Monotonicity and phase transition for the VRJP and the ERRW

Rémy Poudevigne

Université Lyon 1

Definition (Diaconis,Coppersimth)

Let $\mathscr{G} = (V, E)$ be a locally-finite, connected, non-directed graph and $x_0 \in V$ a vertex of this graph. The edge-reinforced random walk with initial weights $(a_e)_e \in E$ is the process $(X_n)_{n\in\mathbf{N}}$ defined by $X_0 = x_0$ and:

$$\mathbb{P}\left(X_{n+1} = y | X_0, \ldots, X_n\right) = 1_{\{X_n, y\}\in E} \frac{a_{\{X_n,y\}} + N_n(\{X_n, y\})}{\sum\limits_{z,\{X_n,z\}\in E} a_{\{X_n,z\}} + N_n(\{X_n, z\})},$$

where

$$N_n(\{x, y\}) = \sum_{i=0}^{n-1} 1_{\{X_i, X_{i+1}\}=\{x,y\}}.$$

Theorem (Pemantle Merkl,Rolles Sabot,Zeng)

For $d \in \{1, 2\}$, the ERRW is recurrent for any initial weight a.

Theorem (Sabot,Tarrès Angel,Crawford,Kozma Disertori,Sabot,Tarrès)

For any $d \geq 3$, there exists $a_r, a_t \in (0, \infty)$ such that for an initial weight a the ERRW in Z^d is recurrent if $a < a_r$ and transient $a > a_t$.

Theorem (P)

For any $d \geq 3$, there exists $a_d \in (0, \infty)$ such that for an initial weight a the ERRW in Z^d is recurrent if $a < a_d$ and transient if $a > a_d$.

Definition (Davis,Volkov)

Let $\mathscr{G} = (V, E)$ be a locally-finite, connected, non-directed graph and $x_0 \in V$ a vertex of this graph. The vertex reinforced jump process with initial weights $(W_e)_{e \in E}$ is the process $(Y_t)_{t \in \mathbf{R}}$ that starts at x_0 and jumps to a neighbour vertex y at a rate

$$1_{\{X_t, y\} \in E} W_{\{X_t, y\}} (1 + \ell_y(t)),$$

where

$$\ell_y(t) = \int_{s=0}^{t} 1_{X_s = y} \mathrm{d}s.$$

The ERRW and the VRJP have similar behaviours. This is explained by the following result.

Proposition (Sabot, Tarrès 2013)

The ERRW on a locally finite $\mathcal{G} = (V, E)$ with initial weights $(a_e)_{e\in E}$ is a mixture of discrete time VRJP where the initial weights $(W_e)_{e\in E}$ are independent gamma random variables of parameter a_e: $W_e \sim \Gamma(a_e)$.

Proposition (Disertori,Spencer,Zirnbauer Sabot,Tarrès)

(i) The probability measure μ_n^W *on* $\mathscr{H}_{i_0}^n = \{u \in \mathbf{R}^n, u_{i_0} = 0\}$ *is defined by the density:*

$$\mu_n^{W,i_0}(\mathrm{d}u) := \left(\frac{1}{2\pi}\right)^{\frac{n-1}{2}} e^{-\sum u_i} e^{-\frac{1}{2}\sum_{i\sim j} W_{\{i,j\}}(e^{u_i-u_j}+e^{u_j-u_i}-2)} \sqrt{|H_{W,u}|_{n-1}}\mathrm{d}u_1 \ldots \mathrm{d}u_{n-1},$$

where $H_{W,u}(i,i) = \sum_{j\sim i} W_{\{i,j\}} e^{u_i+u_j}$, $H_{W,u}(i,j) = -W_{\{i,j\}} e^{u_i+u_j}$ *and* $|H_{W,u}|_{n-1}$ *is the determinant of any minor of* $H_{W,u}$.

(ii) The VRJP on a finite graph (V,E) *with weigths* $(W_e)_{e\in E}$ *is a time-changed random walk in random reversible environments. The environment is given by conductances* $W_{\{x,y\}} e^{U_x+U_y}$ *where the random variable* U *has a probability distribution given by* $\mu_{|V|}^W$.

We can see that for small values of u, the density

$$\mu_n^W(\mathrm{d}u) := \left(\frac{1}{2\pi}\right)^{\frac{n-1}{2}} e^{-\sum u_i} e^{-\frac{1}{2}\sum\limits_{i\sim j} W_{\{i,j\}}(e^{u_i-u_j}+e^{u_j-u_i}-2)} \sqrt{|H_{W,u}|_{n-1}}\mathrm{d}u_1\dots\mathrm{d}u_{n-1},$$

is similar to that of the GFF $(Y_x)_{x\in V}$ where we impose $Y_{i_0}=0$:

$$\mathbf{g}_n^W(\mathrm{d}y) := \left(\frac{1}{2\pi}\right)^{\frac{n-1}{2}} e^{-\frac{1}{2}\sum\limits_{i\sim j} W_{\{i,j\}}(y_i-y_j)^2} \sqrt{|H_W|_{n-1}}\mathrm{d}y_1\dots\mathrm{d}y_{n-1},$$

where $H_W(i,i) := \sum\limits_{j\sim i} W_{\{i,j\}} = H_{W,0}(i,i)$ and $H_W(i,j) := -W_{\{i,j\}} = H_{W,0}(i,j)$.

Proposition

For any weights W and any vertex j we have:

$$\mathbb{E}_{\mu_n^W(\mathrm{d}u)}(e^{u_j}) = 1.$$

Proposition

For any choice of W, the law of $\frac{1}{2}\sum_{i\sim j} W_{\{i,j\}}(e^{U_i-U_j} + e^{U_j-U_i} - 2)$ is a Gamma of parameter $(n-1)/2$.

Similarly for the GFF, the random variable $\frac{1}{2}\sum_{i\sim j} W_{\{i,j\}}(Y_i - Y_j)^2$ is also a Gamma of parameter $(n-1)/2$.

This representation is not always practical so a new one was introduced.

Definition

Let γ be a Gamma random variable of parameters $(1/2, 1/2)$ independent of the rest. The β-field $(\beta_x)_{x\in V}$ is defined by:

$$\forall x \in V, \beta_x := \sum_{y\sim x} W_{\{x,y\}} e^{U_y - U_x} + \gamma 1_{x=i_0}$$

The equivalent for the GFF is the vector $(B_x)_{x\in V}$ defined by:

$$\forall x \in V, B_x := \sum_{y\sim x} W_{\{x,y\}} (Y_y - Y_x).$$

This can also be written has $B = H_W Y$.

The law of the β-field is characterized by the following density.

Definition (Sabot,Tarrès,Zeng Letac)

Set $n \in \mathbf{N}$ and non-negative weights $(W_{i,j})_{1\leq i,j\leq n}$. Let $\eta \in [0,\infty)^n$ be a vector and let $1_n \in R^n$ be the vector with only ones. We can define the probability measure $\nu_n^{W,\eta}$ on $\mathbf{R}^n$ by the density:

$$\nu_n^{W,\eta}(\mathrm{d}\beta) := \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\left({}^t 1_n H_\beta 1_n + {}^t\eta H_\beta^{-1}\eta - 2\,{}^t 1_n \eta\right)} \frac{1}{\sqrt{Det(H_\beta)}} 1_{H_\beta>0}\mathrm{d}\beta_1 \ldots \mathrm{d}\beta_n,$$

where $H_\beta(i,i) = \beta_i$, $H_\beta(i,j) = -W_{\{i,j\}}$ and $H_\beta > 0$ means that H_β is positive definite.

For the GFF, the vector $(B_x)_{x\in V}$ is a degenerate centered gaussian vector of covariance matrix H_W such that $\sum B_x = 0$.

Proposition

The law of the β-field does not depend on the starting point.

Proposition

For any non-negative coefficients $(\lambda_x)_{x\in V}$:

$$\mathbb{E}\left(e^{-\sum\limits_{x\in V}\lambda_x\beta_x}\right) = e^{-\sum\limits_{\{x,y\}\in E} W_{\{x,y\}}(\sqrt{1+2\lambda_x}\sqrt{1+2\lambda_y}-1)}\prod_{x\in V}\frac{1}{\sqrt{1+2\lambda_x}}.$$

Proposition

Let V_1, V_2 be two subset of V such that for any $(x,y)\in V_1\times V_2$, $d(x,y)\geq 2$. The beta-fields in V_1 and V_2 are independent.

Proposition

Let V_1 be a subset of V. The marginal law of the β-field on V_1 is $\nu_{|V_1|}^{W,\eta}$ for some η.

Proposition

Let V_1, V_2 be a partition of V. Let W_e be non-negative weights. Let β be distributed according to $\nu_{|V|}^{W}$, the law of $(\beta_x)_{x\in V_1}$ knowing $(\beta_y)_{y\in V_2}$ is $\nu_{|V_1|}^{W+W'}$ where W' are non-negative weights that do not depend on $(\beta_x)_{x\in V_1}$.

Let V_1, V_2 be a partition of V. Let W_e be non-negative weights. Let B be a centered gaussian vector of covariance matrix H_W with:

$$\begin{pmatrix} H_W^1 & -W^{12} \\ -W^{21} & H_W^2 \end{pmatrix}.$$

Let B_1, B_2 be the restriction of B to V_1 and V_2 respectively. Let $\overline{B}_1$ be defined by :

$$\overline{B}_1 := B_1 + W^{12}\left(H_W^2\right)^{-1} B_2.$$

The vectors $\overline{B}_1$ and B_2 are independent and centered. Furthermore, the value of $\overline{B}_1$ only depends on the GFF on V_1.

Link between β and U

Theorem (Sabot, Tarrès, Zeng)

Set a finite graph $\mathscr{G} = (V, E)$ and non-negative weights $(W_e)_{e\in E}$. Let H_β be distributed according to ν_n^W and let G_β be its inverse. For any $i_0 \in V$, the random walk in random reversible environment given by the random conductances $(\omega_e)_{e\in E}$ defined by:

$$\omega_{x,y} := W_{x,y} G_\beta(i_0, x) G_\beta(i_0, y)$$

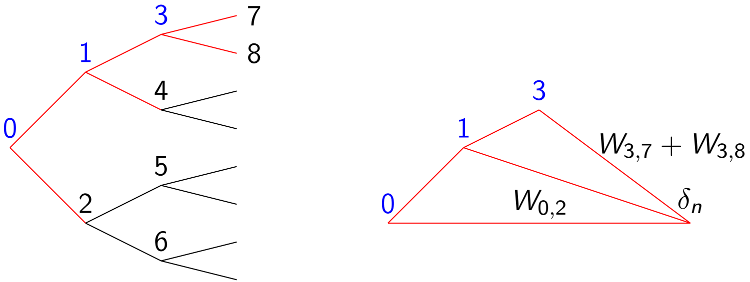
has the same law as a time-change of the VRJP with initial weights $(W_e)_{e\in E}$ and starting point i_0.

This allows us to get back U from β.

Proposition

For any $i_0, x \in V$, $\mathbb{E}\left(\frac{G_\beta(i_0,x)}{G_\beta(i_0,i_0)}\right) = 1$.

For an infinite, connected, locally finite graph $\mathscr{G} = (V, E)$ and initial weights $(W_e)_{e \in E}$ we look at the sequence of graphs $\mathscr{G}_n = (V_n, E_n)$ obtained by keeping a finite subset of $\mathscr{G}$ and collapsing all other vertices into one vertex δ_n. We define $\psi_n(x) := \frac{G_\beta(\delta_n, x)}{G_\beta(\delta_n, \delta_n)}$.



Proposition

For some choice of coupling of the β-fields on a sequence of graphs $\mathcal{G}_n$ that is increasing, for any $x \in V$, for n large enough:

$$\mathbb{E}\left(\psi_{n+1}(x)|\psi_n(x)\right) = \psi_n(x).$$

Since $\psi_n(x) \geq 0$, there exists a random variable $\psi_\infty(x)$ such that a.s

$$\psi_n(x) \to \psi_\infty(x).$$

Theorem (Sabot,Zeng)

If $\psi_\infty(0) = 0$ the VRJP starting at 0 is recurrent, otherwise it is transient.

Theorem ((P))

Let $G=(V,E)$ be a finite graph and $(W_e^-)_{e\in E}$ and $(W_e^+)_{e\in E}$ two families of weights such that for all $e\in E$, $0<W_e^-\leq W_e^+$. For any $x\in V$, there exists $(\beta_i^-)_{i\in V}$ and $(\beta_i^+)_{i\in V}$ distributed according to $\nu_{|V|}^{W^-}$ and $\nu_{|V|}^{W^+}$ respectively such that:

$$\forall y\in V,\ \mathbb{E}\left(\frac{G_{\beta^-}(x,y)}{G_{\beta^-}(x,x)}\Big|\beta^+\right)=\frac{G_{\beta^+}(x,y)}{G_{\beta^+}(x,x)}.$$

This is the same as saying that for any convex function f:

$$\mathbb{E}_{W^-}\left(f\left(\frac{G_\beta(x,y)}{G_\beta(x,x)}\right)\right)\geq\mathbb{E}_{W^+}\left(f\left(\frac{G_\beta(x,y)}{G_\beta(x,x)}\right)\right).$$

The idea is to look at what happens when we decrease a weight $W_{\{x,y\}}$.
We look at the following partition of V: $V_1 := \{x, y\}$ and $V_2 := V \setminus \{x, y\}$.

Proposition

The law of the β-field on V_2 does not depend on the weight $W_{\{x,y\}}$.

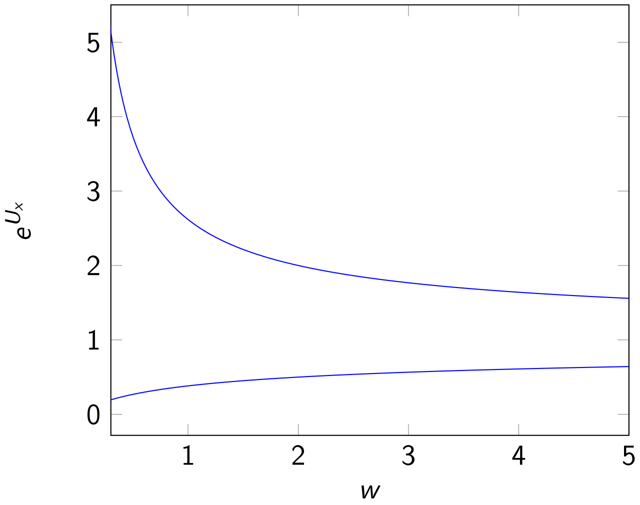
This means that we can condition on the value of the β-field on V_2 and look at what the impact of $W_{\{x,y\}}$ is on V_1.
For the GFF, the covariance matrix of B is given by $H_{W,0}$ with $H_{W,0}(i,i) = \sum_{j \sim i} W_{\{i,j\}}$ and $H_{W,u}(i,j) = -W_{\{i,j\}}$. So the law of B on V_2 does not depend on $W_{\{x,y\}}$.

We will look at the simple case when the graph has only two points i_0, x joined by an edge of weight w. In the general case the starting point is a mixture of the two points.

$$\frac{1}{2\pi} e^{-\sum u_x} e^{-\frac{1}{2} w(e^{u_x}+e^{-u_x}-2)} \sqrt{we^{u_x}} \mathrm{d}u_x.$$

The law of $K := \frac{w}{2}(e^{U_x} + e^{-U_x} - 2)$ is a gamma of parameters $(1/2, 1)$.
Knowing K, the expectation of e^{U_x} is 1.
Knowing K, the smaller w is, the further away from 1 the random variable U_x is.

For a gaussian Y of variance σ^2, the equivalent of k is the random variable $\frac{1}{2\sigma^2} Y^2$ which is also a gamma of parameters $(1/2, 1)$. For the coupling of two centered gaussians Y_+ and Y_- of variances σ_+^2 and σ_-^2, this would be the same has sending Y_- onto $\frac{\sigma_+}{\sigma_-} Y_-$ and $-\frac{\sigma_+}{\sigma_-} Y_-$.



5
4
3
2
1
0
e^{U_x}
1
2
3
4
5
w

A few other consequences

Theorem (P)

If the simple walk on a graph (V,E) with conductances $(W_e)_{e\in E}$ is recurrent then so is the ERRW and the VRJP with initial weights $(W_e)_{e\in E}$.

Theorem (Sabot,Zeng)

On a graph (V,E) the ERRW and the VRJP with initial weights $(W_e)_{e\in E}$ are recurrent with probability 0 or 1 (almost every environment are recurrent or almost every environment are transient) if the graph and the weigths are invariant by translation.

Theorem (P)

On a graph (V,E) the ERRW and the VRJP with initial weights $(W_e)_{e\in E}$ are recurrent with probability 0 or 1.

Recurrence on recurrent graphs