Phase Transitions of Replica Symmetry Breaking for Random Regular NAESAT

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Introduction:
random constraint satisfaction problems;

Combinatorics and Theoretical Computer Science

Constraint satisfaction problem (CSP): is it possible to assign values to a set of *variables* to satisfy a given set of *constraints*?

- Scheduling your appointments for the day
- System of linear equations.
- Colouring a graph or finding a large independent set.
- Satisfying a Boolean formula.

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Random CSPs

Our focus is to investigate properties when the constraints are chosen randomly.

Combinatorial properties of Random Graphs:

- Erdős-Rényi Random Graph: $G(n, \alpha/n)$ with n vertices and edges with probability α/n (average degree α).
- Random α-regular graph: Uniformly chosen from α-regular graphs on n vertices.

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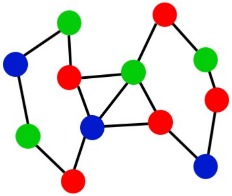
When is there a proper k-colouring?



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Variant NAE-SAT: An assignment $\underline{x}$ is a solution if both $\underline{x}$ *and* $-\underline{x}$ are satisfying. It's **regular** if every variable is in the same number of clauses.

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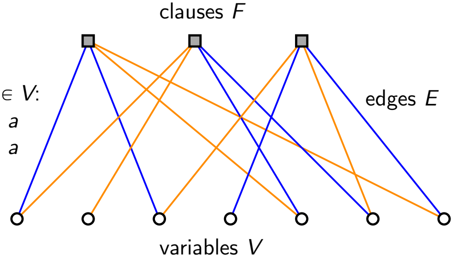
We can encode the formula as a bipartite graph $\mathscr{G} \equiv (V, F, E)$:

clause $a \in F$, variable $v \in V$:
blue edge *(av)* if $+x_v$ in clause a
yellow edge *(av)* if $-x_v$ in clause a



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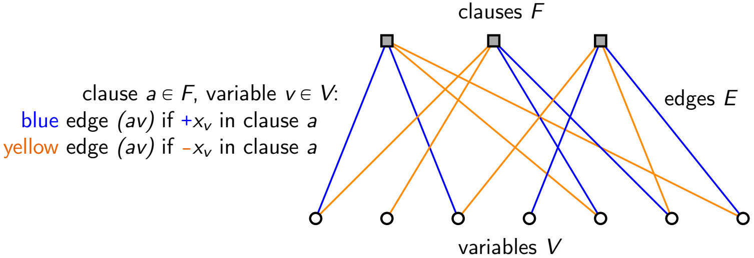

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The resulting random graph is locally tree-like, almost no short cycles and it's local distribution can be described completely.

Main Question:

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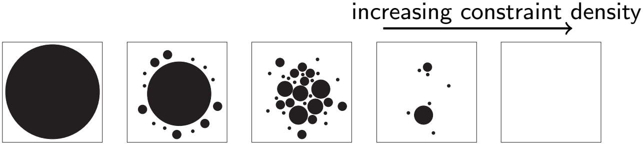
Other Question:

- *Free Energy*: How many solutions are there?
- *Local Statistics*: Properties of solutions such as how many clauses are satisfied only once?
- *Overlaps*: What does the joint distribution of several solutions look like?
- *Algorithmic*: Can solutions be found efficiently?

Theoretical Physics

Disordered systems such as *spin glasses* are models of interacting particles/variables with frustrated interactions. Many random constraint satisfaction problems can be recast as dilute mean-field spin glasses.



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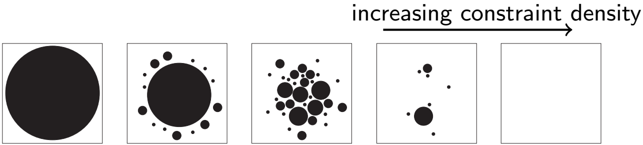

One-step Replica Symmetry Breaking Predictions:
Developed to study dense spin-glasses such as the Sherrington-Kirkpatrick model.

- **Replica Symmetry Breaking**: Clustering of assignments.
- **Cavity Method**: Heuristic for analyzing adding one variable.

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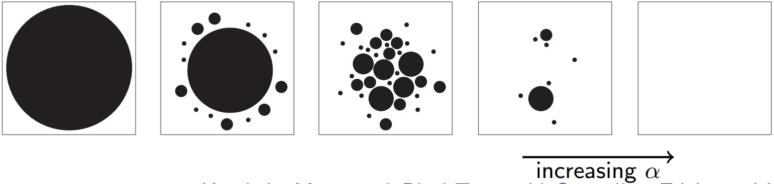
For NAE-SAT, colourings, second moment method succeeds up to $\alpha_2 = \alpha_{\text{sat}} - O(1)$. Fails, for K-SAT for **all** $\alpha > 0$.

Some physics perspective:
condensation and replica symmetry breaking

Phase Diagram

Two solutions are connected if they differ by one bit.

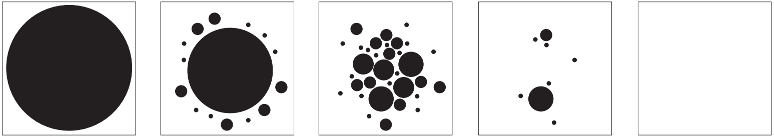


Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07

Montanari–Ricci-Tersenghi–Semerjian '08

Phase Diagram

well-connected



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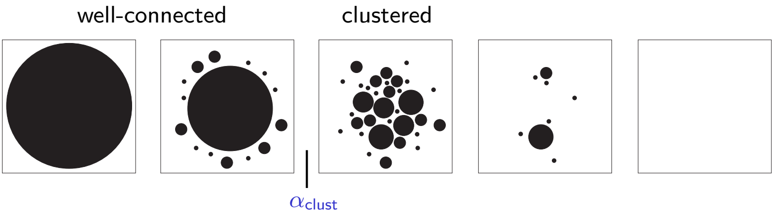

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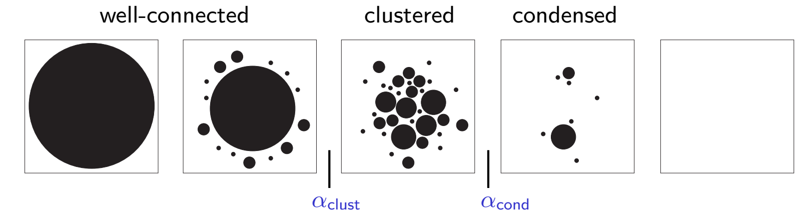

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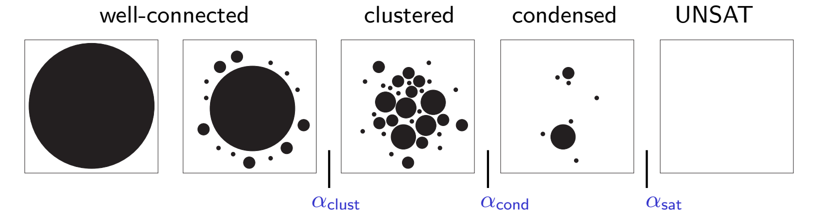

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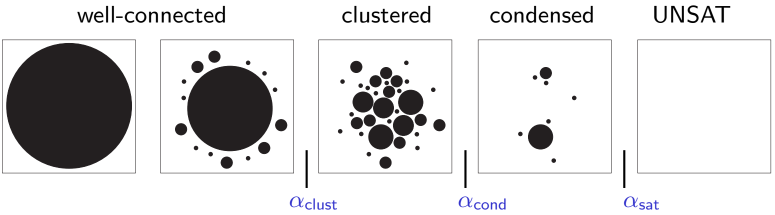

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RSB: These models are called one step replica symmetry breaking (1RSB) because the overlap of two uniformly chosen solutions is concentrated on two points in the condensation regime.

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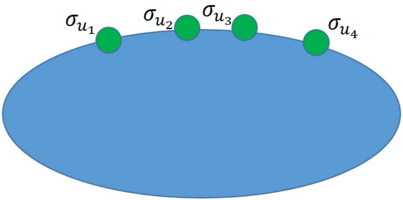
We call this the frozen model. Let Ω_n be the number of $\{+, -, \mathtt{f}\}^{V(\mathscr{G})}$ configurations. Locally rigid resulting in no clustering.

Cavity Method: Adding a new vertex v (or clause).



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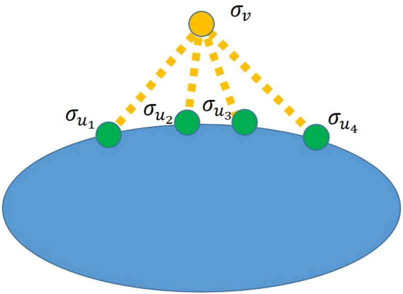

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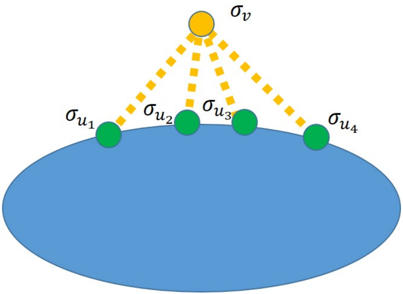

If we know the joint distribution of σ_{u_i} we can:

1. Calculate the law of σ_v
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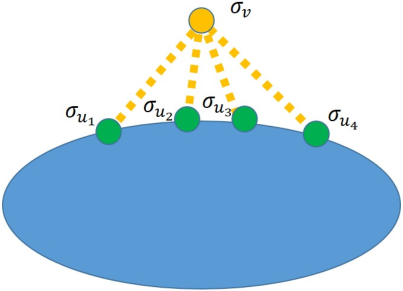

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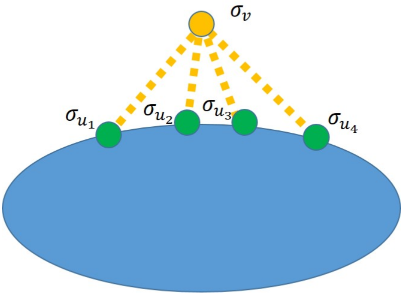

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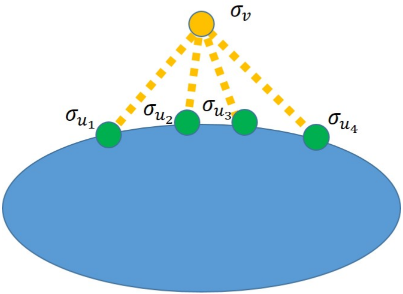

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Self-consistency: The law of σ_v should also be drawn from μ which means μ must satisfy a fixed point equation.

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Expected change in $\log \Omega_n$ *to* $\log \Omega_{n+1}$.

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Then the 1RSB prediction $d_{SAT} \approx 2^{k-1} k \log 2 + O(1)$ is the root of $\Phi(\alpha) = 0$. –Ding, S., Sun '16

Beyond the Satisfiability Threshold

Condensation

Complexity function $\Sigma \equiv \Sigma_\alpha(s)$ such that:

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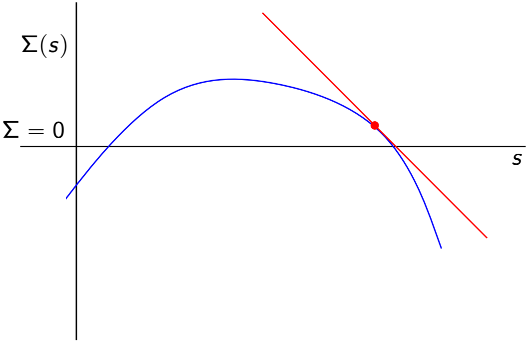

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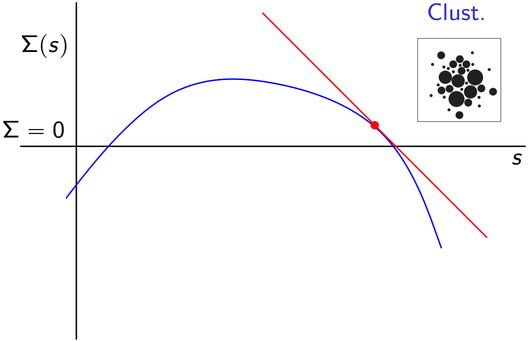

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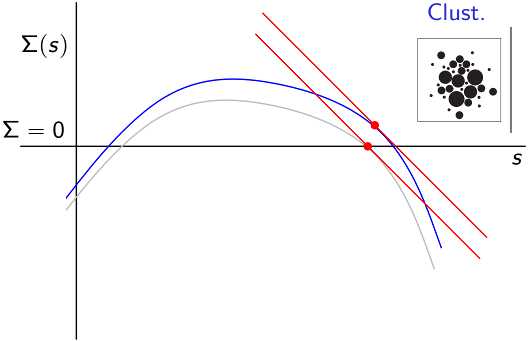

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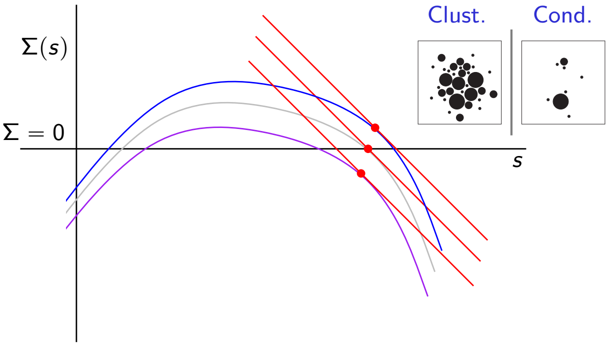

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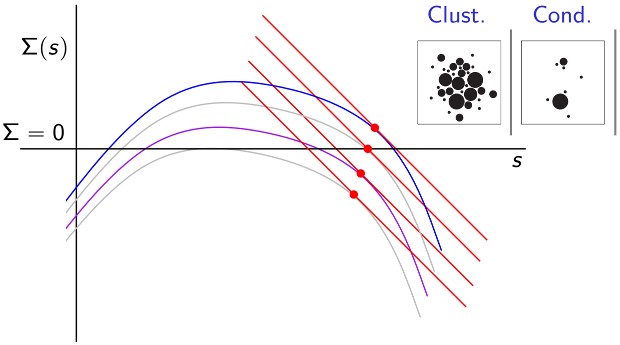

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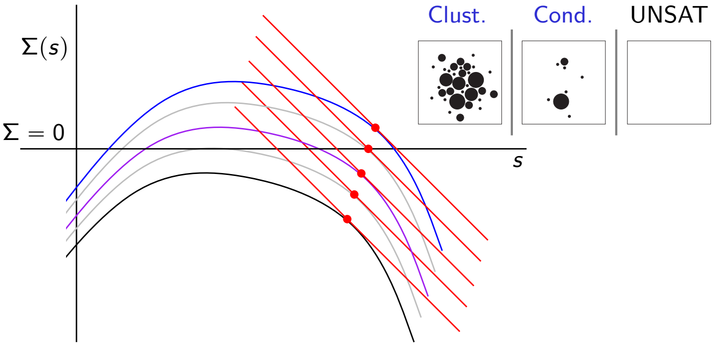

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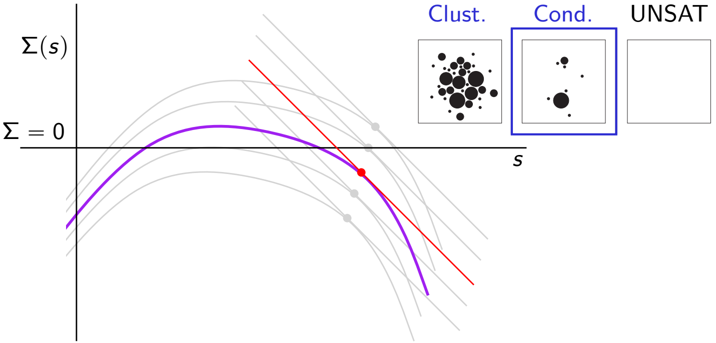

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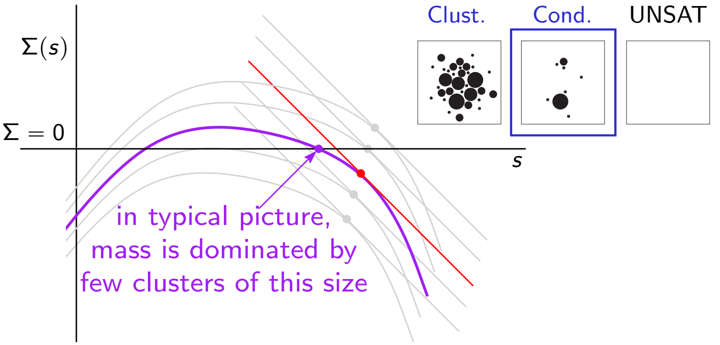
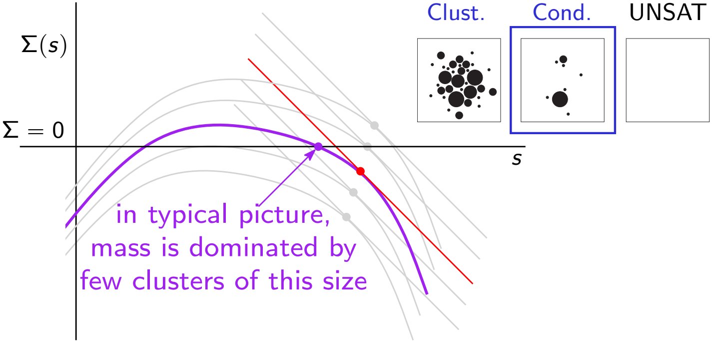


Condensation and non-concentration



The 1-RSB prediction:

- Satisfiability Threshold

$$\alpha_{\text{sat}} = \sup\{\alpha : \sup_s \Sigma(s) \geqslant 0\}$$

- Condensation Threshold and free energy

$$\alpha_{\text{cond}} = \sup\left\{\alpha : \sup_s s + \Sigma(s) = \sup_{s:\Sigma(s)\geqslant 0} s + \Sigma(s)\right\}$$

$$\Phi = \lim_{n\to\infty} \frac{1}{n} \log Z = \sup\{s + \Sigma(s) : \Sigma(s) > 0\} = \sup\{s : \Sigma(s) > 0\}$$

Results beyond the condensation threshold:

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Condensation Threshold:

Random *k*-Colourings G(n,p) large *k* [Bapst, Coja-Oghlan, Hetterich, Rassmann, Vilenchik]

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Condensation Regime Free Energy:

Regular k-NAESAT large k [S', Sun, Zhang]

Satisfiability Threshold:

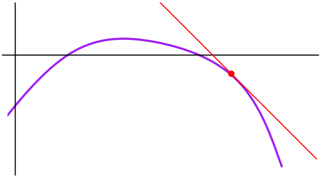
Regular NAESAT large k [Ding, S', Sun]

Maximum Independent Set *d*-Regular, large *d* [Ding, S', Sun]

Regular SAT, large *k* [Coja-Oghlan, Panagiotou]

Random *k*-SAT, large *k* [Ding, S', Sun]

Free Energy



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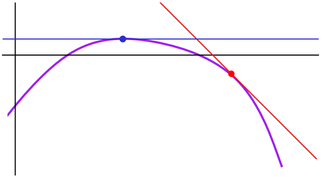

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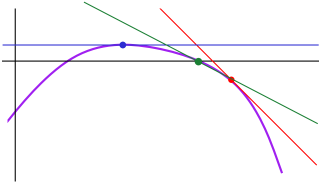

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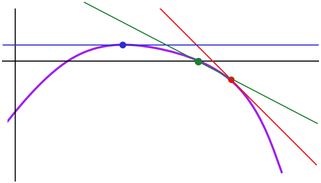

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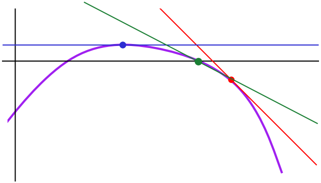

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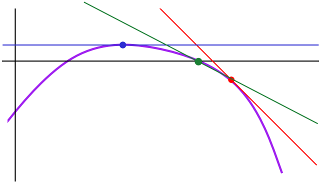

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The moments of Z_λ may be computed by adding local weights to the free variables in the $\{+, -, \mathtt{f}\}$ configurations.

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For regular NAE-SAT and $k \geqslant k_0$, the limit $\Phi(\alpha)$ exists for $\alpha_{\text{cond}} < \alpha < \alpha_{\text{sat}}$, given by an explicit formula matching the 1-RSB prediction from statistical physics. S., Sun, Zhang '16

New Results

Theorem (Nam, S., Sohn 2020) *For $k \geqslant k_0$ (absolute constant), random regular k-NAESAT, WHP the largest and second largest clusters both have a constant fraction of the set total solutions. Two uniformly chosen solutions have normalized hamming distance concentrated on* **two** *points*.

- Requires estimating the partition function up to multiplicative $O(1)$ factor.
- States space of free trees is unbounded.

Thanks!