a_p} \right\} \ln \left(\int [d\phi] \, \exp\left\{ -\frac{1}{2}\phi^2 + \frac{1}{p\,w} T\phi^p \right\} \right)$$
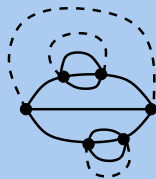
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Feynman expand in $\phi \rightarrow$ maps
Logarithm $\rightarrow$ connected maps.

$\int [dT]$, $N \rightarrow \infty \rightarrow$ melons, each brings 1.

$\omega(w) \rightarrow$ generating function of rooted melons.
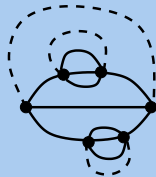
Quenched is

$$\int [dT] \, \exp \left\{ - \frac{N^{p-1}}{2p} \sum T^2_{a_1 \ldots a_p} \right\} \ln \left(\int [d\phi] \, \exp \left\{ - \frac{1}{2}\phi^2 + \frac{1}{p \, w} T\phi^p \right\} \right)$$

Feynman expand in $\phi \to$ maps
Logarithm $\to$ connected maps.

$\int [dT]$, $N \to \infty \to$ melons, each brings 1.

$\omega(w) \to$ generating function of rooted melons.

- build the graphical representation for the spiked model.
- what is b_t in this language?

Problem 2: best rank 1 approximation

The real eigenpair with maximal eigenvalue yields the best rank one approximation of T

$$\inf_{\alpha, x} \left\{ ||T - \alpha x^{\otimes p}||_F \,\middle|\, x^2 = 1 \right\}$$

Substituting $T \to \lambda v^{\otimes p}$ in any invariant $\mathcal{B}_n$ with n vertices $\text{Tr}_{\mathcal{B}_n}(T) = \lambda^n$

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For $p = 2$

$$\mathrm{Tr}(T^n) \sim \lambda^n + \mathrm{small}$$

Is for $p \geq 3$

$$\mathrm{Tr}_{\mathcal{B}_n}(T) \sim \lambda^n + \mathrm{small} \quad ?$$

Identify the signal v in the spiked model

$$A_{a_1 \ldots a_p} = \frac{b}{N^{\frac{p}{2}-1}} \, v_{a_1} \ldots v_{a_p} + T_{a_1 \ldots a_p} \, ,$$

for $b > b_t$