## Inhomogeneous percolation on ladder graphs

#### Réka Szabó

joint work with Daniel Valesin

May 28, 2020

#### Online probability seminar, Lyon

R. Szabó, D. Valesin

Inhom. percolation on ladder graphs

May 28, 2020 1/60

# The inhomogeneous percolation framework

 $\mathbb{G}=(\mathbb{V},\mathbb{E})$  oriented/non-oriented graph, split the edge set into two disjoint sets:  $\mathbb{E}=\mathbb{E}'\cup\mathbb{E}''$ 

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Let  $C_\infty$  denote the event that there is an infinite open cluster, then for any  $q \in [0,1]$  define

$$p_c(q) := \sup\{p : \mathbb{P}_{p,q}(C_\infty) = 0\}.$$

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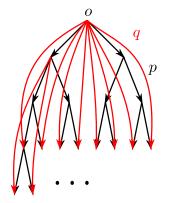
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$$p_c(q) := \sup\{p : \mathbb{P}_{p,q}(C_\infty) = 0\}.$$

What can we say about  $q \mapsto p_c(q)$ ?

## Related work

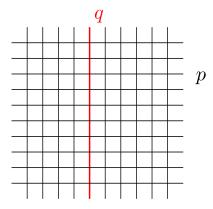
de Lima, Rolla, Valesin (17): oriented d-regular tree with additional edges of length k



 $p_c(q)$  is continuous and strictly decreasing in the region where it is positive

## Related work

Zhang (94): bond percolation on  $\mathbb{Z}^2$ 



 $p_c(q)$  is constant on (0,1)

no infinite cluster at  $p_c(q)$ 

May 28, 2020 4 / 60

# Construction of ladder graphs

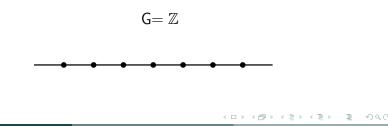
- $\boldsymbol{G}$  is an arbitrary 'base graph' that is
  - connected
  - locally finite
  - (infinite)

We construct a non-oriented and an oriented ladder graph from  ${\it G}$ 

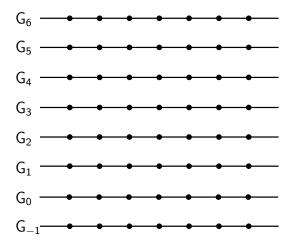
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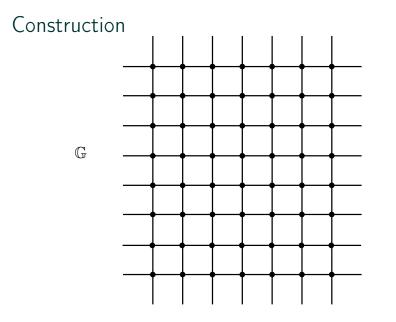


#### Construction



May 28, 2020 6 / 60

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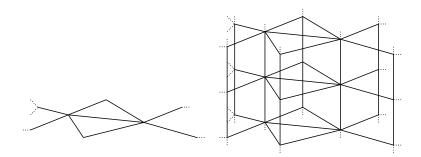


May 28, 2020 7 / 60

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### Construction



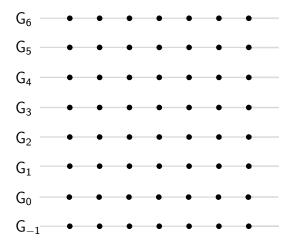
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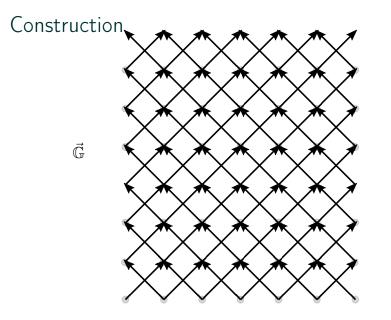
#### Construction



May 28, 2020 9 / 60

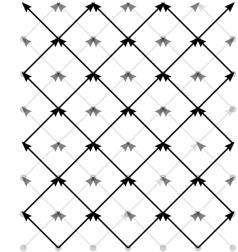
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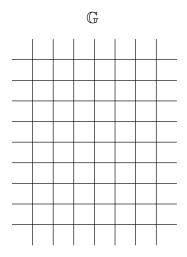
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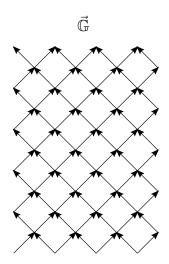
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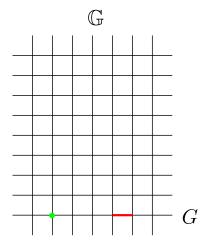
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### Construction

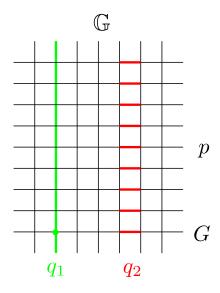




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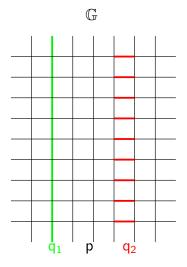
Fix finitely many vertices and edges of the base graph G

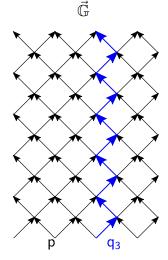


Fix finitely many vertices and edges of the base graph G

Let the edges of the coloumns corresponding to these edges and vertices be open with probability  $q_1, q_2, \ldots$ 

May 28, 2020

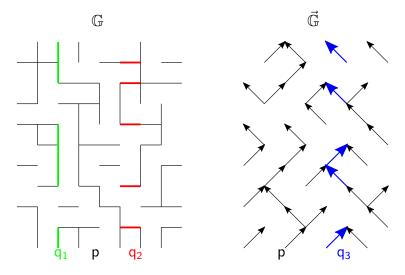




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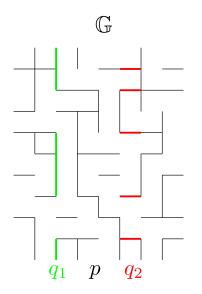
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# Result



$$\mathbf{q} := (q_1, q_2, \ldots, q_K)$$

#### Theorem

The function  $\mathbf{q} \mapsto p_c(\mathbf{q})$  is continuous on  $(0, 1)^K$ .

May 28, 2020

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17 / 60

## Motivation

Contact Process: model of epidemics on a graph with vertices in state 0 (healthy) or 1 (infected)

Transition rules:

- $\bullet \ 1 \longrightarrow 0 \text{ at rate } 1$
- 0  $\longrightarrow$  1 at rate  $\lambda \cdot \#\{\text{infected neighbors}\}$

 $\lambda_c = \sup\{\lambda : CP \text{ with parameter } \lambda \text{ a.s. dies out}\}$ 

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graphical representation of CP on  ${\it G}$ 

 $\Leftrightarrow$  percolation on the oriented ladder graph with base graph G

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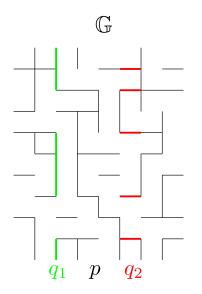
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graphical representation of CP on G

 $\Leftrightarrow$  percolation on the oriented ladder graph with base graph G

Conjecture (Pemantle and Stacey, 2001) Changing the infection parameter on a finite set of edges does not affect  $\lambda_c$ .

# Result



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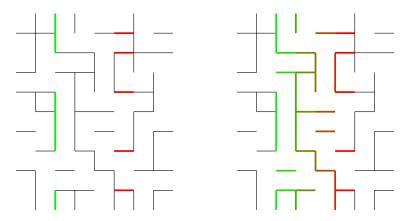
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19/60

## Result

If the theorem holds for a finite set of columns, than it holds for any subset of these columns.

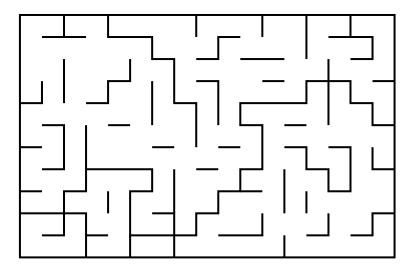


# Coupling lemma

#### Lemma (de Lima, Rolla, Valesin '17)

Let  $\{\mathbb{P}_{\theta}\}_{\theta\in\Theta}$  denote probability measures on a finite set S, parametrized by  $\theta$ , such that  $\theta \mapsto \mathbb{P}_{\theta}(x)$  is continuous for every  $x \in S$ . Assume that for some  $\theta_1 \in \Theta$  and  $\bar{x} \in S$  we have  $\mathbb{P}_{\theta_1}(\bar{x}) > 0$ . Then, for any  $\theta_2$  close enough to  $\theta_1$ , there exists a coupling of two random elements X and Y of S such that  $X \sim \mathbb{P}_{\theta_1}$ ,  $Y \sim \mathbb{P}_{\theta_2}$  and

$$\mathbb{P}\left(\{X=Y\}\cup\{X=\bar{x}\}\cup\{Y=\bar{x}\}\right)=1.$$

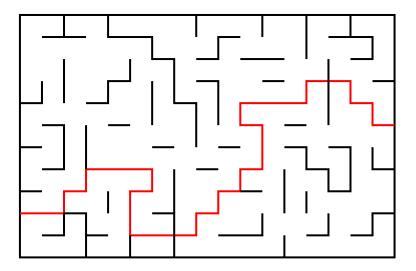


May 28, 2020 22 / 60

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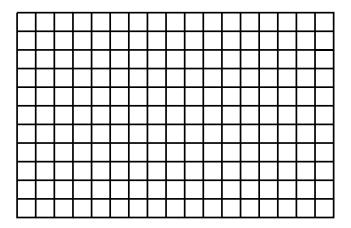


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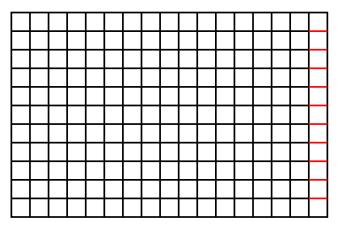


 $\mathbb{P}_{p,p}(\text{crossing})$ 

May 28, 2020

24 / 60

 $\mathbf{p} + \epsilon$ 

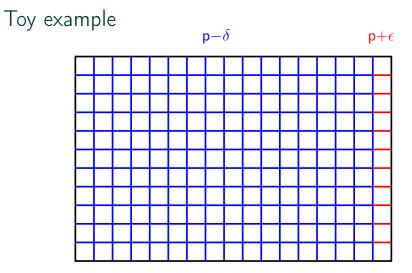


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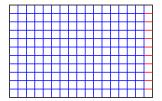
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May 28, 2020 25 / 60



 $\forall \boldsymbol{p}, \epsilon \; \exists \delta : \mathbb{P}_{\boldsymbol{p}, \boldsymbol{p}}(\text{crossing}) \leq \mathbb{P}_{\boldsymbol{p}-\delta, \boldsymbol{p}+\epsilon}(\text{crossing})$ 



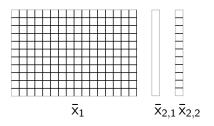


Sets of all possible configurations:  $S_1$ ,  $S_2$ 

Goal: introduce a coupling of configurations (s, s') on  $(S_1 \times S_2)^2$  such that  $s \sim \mathbb{P}_{p,p}$ ,  $s' \sim \mathbb{P}_{p-\delta,p+\epsilon}$  and

crossing on  $s \Rightarrow$  crossing on s'

$$\begin{split} S &:= S_1 \times S_2 \times S_2 \\ \theta_1 &:= \left(p, p, \frac{\epsilon}{1-p}\right) \\ \theta_2 &:= \left(p - \delta, p, \frac{\epsilon}{1-p}\right) \\ \bar{x} &:= \left(\bar{x}_1, \bar{x}_{2,1}, \bar{x}_{2,2}\right) \end{split}$$

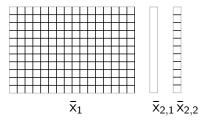


#### Lemma

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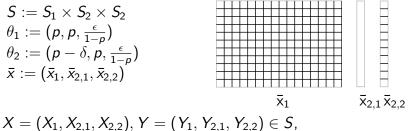
Then 
$$X = (X_1, X_{2,1}, X_{2,2}), Y = (Y_1, Y_{2,1}, Y_{2,2}) \in S$$
,  
 $X \sim \mathbb{P}_{p,p,\frac{\epsilon}{1-p}}, Y \sim \mathbb{P}_{p-\delta,p,\frac{\epsilon}{1-p}}$  and  
 $\mathbb{P}(\{X = Y\} \cup \{X = \bar{x}\} \cup \{Y = \bar{x}\}) = 1.$ 

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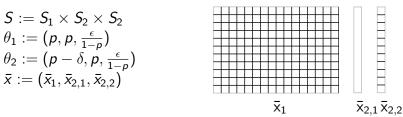
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$$s = (s_1, s_2) := (X_1, X_{2,1}) \sim \mathbb{P}_{p,p}$$

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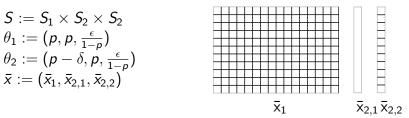
$$egin{aligned} s &= (s_1, s_2) := (X_1, X_{2,1}) \sim \mathbb{P}_{p,p} \ s' &= (s'_1, s'_2) := (Y_1, Y_{2,1} \lor Y_{2,2}) \sim \mathbb{P}_{p-\delta, p+\epsilon} \end{aligned}$$

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May 28, 2020 29 / 60

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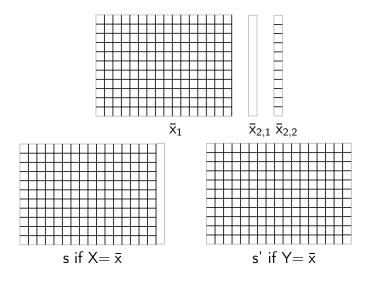


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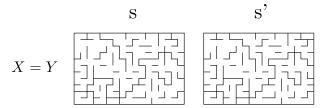
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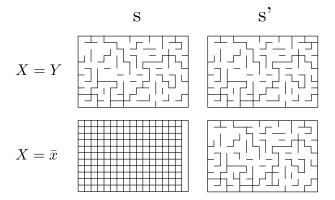


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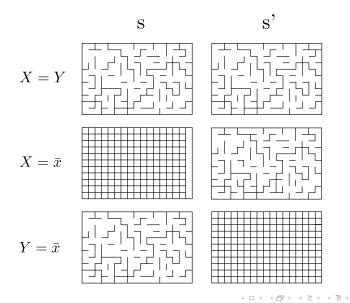


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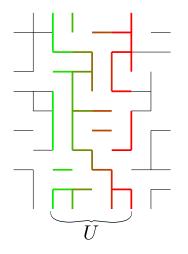
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Inhom. percolation on ladder graphs

May 28, 2020 33 / 60

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# Proof of Theorem



$$\mathbf{q} := (q_1, q_2, \ldots, q_K)$$

# Theorem The function $\mathbf{q} \mapsto p_c(\mathbf{q})$ is continuous on $(0, 1)^K$ .

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#### Proof of Theorem

#### Claim

For all  $p \in (0, 1)$ ,  $\mathbf{q} \in \mathbb{R}^{K}$  and  $\epsilon \in (0, 1 - p)$  if  $\delta \in \mathbb{R}^{K}$  is small enough, then  $\mathbb{P}_{p,\mathbf{q}}(C_{\infty}) \leq \mathbb{P}_{p+\epsilon,\mathbf{q}-\delta}(C_{\infty}).$ 

# Proof of Theorem

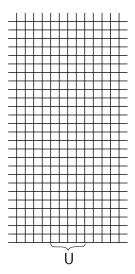
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Proof:

Goal: Introduce a coupling of configurations  $(\omega, \omega')$  such that  $\omega \sim \mathbb{P}_{p,\mathbf{q}}$ ,  $\omega' \sim \mathbb{P}_{p+\epsilon,\mathbf{q}-\delta}$  and

$$C_{\infty}$$
 in  $\omega \Rightarrow C_{\infty}$  in  $\omega'$ .

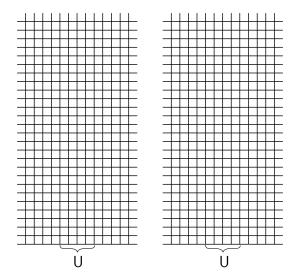


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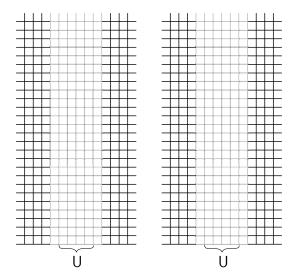
Inhom. percolation on ladder graphs

May 28, 2020 3

36 / 60

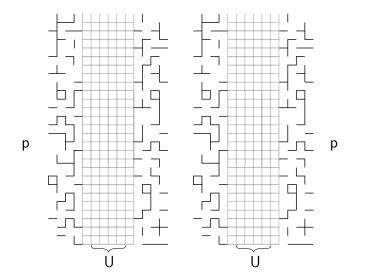


May 28, 2020 37 / 60



May 28, 2020

38 / 60



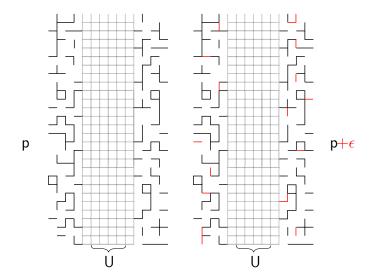
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May 28, 2020

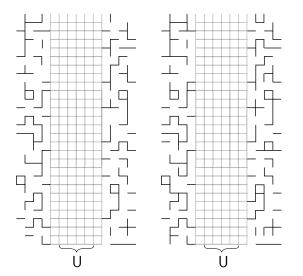
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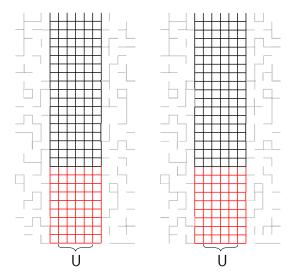
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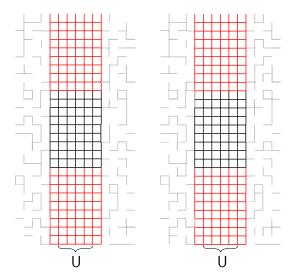
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Inhom. percolation on ladder graphs

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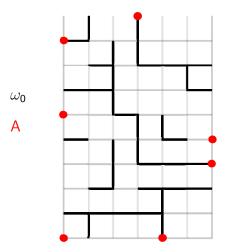
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Proof of Claim

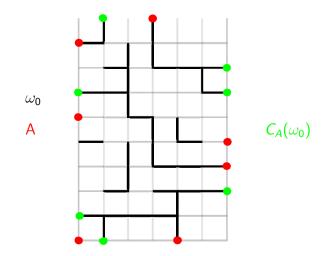


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Proof of Claim



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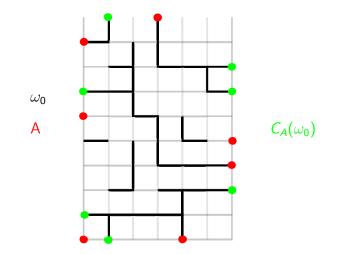
May 28, 2020 47 / 60

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Proof of Claim



Goal: coupling  $(\omega_0, \omega'_0)$  satisfying  $C_A(\omega_0) \subseteq C_A(\omega'_0) \forall A$ 

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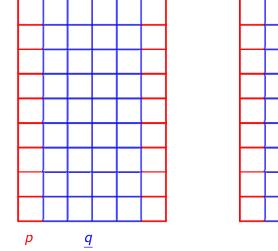
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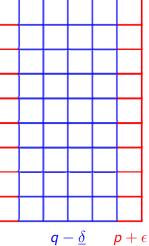
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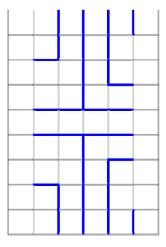
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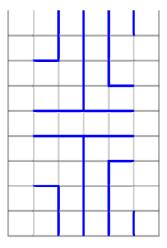
May 28, 2020 49 / 60

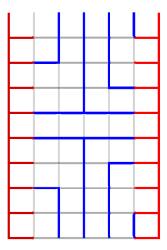


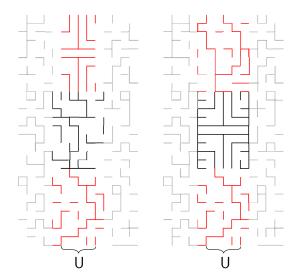
R. Szabó, D. Valesin

May 28, 2020 50 / 60

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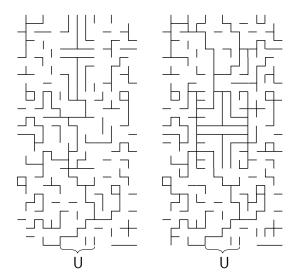




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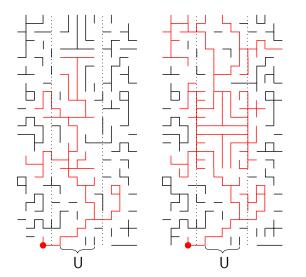
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# Thank you!

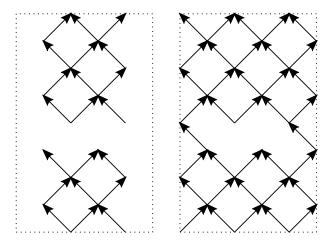
R. Szabó, D. Valesin

Inhom. percolation on ladder graphs

May 28, 2020

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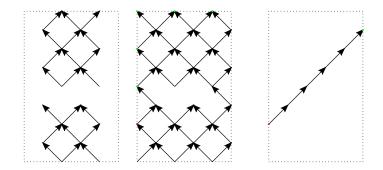
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May 28, 2020 56 / 60

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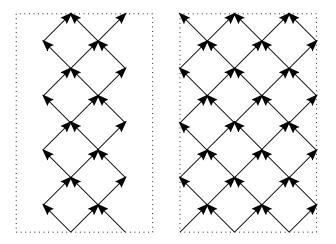


May 28, 2020 57 / 60

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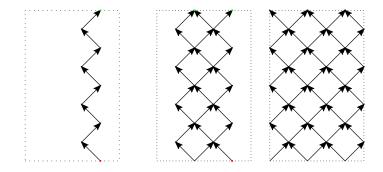
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May 28, 2020 58 / 60

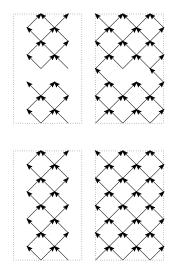
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May 28, 2020 59 / 60

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R. Szabó, D. Valesin

Inhom. percolation on ladder graphs

May 28, 2020

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