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FINITELY AXIOMATIZABLE STRONGLY MINIMAL GROUPS

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Abstract. We show that if G is a strongly minimal finitely axiomatizable group, the division ring of quasi-endomorphisms of G must be an infinite finitely presented ring.

§1. Introduction. Questions about finite axiomatizability of first order theories are nearly as old as model theory itself and seem at first glance to have a fairly syntactical flavor. But it was in order to show that totally categorical theories cannot be finitely axiomatized that, in the early eighties, Boris Zilber started developing what is now known as "Geometric stability theory". Indeed, as is often the case, in order to answer such a question, one needs to develop a fine analysis of the structure of models in the class involved and to understand exactly how each model is constructed.

The easiest way to force a structure to be infinite by one first order sentence is to impose an ordering without end points, or a dense ordering, thus making the structure unstable. It was hence rather natural to wonder about theories at the other extremity of the stability spectrum, and, in the early 60's, to ask whether there existed finitely axiomatizable totally categorical theories or simply uncountably categorical theories [22, 17].

Each model of a totally categorical theory is prime over a strongly minimal set. It is not too difficult to see that a totally categorical strongly minimal set cannot be finitely axiomatizable [15]. Much more complicated, the proof of the non finite axiomatizability for the whole class goes through a characterization of the geometries associated to totally categorical strongly minimal sets (locally modular and locally finite) and then an analysis of how any model is "built" around the strongly minimal set ([23], [24] and [6] where the result is proved for all ω-stable ω-categorical theories).

Around the same time as Zilber's negative answer for the totally categorical case, Peretjat'kin produced an example of a finitely axiomatized $\aleph_1$-categorical theory [19]. This example was in the following years simplified by Baisalov (see [9, §12.2, Example 5]). This final example has Morley Rank equal to 2, thus still leaving open the question of the existence of a finitely axiomatizable strongly minimal set (Morley rank and degree equal to 1). Furthermore all the known examples of finitely axiomatizable $\aleph_1$-categorical theories are rather similar and constructed around a strongly minimal set with trivial pregeometry, also leaving open the question

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of the existence of a finitely axiomatizable $\aleph_1$-categorical theory with non trivial pregeometry.

In 1994, Hrushovski [10] showed that any finitely axiomatizable $\aleph_1$-categorical theory must have locally modular pregeometry, thus reducing the remaining open questions to two very different cases:

- the existence of a finitely axiomatizable trivial strongly minimal set
- the existence of *any* finitely axiomatizable $\aleph_1$-categorical theory which contains a locally modular strongly minimal group.

The canonical example of a strongly minimal locally modular non trivial theory is that of infinite K-vector spaces, for a fixed division ring K. It is open whether there exists any finitely axiomatizable complete theory of R-modules, for R any ring, but it is very easy to check that if K is an infinite division ring which is finitely presented as a ring, then the theory of K-vector spaces can be finitely axiomatized. Unfortunately, the existence of such a division ring is open (see Section 4). Conversely, it was originally shown by Paljutin [18], in a paper where he characterizes finitely axiomatizable uncountably categorical quasi-varieties, that, if the theory of infinite K-vector spaces is finitely axiomatizable, then K is finitely presented as a ring (see Section 4.2).

In the paper cited above, Hrushovski conjectures that, more generally, a finitely axiomatizable $\aleph_1$-categorical non trivial theory exists if and only if such an infinite finitely presented division ring exists. Any $\aleph_1$-categorical non trivial locally modular theory must contain a locally modular strongly minimal group G, and the geometry associated to such a group is that of infinite K-vector spaces, where K is the division ring of quasi-endomorphisms of G (see Section 3 for the definitions). The precise conjecture in [10] is that, in any finitely axiomatizable $\aleph_1$-categorical non trivial theory, the associated division ring of quasi-endomorphisms is infinite and finitely presented as a ring.

One should remark that although every $\aleph_1$-categorical non trivial locally modular theory must contain a definable strongly minimal group, one cannot use general arguments to transfer down the finite axiomatizability to the strongly minimal group. We will see in Section 4.1 some general assumptions under which finite axiomatizability can be transferred (bi-interpretability, definable finite partition). But, it is not even true in general that, if M is finitely axiomatizable and contained in the algebraic closure of a strongly minimal set D (M is then said to be almost strongly minimal), the strongly minimal set D, with the induced structure from M, must be finitely axiomatizable. In the finitely axiomatized $\aleph_1$-categorical Morley rank 2 theory which was mentioned above, for example, the whole structure M is contained in the algebraic closure of a strongly minimal subset $D \subset M \times M$ (the diagonal), whose induced structure is that of the integers with the successor function, which is not finitely axiomatizable.

In past years, work around strongly minimal finitely axiomatizable trivial sets has also centered around a conjecture relating their existence to the existence of an infinite group with specific properties (see Section 4 for some further details).

In this paper we show that Hrushovski's conjecture holds for strongly minimal groups, and more generally for Morley Rank one groups: If G is a finitely axiomatizable strongly minimal group, then the division ring of quasi-endomorphisms of G must be infinite and finitely presented (Theorem 4.19).

By Hrushovski's result, we know that such a group must be locally modular. This enables us to reduce to the case when G is a strongly minimal abelian structure. Then we show (Proposition 4.16) that if G is a finitely axiomatizable strongly minimal abelian structure, the division ring K of quasi-endomorphisms of G must be infinite and that the theory of K-vector spaces must also be finitely axiomatizable (Lemma 4.15).

We begin in Section 2 by recalling or proving some general facts about abelian structures, under the precise form they will be needed later. In particular, we describe, in Section 2.2, the theory which will end up being both finitely axiomatizable and interdefinable with the theory of K-vector spaces. In Section 3 we recall the basic facts about the ring of quasi-endomorphisms of a locally modular strongly minimal group and we look at strongly minimal abelian structures. In Section 4, we consider the question of finite axiomatization. We begin by a somewhat technical section (4.1) where we give precise definitions of finite axiomatizability in the case of infinite languages and we show how this notion transfers when changing languages or structures. In order to be as self-contained as possible on the subject of finite axiomatizability, in Section 4.2, we recall very precisely the two classical examples (regular group actions and vector spaces). In the next section (Section 4.3), we prove the main theorem, that if a strongly minimal abelian structure is finitely axiomatizable, then its division ring of quasi-endomorphisms, K, must be infinite and the theory of K-vector spaces must be finitely axiomatizable. Finally in the last section (4.4) we conclude for strongly minimal groups and more generally for groups of Morley Rank one.

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§2. Abelian structures.

2.1. Axiomatization and quantifier elimination. In this first section, we recall the precise statements about axiomatization and pp-elimination of quantifiers for abelian structures.

We define an *abelian structure* $\mathbb{G}$, to be a commutative group

$$\mathbb{G} = (G, +, -, 0, (H_i)_{i \in I}),$$

where each H_i is a subgroup of some G^{n_i}. We denote by L_0 the following language: $\{+, -, 0, (H_i)_{i \in I}\}$. We are going to consider expansions of abelian structures by constants and we will denote by L_c the language of an expansion of $\mathbb{G}$ by some constants in a subset C, i.e., $L_c = L_0 \cup \{c : c \in C\}$.

Recall that the set of positive primitive formulas is the closure of the atomic formulas by conjunction and existential quantifiers.

It has been well-known for years that in a complete theory of modules, every formula is equivalent to a Boolean combination of pp-formulas and that a complete theory of modules is axiomatized by so-called invariant statements describing the index of pairs of positive primitive definable subgroups [3, 21].

The similar result for abelian structures has also been known for a long time (abelian structures were originally introduced by E. Fisher in [7]) but was never published in any "official" form until it appeared as a special case in the general treatment of theories given by cosets in [8]. As we are dealing with questions of finite axiomatization, it is important for us to be precise about the form of the axioms and the language we are working in. For this reason we will recall briefly the precise definitions we need and state, mostly without proofs, the results under the exact form we require. Some similar considerations appear also in [5].

LEMMA 2.1. *Let $\phi(x_1, \dots, x_n)$ be a consistent positive primitive formula in L_0 in n variables $(n \geq 1)$ and without parameters. Then $\{a \in G^n : \mathbb{G} \models \phi(a)\}$ is a subgroup of G^n.*

Note that the set of pp-definable subgroups in $\mathbb{G}$ corresponds to the closure of the groups $(H_i : i \in I)$, the trivial groups ($\{0\}$ and G), the diagonal of G^2 and the graph of the addition, by cartesian product, permutation of coordinates, intersection and projection.

LEMMA 2.2. *Let $\phi(\bar{x}, \bar{0})$ be a pp-formula from L_0 without parameters, which defines a subgroup in G^n. Let $\bar{d}$ be a tuple from G. Then $\phi(\bar{x}, \bar{d})$ is empty or is a coset of the pp-definable subgroup defined by $\phi(\bar{x}, \bar{0})$.*

A pp-formula in the language L_c is equivalent to $\phi(\bar{x}, \bar{c})$ where ϕ is a pp-formula from L_0 and $\bar{c}$ is a tuple of constants. *In particular a subgroup of G^n which is definable by a pp-formula from L_c is in fact already pp-definable in L_0 without parameters.*

Let $T(\mathbb{G})$ be the following set of sentences from L_0:

- G is a commutative group,
- for each original predicate H_i from L_0, H_i is a subgroup of G^{n_i},
- *the equivalence sentences*: all sentences of the form $\forall \bar{x}\ (\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$, for ϕ and ψ pp-formulas which define the same subgroup of G^n (note that these sentences give the following relations between pp-definable subgroups: inclusion, intersection, projection and equality up to a permutation of variables),
- *the dimension sentences*: for each pair $H \subset H'$ of pp-definable subgroups of G, such that the index of H in H' is equal to n, the sentence "$[H' : H] = n$"; for each pair $H \subset H'$ of pp-definable subgroups of G, such that the index of H in H' is infinite in G, the infinite scheme of sentences "$[H' : H] \geq k$" , for every $k \geq 1$.

FACT 2.3. *The theory $T(\mathbb{G})$ is complete in the language L_0 and admits quantifier elimination to the pp-formulas, that is, every formula is equivalent modulo $T(\mathbb{G})$ to a Boolean combination of pp-formulas.*

Note that it follows directly that every abelian structure is stable and one-based (see Fact 4.5).

COROLLARY 2.4. *The theory of $\mathbb{G}$ in the language L_c is given by $T(\mathbb{G})$ together with the pp-types of the constants (i.e., for each pp-definable group H, we have to describe the H-congruences on the set of constants).*

COROLLARY 2.5. *Let $a \in G$ and $B \subset G$, then a is algebraic over B in the L_c-structure $\mathbb{G}$ if and only if a is in a B-definable coset of some finite pp-definable subgroup of G.*

PROOF. By elimination to pp-formulas, the type of a over B is given by the set X of B-definable cosets of pp-definable subgroups to which a belongs, and the set Y of B-definable cosets of pp-definable subgroups to which a does not belong. Note that X

is closed under finite intersections. By compactness, as a is algebraic over B, there is some coset A in X and some cosets $B_1, B_2, \ldots, B_n$ from Y such that $A \setminus (B_1 \cup \cdots \cup B_n)$ is finite non empty. We can suppose that each B_i is contained in A, by taking its intersection with A. Then $A = a + H = \{a_1\} \cup \cdots \cup \{a_m\} \cup (d_1 + H_1) \cup \cdots \cup (d_n + H_n)$, where $\{a_i\}$ is considered as a coset of the trivial group, $d_i + H_i = B_i$ and $H_i \subset H$. By Neumann's Lemma, if some coset $a + H$ is covered by a finite number of cosets, then it is covered by those cosets which correspond to subgroups of finite index in H. It follows that A itself is finite (as $A \setminus (B_1 \cup \cdots \cup B_n)$ is not empty, $\{0\}$ must be of finite index in A). ⊣

We finish with a remark that will be very useful in the sequel. As abelian structures are stable and one-based, it follows by general results (see Fact 4.5) that any definable connected subgroup is definable over $acl^{eq}(\emptyset)$. But for abelian structures, one can in fact show more:

Proposition 2.6. *Let $\mathbb{G}$ be an abelian structure in L_c. Let $H \subset G^n$ be any definable connected subgroup (with parameters) in L_c. Then H is pp-definable. In particular, H is definable in L_0 over $\emptyset$.*

Proof. Let H be a connected definable subgroup of G^n. By pp-elimination, there is a set A such that H is equivalent to a Boolean combination of pp-formulas with parameters in A. We can suppose that the unique generic type of H, q, is defined and stationary over A. As q is a complete type over A, we can suppose that there are $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m \in A$ and pp-definable (in L_0 over $\emptyset$) subgroups ϕ_0, $\phi_1, \ldots, \phi_m$ of G^n such that:

1. for each $i > 0$, ϕ_i is a subgroup of ϕ_0,
2. $(\bar{a}_0 + \phi_0) \setminus (\bar{a}_1 + \phi_1 \cup \cdots \cup \bar{a}_m + \phi_m) \subseteq H$,
3. every generic of H over A is in $(\bar{a}_0 + \phi_0) \setminus (\bar{a}_1 + \phi_1 \cup \cdots \cup \bar{a}_m + \phi_m)$,
4. for each $i > 0$, each ϕ_i is of infinite index in ϕ_0: indeed if ϕ_i has finite index in ϕ_0, by enlarging A if necessary, we can suppose that $\bar{a}_0 + \phi_0 = \bigcup_{1 \leq j \leq k} \bar{e}_j + \phi_i$, with $\bar{e}_j \in A$, and replace $\bar{a}_0 + \phi_0$ with one of the $\bar{e}_j + \phi_i$.

It follows that $H = \phi_0$: the difference of two generics of H (over A) is in ϕ_0, so $H \subset \phi_0$. Conversely, let h be a generic of H over A. Take a generic x of ϕ_0 over Ah. Since for each $i > 0$, ϕ_i is a subgroup of ϕ_0 of infinite index, $x \notin (\bar{a}_i - h) + \phi_i$. Thus $x + h \in (\bar{a}_0 + \phi_0) \setminus (\bar{a}_1 + \phi_1 \cup \cdots \cup \bar{a}_m + \phi_m)$ and $x \in H$. ⊣

2.2. Direct sums and pure injectives. In this section we consider an abelian structure $\mathbb{G} = (G, +, -, 0, (H_i)_{i \in I}, \{c \in C\})$ with constants, in the language L_c, and we denote its complete theory by $\mathbb{T}$.

The following direct sum construction plays an essential role in the paper.

Let $\mathbb{S}$ be an abelian structure in the language L_0. Let $\mathbb{G}_S := \mathbb{G} \oplus \mathbb{S}$, be the L_c-structure with universe the group $G_S = G \oplus S$ and with the obvious interpretation of the symbols in the language: each constant c is interpreted by $(c, 0)$; for H of arity n , we interpret H in G_S by $H(G) \oplus H(S)$ where $H(G) := \{(h, 0) \in {G_S}^n : \mathbb{G} \models H(h)\}$ and $H(S) := \{(0, h) \in {G_S}^n : \mathbb{S} \models H(h)\}$. If $g = (g_1, \ldots, g_n) \in G^n$ and $s = (s_1, \ldots, s_n)$, we will use both notations $g + s$ or (g, s) to denote the element $(g_1 + s_1, \ldots, g_n + s_n) \in {G_S}^n$.

The following lemmas (2.7 to 2.10) are well known for the case of modules (in the usual language for modules, see [21]) and mostly folklore for abelian structures.

The proofs are similar to the ones in the case of modules. We give here only the ones that provide information which can be useful.

LEMMA 2.7. *For each pp-formula* $\phi(x)$ $(x = (x_1, \ldots, x_n))$ *from* L_0, $\phi(G_S) = \phi(G) \oplus \phi(S)$, *i.e., for* $h_1 \in G^n$, *and* $h_2 \in {G_S}^n$, $\mathbb{G}_S \models \phi(h_1 + h_2)$ *if and only if* $\mathbb{G} \models \phi(h_1)$ *and* $\mathbb{S} \models \phi(h_2)$.

PROOF. By induction on pp-formulas. For atomic formulas, it follows from the way $\mathbb{G}_S$ is defined as being the direct sum of $\mathbb{G}$ and $\mathbb{S}$ as L_0-structures. For a conjunction of two pp-formulas, it follows easily from the fact that G_S is the direct sum of G and S as groups. There remains to check the case of a projection. Let $\phi(x_1, \ldots, x_n)$ be a pp-formula such that $\phi(G_S) = \phi(G) \oplus \phi(S)$ and consider the pp-formula $\psi := \exists x_1 \phi$. We have trivially that $\psi(G) \oplus \psi(S) \subset \psi(G_S)$. Let $(a_2, \ldots, a_n) \in \psi(G_S)$. Then there is $a_1 \in G_S$ such that $(a_1, \ldots, a_n) \in \phi(G_S) = \phi(G) \oplus \phi(S)$. So $(a_1, \ldots, a_n) = (b_1, \ldots, b_n) + (c_1, \ldots, c_n)$, with $(b_1, \ldots, b_n) \in \phi(G)$ and $(c_1, \ldots, c_n) \in \phi(S)$. But then $(a_2, \ldots, a_n) = (b_2, \ldots, b_n) + (c_2, \ldots, c_n)$ with $(b_2, \ldots, b_n) \in \psi(G)$ and $(c_2, \ldots, c_n) \in \psi(S)$. ⊣

REMARK. In particular, if $\phi(x, y)$ is a pp-formula in L_0 and $c = (c_1, \ldots, c_n)$ are some constants from C, then $\mathbb{G}_S \models \phi(g + s, c)$ if and only if $\mathbb{G} \models \phi(g, c)$ and $\mathbb{S} \models \phi(s, 0)$.

DEFINITION. Let $\mathbb{M}$ be an abelian structure. We say that $\mathbb{M}$ is *pure injective* if every set of pp-formulas with parameters in M (with possibly infinitely many variables) which is finitely realized in M is realized in M.

PROPOSITION 2.8. *If* $\mathbb{G}_0$ *is an elementary substructure of* $\mathbb{G}$ *and is pure injective, then there exists* f, *an* L_c*-homomorphism from* G *to* G_0, *such that* f *is the identity on* G_0.

COROLLARY 2.9. *If* $\mathbb{G}_0$ *is an elementary substructure of* $\mathbb{G}$ *and is pure injective, then* $G = G_0 \oplus G_1$ *and for each* pp*-definable subgroup* $H \subset G^n$ *in* L_0, $H = H \cap ({G_0}^n) \oplus H \cap ({G_1}^n)$.

PROOF. Let $G_1 = \ker f$ where f is given by the previous proposition. Then $a = f(a) + (a - f(a))$ and $a - f(a) \in Ker f$ for each $a \in G$: indeed $f(a - f(a)) = f(a) - f(f(a)) = f(a) - f(a) = 0$. Similarly, for any $n \geq 2$, $G^n = G_0^n \oplus {G_1}^n$. So we can suppose that $H \subset G$ to simplify the notation. Since H is a pp-definable subgroup and f is an L_0-homomorphism, $f(H) \subset H \cap G_0$ and so $f(H) = H \cap G_0$. Thus $H = H \cap G_0 \oplus H \cap G_1$: for each $h \in H$, $f(h) \in H \cap G_0$ and $(h - f(h)) \in \ker f \cap H$. ⊣

Let $G = G_0 \oplus G_1$, as above in Corollary 2.9, where $\mathbb{G}_0$ is a pure injective elementary substructure of $\mathbb{G}$ for the language L_c, and where $G_1 = \ker f$, f given by Proposition 2.8. Let $\mathbb{G}_1$ be the following abelian structure on G_1 in the language L_0:

$$\mathbb{G}_1 = (G_1, +, -, 0, (H_i \cap G_1^{n_i})_{i \in I}).$$

Then $\mathbb{G} = \mathbb{G}_0 \oplus \mathbb{G}_1$ as abelian structures and by Lemma 2.7:

LEMMA 2.10. *For every subgroup* H *of* G^n *definable by a* pp*-formula* ϕ *in* $\mathbb{G}$, *the group* $\phi(G_1) := \{a \in {G_1}^n \, : \, \mathbb{G}_1 \models \phi(a)\}$ *is equal to* $H \cap G_1^n$.

We will now describe an axiomatization for the theory of $\mathbb{G}_1$.

Let $\mathbb{T}_1(\mathbb{G})$ be the following modification of the axioms $T(\mathbb{G})$:

- the axioms for abelian groups,
- for each original predicate H from L_0, "H is a subgroup",
- the equivalence sentences from $T(\mathbb{G})$,
- "$[\phi : \psi]$ is infinite" for every pair of pp-formulas such that in $\mathbb{G}$, $\psi(G) \subset \phi(G) \subset G$ and $[\phi(G) : \psi(G)]$ is infinite,
- $[\phi : \psi] = 1$ for every pair of pp-formulas such that in $\mathbb{G}$, $\psi(G) \subset \phi(G) \subset G$ and $[\phi(G) : \psi(G)]$ is finite.

PROPOSITION 2.11. *If* $\mathbb{G}$ *is* $|G_0|^+$*-saturated,* $\mathbb{T}_1(\mathbb{G})$ *axiomatizes the complete theory of the abelian structure* $\mathbb{G}_1$*, that is,* $\mathbb{T}_1(\mathbb{G}) \vdash T(\mathbb{G}_1)$.

PROOF. We show first that $\mathbb{G}_1$ is a model of $\mathbb{T}_1 := \mathbb{T}_1(\mathbb{G})$. By Lemma 2.10, $\mathbb{G}_1$ satisfies the equivalence sentences from $T(\mathbb{G})$. Let $K \subset H$ be a pair of pp-definable subgroups of G. If $[H : K]$ is infinite then by $|G_0|^+$-saturation of $\mathbb{G}$, $[H : K] > |G_0|$. Thus $[H \cap G_1 : K \cap G_1]$ is infinite. If $[H : K]$ is equal to k, since G_0 is an elementary substructure, there are $a_1, \dots, a_k \in G_0$ such that $H = a_1 + K \cup \dots \cup a_k + K$. Let $x \in H \cap G_1$. Then $x = a_i + b$ where $b \in K$. Let f be as in Proposition 2.8, then $0 = f(x) = f(a_i) + f(b) = a_i + f(b)$. But $f(b) \in K$, so $x \in K$ and $H \cap G_1 = K \cap G_1$.

Now we show that $T(\mathbb{G}_1)$ is a consequence of $\mathbb{T}_1$. Let ϕ and ψ be a pair of pp-formulas from L_0 which define subgroups H and K of G^n such that $H \cap G_1^n = K \cap G_1^n$. We have to show that the sentence $\forall \bar{x}\, (\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$ is a consequence of $\mathbb{T}_1$. Note first that if $K \subset H$ and $[H : K]$ is finite, this follows from the last axioms by an easy induction on n. But we can consider the pair $K \cap H \subset H$ and remark that $[H : K \cap H]$ is finite: indeed, if $[H : K \cap H]$ was infinite, then $[H \cap G_1^n : K \cap H \cap G_1^n]$ would be infinite. For the dimension sentences, suppose now that H and K are pp-definable subgroups of G such that $K \cap G_1 \subset H \cap G_1$. Then the sentence for the index of $[H \cap G_1 : K \cap G_1]$ is a consequence of the sentence from $\mathbb{T}_1$ corresponding to the pair of pp-formulas $\psi \cap \phi$ and ϕ. ⊣

REMARK 2.12. Note that G_1 contains no non trivial finite pp-definable subgroup. *So, in* $\mathbb{G}_1$*, the algebraic closure of the empty set is reduced to* 0 *and algebraic closure corresponds to definable closure.*

The assumption that $\mathbb{G}$ is saturated is essential in the previous proposition in order to obtain that $\mathbb{G}_1$ is a model of the right theory. But if one considers abelian structures which are models of $\mathbb{T}_1$, saturation is no longer relevant:

PROPOSITION 2.13. *If* $\mathbb{S}$ *is a model of* $\mathbb{T}_1$ *and* $\mathbb{G}$ *is a model of* $\mathbb{T}$*, then* $\mathbb{G}_S$ *(the abelian structure* $\mathbb{G} \oplus S$*) is a model of* $\mathbb{T}$ *and the map i from G into* G_S*,* $i(g) = (g, 0)$ *is an* L_c*-elementary isomorphism (i.e.,* $G \oplus \{0\}$ *is an* L_c*-elementary substructure of* $\mathbb{G}_S$*).*

PROOF. Let ϕ and ψ be two pp-formulas such that $\phi(G) \subset \psi(G)$. Then $\phi(S) \subset \psi(S)$ (by equivalence sentences). If $[\psi(G) : \phi(G)] = k$ then $\psi(S) = \phi(S)$ (by $\mathbb{T}_1$), so $[\psi(G_S) : \phi(G_S)] = k$. If $[\psi(G) : \phi(G)] = \infty$ then $[\psi(G_S) : \phi(G_S)] = \infty$. Moreover, by definition of $\mathbb{G}_S$, the pp-types of the constants are preserved. Hence $\mathbb{G}_S$ is a model of $\mathbb{T}$, so $G \oplus \{0\}$ and $\mathbb{G}_S$ are elementarily equivalent, and it follows by pp-elimination in $\mathbb{T}$ and Lemma 2.7 that $G \oplus \{0\} \prec \mathbb{G}_S$. ⊣

The above construction will be particularly relevant when the theory $\mathbb{T}$ is totally transcendental.

Recall that in a totally transcendental group, there is no infinite strictly decreasing sequence of definable subgroups. It follows easily that the same is true for cosets and that exactly as for modules (see [21]):

PROPOSITION 2.14. *If* $\mathbb{T}$ *is totally transcendental then every model of* $\mathbb{T}$ *is pure injective.*

§3. The strongly minimal case.

We remind the reader that a definable set D in a structure M (defined by a formula $\phi(\bar{x})$ with parameters from M) is said to be *strongly minimal* if, in every elementary extension N of M, every relatively definable subset of D is finite or co-finite (i.e., for every formula $\psi(\bar{x})$ with parameters from N, $\phi(\bar{x}) \wedge \psi(\bar{x})$ is finite or $\phi(\bar{x}) \wedge \neg\psi(\bar{x})$ is finite). A structure M is said to be strongly minimal if the formula $(x = x)$ is strongly minimal. If M is strongly minimal, model theoretic algebraic closure (denoted acl) defines a pregeometry on M. In particular, for any $X \subset M$, the dimension of X (the cardinality of a maximal algebraically free subset in X) is well defined. We say that M is *trivial*, or has trivial pregeometry if, for all $A \subset M$, $acl(A) = \bigcup_{a \in A} acl(\{a\})$. We say that M is *locally modular* if for all algebraically closed $X, Y \subset M$, such that $dim(X \cap Y) > 0$, $dim(X \cup Y) = dim(X) + dim(Y) - dim(X \cap Y)$. We will explicitly state the results we use about locally modular strongly minimal groups. For proofs and details we refer to [20], [4] or [16].

By a *strongly minimal group*, we mean, as usual, a group G with possibly extra structure in a language L, which is strongly minimal as an L-structure.

Let $\mathscr{G} = (G, L)$ be a strongly minimal group in a language L. Let $G_0 := G \cap acl(\emptyset)$. A *quasi-endomorphism* of $\mathscr{G}$ is a connected subgroup H of G^2, definable over $acl^{eq}(\emptyset)$, different from $G \times G$ and such that the first projection of H is equal to G. It follows that H is strongly minimal. We define the kernel of H and the cokernel of H to be respectively:

$$KerH := \{a \in G : (a, 0) \in H\}, \; Coker(H) := \{a \in G : (0, a) \in H\}.$$

The cokernel of H is always finite, and if H is not trivial, that is, if $H \neq G \times \{0\}$, the kernel of H is finite. We denote by $QS(\mathscr{G})$ the set of quasi-endomorphisms of $\mathscr{G}$.

Remark: By strong minimality, if G_0 is infinite, G_0 is an elementary substructure of $\mathscr{G}$ and in that case, all quasi-endomorphisms of $\mathscr{G}$ are actually definable over G_0. In any case, if $\mathscr{M}_0$ is a prime model for $Th(\mathscr{G})$, all quasi-endomorphisms are definable over M_0.

A quasi-endomorphism induces an endomorphism of G/G_0: if $H \in QS(\mathscr{G})$, then $\{(a + G_0, b + G_0) : (a, b) \in H\}$ is the graph of an endomorphism f_H of G/G_0. Furthermore the map which to every $H \in QS(\mathscr{G})$ assigns the endomorphism f_H is a bijection from $QS(\mathscr{G})$ onto the ring of the "quasi-definable" endomorphisms of G/G_0. The ring of endomorphisms of G/G_0 induces the *structure of a division ring on* $QS(\mathscr{G})$.

In the case of a locally modular group, the pregeometry on $\mathscr{G}$ defined by the relation of algebraic closure corresponds to the geometry of $QS(\mathscr{G})$-vector spaces. More precisely:

Fact 3.1. *Let $\mathscr{G}$ be a locally modular strongly minimal group, let $b, a_1, \dots, a_n \in G$. Then $b \in acl(a_1, \dots, a_n)$ if and only if there are quasi-endomorphisms $S_1, \dots, S_n$ and elements $h_1, \dots, h_n \in G$ such that for each i, $1 \leq i \leq n$,*

$$(a_i, h_i) \in S_i \text{ and } b - (h_1 + \cdots + h_n) \in G_0.$$

Any strongly minimal abelian structure $\mathbb{A}$ is locally modular and by 2.6 *all quasi-endomorphisms of $\mathbb{A}$ are definable over $\emptyset$.*

By general results about one-based groups (see Section 4.4), any locally modular strongly minimal group is "almost interdefinable" with a strongly minimal abelian structure. This will enable us at the end to reduce to the case of finitely axiomatizable abelian structures.

From now on in this section, $\mathbb{G}$ is a strongly minimal abelian structure with constants, in the language L_c, and $\mathbb{T}$ denotes its theory in L_c.

Consider, $T(\mathbb{G})$, the axiomatization of the theory of $\mathbb{G}$ in L_0 given in the previous sections, and $\mathbb{T}_1 = \mathbb{T}_1(\mathbb{G})$, the associated theory.

Note that one can see directly from the axiomatizations of the form $T(\mathbb{G})$ when an abelian structure is strongly minimal: by pp-elimination $\mathbb{G}$ will be strongly minimal if and only if $\mathbb{G}$ is infinite and for any pp-definable subgroup H of G, H is finite or equal to G.

Lemma 3.2. *The theory $\mathbb{T}_1$ is strongly minimal.*

Proof. First a model of $\mathbb{T}_1$ must be infinite (the formula $x = x$ is a pp-formula). Let ϕ be a pp-definable subgroup of G. By strong minimality of $\mathbb{G}$, either ϕ is finite, or ϕ is equal to G. Hence in any model $\mathbb{H}$ of $\mathbb{T}_1$, ϕ is trivial or $\phi = H$. By pp-elimination, $\mathbb{T}_1$ is also strongly minimal. ⊣

We also know (see Remark 2.12) that in any model of $\mathbb{T}_1$, $acl(\emptyset) = \{0\}$ and $acl = dcl$. It follows easily that, if $\mathbb{G}_1$ is a model of $\mathbb{T}_1$ and if K_1 is the division ring of quasi-endomorphisms of $\mathbb{G}_1$, the structure on $\mathbb{G}_1$ is exactly the K_1-vector space structure. But we want to check that $K_1 = K$, where K is the division ring of quasi-endomorphisms of $\mathbb{G}$.

Let us recall the definition of *interdefinability*:

Definition 3.3. Let $\mathscr{M}_1 = (M, L_1)$ and $\mathscr{M}_2 = (M, L_2)$ be, respectively, L_1 and L_2-structures with the same universe M. Let $A \subset M$, we say that $\mathscr{M}_1$ is A-definable in $\mathscr{M}_2$ if every A-definable subset in $\mathscr{M}_1$ is A-definable in $\mathscr{M}_2$, equivalently, if every symbol in the language L_1 is A-definable in $\mathscr{M}_2$. We say that $\mathscr{M}_1$ and $\mathscr{M}_2$ are *A-interdefinable* if each $\mathscr{M}_i$ is A-definable in the other, equivalently, if $\mathscr{M}_1$ and $\mathscr{M}_2$ have the same A-definable subsets.

Proposition 3.4. *Let K be the division ring of quasi-endomorphisms of $\mathbb{G}$ and let $\mathbb{S}$ be any model of $\mathbb{T}_1$. Every quasi-endomorphism r of $\mathbb{G}$ induces an endomorphism of $\mathbb{S}$ and $\mathbb{S}$ is $\emptyset$-interdefinable with the K-vector space structure $\mathbb{S}_K := (S, +, -, 0, \{r, : r \in K\})$.*

Proof. Let $\mathbb{G}'_S = \mathbb{G}' \oplus \mathbb{S}$, where $\mathbb{G}'$ is a countable elementary substructure of $\mathbb{G}$ (take for G', $G_0 = acl(\emptyset)$, if it is infinite) and $\mathbb{S}$ is any model of $\mathbb{T}_1$. Then $\mathbb{G}'_S$ is a model of $\mathbb{T}$, the theory of $\mathbb{G}$ (2.13). Let $\phi_r(x, y)$ be a pp-formula which defines the quasi-endomorphism $r \in K$, $r \neq Id$. By the axioms in $\mathbb{T}_1$, in $S \times S$, $\phi_r(x, y)$ defines a subgroup with first projection equal to S, which is strict and connected, hence which is a quasi-endomorphism of S. The kernel, $\{y \in S : S \models \phi_r(y, 0)\}$, must be trivial, as well as the co-kernel. So $\phi_r(x, y)$ defines a bijective endomorphism of S. In particular, K is a subring of the ring of definable endomorphisms of S. Consider the following K-vector space structure on S: $\mathbb{S}_K := (S, +, -, 0, \{r : r \in K\})$, where we define the action of r on S by letting $r.x = y$ iff $S \models \phi_r(x, y)$.

As the structure $\mathbb{S}_K$ is clearly definable in the L_0-structure $\mathbb{S}$, in order to prove that $\mathbb{S}$ is definable in $\mathbb{S}_K$, it suffices to show that, in any model $\mathbb{S}$ of $\mathbb{T}_1$, if any two tuples, $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in S have the same K-vector space type, then they also have the same L_0-type.

Claim. Let $a, b, a_1, \ldots, a_n \in S$ be such that a is K-linearly independent from $a_1, \ldots, a_n$ and b is also K-linearly independent from $a_1, \ldots, a_n$, then there exist an L_0-automorphism of S which sends a to b and fixes $a_1, \ldots, a_n$.

Proof. In the structure $\mathbb{G}'_S$, $(0, a) \notin \operatorname{acl}((G' \oplus \{0\}) \cup \{(0, a_1), \ldots, (0, a_n)\})$ (in the language L_c): otherwise, there would be $r_1, \ldots, r_n \in K$, $(x_1, y_1), \ldots, (x_n, y_n) \in G'_S$ and $g \in G'$ such that

$$(0, a) = (x_1, y_1) + \cdots + (x_n, y_n) + (g, 0) \text{ and } G'_S \models \phi_{r_i}((0, a_i), (x_i, y_i)).$$

But then, $a = y_1 + \cdots + y_n = r_1 a_1 + \cdots + r_n a_n$. Similarly $(0, b) \notin \operatorname{acl}((G' \oplus \{0\}) \cup \{(0, a_1), \ldots, (0, a_n)\})$.
Since $\mathbb{G}'_S$ is strongly minimal and $G' \oplus \{0\} \prec \mathbb{G}'_S$, there exists an automorphism τ of $\mathbb{G}'_S$ which sends $(0, a)$ to $(0, b)$ and fixes pointwise $G' \oplus \{0\}$ and $(0, a_1), \ldots, (0, a_n)$. From τ we construct easily an L_0-automorphism of S which sends a to b and fixes $a_1, \ldots, a_n$: just let $\sigma(s) := s'$ if $\tau(0, s) = (x, s')$. $\dashv_{\text{Claim}}$

Let $a_1 \ldots a_n$ and $b_1 \ldots b_n$ be two tuples of S which have the same K-vector space type. Assume that $a_1 \ldots a_k$ are K-linearly independent and for every $j > k$, a_j is dependent on $a_1 \ldots a_k$. By the previous claim, there is an L_0-automorphism σ of S which sends $a_1 \ldots a_k$ onto $b_1 \ldots b_k$. Now, as $\mathbb{S}_k$ is definable in $\mathbb{S}$, it follows that for every $j > k$, $\sigma(a_j) = b_j$. $\dashv$

Remark. If $\mathbb{G}$ is such that $G_0 = acl(\emptyset) = 0$, then $\mathbb{T} = T(\mathbb{G}) = \mathbb{T}_1$ and $\mathbb{G}$ itself has the structure of a K-vector space. In that case, it will follow directly that $\mathbb{T}$ is finitely axiomatizable if and only if the theory of infinite K-vector spaces is finitely axiomatizable, and hence if and only if (see Section 4.2.2) K is finitely presented as a ring.

§4. Application to finite axiomatizability.

In [18] Paljutin shows that "There exists a finitely axiomatizable, not locally finite categorical quasi-variety if and only if one of the following conditions hold:

(1) there exists an infinite finitely presented ring which is a division ring;
(2) there exists an infinite finitely presented group with a finite number of elements $g_1, \ldots, g_n$ such that every non trivial cyclic subgroup of G intersects one of the conjugacy classes of the elements $g_1, \ldots, g_n$."

The proof proceeds by showing first that, if there is a such a finitely axiomatizable quasi-variety (a quasi-variety is the class of models of a set of universal Horn sentences), then there exists one which is "standard", where the standard quasi-varieties are either K-vector spaces for a division ring K, or the Cayley graph of a group. Then he shows that if the quasi-variety of K-vector spaces is finitely axiomatizable and not ω-categorical, K must satisfy condition (1), and that if the Cayley graph of G is finitely axiomatizable and not ω-categorical, then G must satisfy (2) (a proof of this part, due to M. A. Taitslin and Yu. E. Shimarev had already appeared in [1]).

The existence of such a ring and such a group are both still open. The existence of an infinite finitely presented group with finitely many conjugacy classes is a well-known long standing open question, but the existence of the a priori weaker condition required in (2) is also open.

Concerning the existence of a ring satisfying (1), it seems that it is already unknown whether there exists an infinite finitely generated ring which is a division ring. One can only easily see that such a division ring cannot be commutative (see Section 4.2.2 below).

Some years ago, A. Ivanov obtained some partial results on the conjecture that the existence of any finitely axiomatizable strongly minimal trivial set must imply the existence of a group satisfying (2) [12, 13]. The idea behind this conjecture is that such a group should appear as a subgroup or a quotient of the automorphism group of a connected component of the trivial strongly minimal set.

As explained in the introduction, Hrushovski showed in [10] that any finitely axiomatizable $\aleph_1$-categorical theory must be locally modular and suggested the conjecture that if the theory has non trivial pregeometry, then the associated division ring of quasi-endomorphisms must satisfy (1).

In order to be quite self-contained on the subject of finite axiomatizability, and because sometimes a certain confusion arises on what exactly is meant by finite axiomatizability (in the case of an infinite language, for example), we will present in the next section precise definitions and basic transfer properties. For the same reasons, in Section 4.2, we will present a detailed exposition of the two "standard" cases.

4.1. Transferring finite axiomatizability. We are going to need to transfer the property of being finitely axiomatizable through various changes of languages and interpretations, and to be quite precise when we do it. We have unfortunately not found a completely adequate reference for our purpose, which we could have simply quoted or referred to. Most of what follows appears in various places under slightly different forms, usually with the added assumption of ω-categoricity. The closest references for bi-interpretability can be found in [2] or, more recently, in [20]. In both cases the notions were used in the context of quasi-finitely axiomatizable ω-categorical theories, a context where one can replace syntactical arguments by topological considerations about the automorphism groups of the structures.

First, recall that a theory T in a language L is said to have a finite axiomatization if there is a finite set F of sentences from L such that $F \vdash T$. A classical result states that if $\mathscr{K}$ is a class of L-structures, then $\mathscr{K}$ is the class of models of a finite set of axioms if and only if both the class $\mathscr{K}$ and its complement are closed under isomorphisms and ultraproducts. No class of structures in a genuinely infinite language can satisfy these conditions. But one naturally comes across structures in such infinite languages, for example modules over infinite rings, for which the question of finite axiomatizability also makes sense.

So we will begin by recalling the definition of finite axiomatizability which applies to any language.

From now on, when we use the word *theory*, we mean a consistent set of axioms *which is closed under deduction*.

DEFINITION 4.1. Let L be a language and T a theory in L. We say that T is *finitely axiomatizable* if there is a finite sub-language L_0 of L such that any model M of T is interdefinable (Definition 3.3) with its reduct to L_0 and the theory T_{L_0}, the restriction of T to the language L_0, has a finite axiomatization in L_0. We then say that T is *finitely axiomatizable in the finite language* L_0.

Let us first fix some notation. Remark that, if the structure (M, L_0) is definable in the structure (M, L_1), then functions symbols from L_0 are replaced in the "translation" by their graphs. One can hence without any loss of generality, and in order to avoid extra cumbersome notation, when dealing with interdefinability, or bi-interpretability further down below, replace functions by their graphs and suppose that the languages involved contain only relation symbols and constants.

For every symbol s (relation or constant) in a language L let $\delta_s(\bar{x})$ denote the atomic formula defining the symbol:

- if s is a relation symbol R, $\delta_R(\bar{x}) := R(\bar{x})$,
- if s is a constant symbol, c, $\delta_c(x) := (x = c)$.

Let $\mathscr{M}_0 = (M, L_0)$ and $\mathscr{M}_1 = (M, L_1)$ be respectively L_0 and L_1-structures on M. Then $\mathscr{M}_0$ is $\emptyset$-definable in $\mathscr{M}_1$ if and only if, for each s in L_0, there is a formula $\phi_1[\delta_s](\bar{x})$ in L_1, such that for all $\bar{a} \subset M$,

$$\mathscr{M}_0 \models \delta_s(\bar{a}) \text{ iff } \mathscr{M}_1 \models \phi_1[\delta_s](\bar{a}).$$

We then define by induction a "translation" for every formula of L_0:

If $\theta(\bar{x})$ is an atomic formula, i.e., $\theta(\bar{x}) = R(\bar{x}, c_1, \dots, c_n)$, where R is a relation symbol, and $c_1, \dots c_n$ are constant symbols in L_0, we let

$$\phi_1[\theta](\bar{x}) := \exists y_1 \dots \exists y_n\, \phi_1[\delta_R](\bar{x}, y_1, \dots, y_n) \wedge \phi_1[\delta_{c_1}](y_1) \wedge \cdots \wedge \phi_1[\delta_{c_n}](y_n).$$

If $\theta(\bar{x}) = \theta_1(\bar{x}) \wedge \theta_2(\bar{x})$ in L_0, we let $\phi_1[\theta](\bar{x}) := (\phi_1[\theta_1](\bar{x}) \wedge \phi_1[\theta_2](\bar{x}))$.
If $\theta(\bar{x}) = \neg\psi(\bar{x})$, we let $\phi_1[\theta](\bar{x}) := \neg(\phi_1[\psi](\bar{x}))$.
If $\theta(\bar{x}) = \exists y\, \psi(\bar{x}, y)$, we let $\phi_1[\theta](\bar{x}) := \exists y\, (\phi_1[\psi](\bar{x}, y))$.
It follows that for all $\bar{a} \subset M$, for all $\theta(\bar{x})$ in L_0,

$$\mathscr{M}_0 \models \theta(\bar{a}) \text{ iff } \mathscr{M}_1 \models \phi_1[\theta](\bar{a}).$$

Using the following lemma and a compactness argument, one can check easily that if the language L is finite, T is finitely axiomatizable in the sense above if and only if it has a finite axiomatization in L.

LEMMA 4.2. *Let T be a theory in a language L, let L_0 be a sub-language of L and let Σ be a subset of sentences from T such that, for every symbol s in L, Σ includes a sentence of the form*:

$$(\forall \bar{x}\, (\delta_s(\bar{x}) \leftrightarrow \phi_s(\bar{x})),$$

where $\phi_s(\bar{x})$ is a formula in L_0. Then T is axiomatized by $\Sigma \cup T_{L_0}$.

PROOF. As above, by induction, we see that the condition on Σ exactly says that every L-structure $\mathcal{M}$ which is a model of Σ is definable in its restriction to L_0, the translation being uniform for all models of Σ, that is: for every formula $\theta(\bar{x})$ in L there exists a formula $\phi_0[\theta](\bar{x})$ in L_0 such that $\Sigma \vdash \forall x\,(\phi_0[\theta](x) \leftrightarrow \theta(x))$. The conclusion then follows directly. ⊣

Finite axiomatizability transfers through interdefinability:

PROPOSITION 4.3. *Let $\mathcal{M}_0 = (M, L_0)$ and $\mathcal{M}_1 = (M, L_1)$ be two structures on the same domain M which are interdefinable. Then $Th(\mathcal{M}_0)$ is finitely axiomatizable if and only if $Th(\mathcal{M}_1)$ is finitely axiomatizable.*

PROOF. Note first that we can assume that both L_0 and L_1 are finite: indeed, suppose that $Th(\mathcal{M}_1)$ is finitely axiomatizable. As interdefinability is transitive, we can suppose that L_1 is finite. Then there is a finite sub-language of L_0, such that the reduct of $\mathcal{M}_0$ to this finite sub-language is interdefinable with L_1, hence also with $\mathcal{M}_0$ in L_0.

Using the notation described above, it follows from interdefinabilty that we have translations ϕ_0 and ϕ_1 defined by induction, such that:

(i) for all $\bar{a} \subset M$, for all $\theta(\bar{x})$ in L_i, $i \in \{0, 1\}$,

$$\mathcal{M}_i \models \theta(\bar{a}) \text{ iff } \mathcal{M}_{1-i} \models \phi_{1-i}[\theta](\bar{a}).$$

One can also easily check the following:

(ii) for every formula $\theta(\bar{x})$ in L_i,

$$\mathcal{M}_i \models \forall\bar{x}\,(\theta(\bar{x}) \leftrightarrow \phi_i[\phi_{1-i}[\theta]](\bar{x})).$$

Let $\Sigma_0 := \bigwedge_{s\in L_0}(\forall\bar{x}\,(\delta_s(\bar{x}) \leftrightarrow \phi_0[\phi_1[\delta_s]](\bar{x}))$.

Consider now any L_0-structure, $\mathcal{N}_0$, which is a model of Σ_0. On N, the domain of $\mathcal{N}_0$, define an L_1-structure, $\mathcal{N}_1$, by interpreting the symbols of L_1 according to the translation ϕ_0, that is: for every symbol s of L_1, for every $\bar{b} \subset N$, $\mathcal{N}_1 \models \delta_s(\bar{b})$ iff $\mathcal{N}_0 \models \phi_0[\delta_s](\bar{b})$. Then

(iii) for every formula $\theta(\bar{x})$ in L_1, for all $\bar{b} \subset N$,

$$\mathcal{N}_1 \models \theta(\bar{b}) \text{ iff } \mathcal{N}_0 \models \phi_0[\theta](\bar{b}).$$

(iv) for every formula $\psi(\bar{x})$ in L_0,

$$\mathcal{N}_0 \models (\forall\bar{x}\,(\psi(\bar{x}) \leftrightarrow \phi_0[\phi_1[\psi]](\bar{x})).$$

Suppose now that $Th(\mathcal{M}_1)$ is finitely axiomatizable, by a sentence θ_1 in L_1. We claim that $Th(\mathcal{M}_0)$ is then axiomatized by $\phi_0[\theta_1]$ together with the sentence Σ_0 defined above. Let $\mathcal{N}_0$ be any L_0-structure model of $\phi_0[\theta_1]$ and of Σ_0. Consider $\mathcal{N}_1$ the L_1-structure on N associated to $\mathcal{N}_0$ as above. Then, by (iii), $\mathcal{N}_1$ is a model of θ_1, and hence of $Th(\mathcal{M}_1)$. Let ψ be any sentence in L_0, such that $\mathcal{N}_0 \models \psi$. Then by (iv), $\mathcal{N}_0 \models \phi_0[\phi_1[\psi]]$, and hence by (iii), $\mathcal{N}_1 \models \phi_1[\psi]$. As $\mathcal{N}_1 \equiv \mathcal{M}_1$, $\mathcal{M}_1 \models \phi_1[\psi]$, hence by (i), $\mathcal{M}_0 \models \psi$. This shows that $\mathcal{N}_0$ is a model of $Th(\mathcal{M}_0)$. ⊣

Now some very basic remarks about expanding the language while keeping finite axiomatizability:

LEMMA 4.4. (i) *Let $\mathcal{M} = (M, L)$ be a structure in a finite language L, let E be an $\emptyset$-definable equivalence relation on M^n and $U \subset M^n$ an $\emptyset$-definable subset. Let $\mathcal{M}_E$ be the following reduct of $\mathcal{M}^{eq}$, $\mathcal{M}_E := (M, U/E, L, f_E)$, where f_E*

is the restriction of the quotient map to U. Then the theory of $\mathcal{M}$ has a finite axiomatization if and only if the theory of $\mathcal{M}_E$ has a finite axiomatization.

(ii) *Let $\mathcal{M} = (M, L)$ be a structure in a finite language L. Let $\bar{a}$ be a tuple of M such that the type of $\bar{a}$ is isolated. Then the theory of $\mathcal{M}$ has a finite axiomatization if and only if the theory of $\mathcal{M}_{\bar{a}} = (M, L, \bar{a})$ has a finite axiomatization.*

Proof. (ii) is clear, and for (i), one just needs to note that the theory of $\mathcal{M}_E$ can be axiomatized by the theory of $\mathcal{M}$ in L together with the sentence ϕ:

$$(\forall y \in U/E \ \exists x \in U \ f_E(x) = y) \wedge (\forall x_1 \in U \ \forall x_2 \in U \ (E(x_1, x_2) \leftrightarrow (f_E(x_1) = f_E(x_2)))). \qquad \dashv$$

We now check that one can reduce questions of finite axiomatizability for $\aleph_1$-categorical one-based groups to the case of abelian structures.

Recall the fundamental properties of one-based groups (the reader can take Property 1 as a definition for stable one-based groups).

Fact 4.5. [11]

1. *A group $\mathcal{G}$ is stable and one-based if and only if, for every $n \geq 1$, every definable subset of G^n is a Boolean combination of definable cosets of definable subgroups of G^n.*
2. *A stable one-based group is definably abelian by finite, that is, has a definable normal abelian subgroup of finite index.*
3. *Let $\mathcal{H}$ be a stable one-based group. Let $S \subset H^n$ be a definable connected subgroup. Then S is definable over $acl^{eq}(\emptyset)$.*
4. *If G is ω-stable and one-based, then every definable subset of G^n is a Boolean combination of definable cosets of connected definable subgroups of G^n.*

Corollary 4.6. *Let $\mathcal{G}$ be an ω-stable one-based group. Fix $\mathcal{M}_0$ a countable elementary submodel of $\mathcal{G}$. Let $A \subset G$; any A-definable subset of G^n is a Boolean combination of A-definable cosets of some M_0-definable connected subgroups.*

(Remark that in the above corollary $\mathcal{G}$ could be equal to $\mathcal{M}_0$.)

Now let $\mathcal{G} = (G, L)$ be a an ω-stable one-based connected group. Fix $\mathcal{M}_0$ some countable elementary submodel of $\mathcal{G}$. Let $(H_i)_{i \in I}$ be the family of all connected M_0-definable subgroups in $\bigcup_{n \geq 1} G^n$. Let $\mathbb{G}$ be the following abelian structure:

$$\mathbb{G} = (G, +, -, 0, (H_i)_{i \in I}, (m)_{m \in M_0}).$$

It follows from Fact 4.5 and Corollary 4.6 that $\mathcal{G}$ and $\mathbb{G}$ are M_0-interdefinable.

Corollary 4.7. *Let $\mathcal{G}$ be an ω-stable one-based connected group such that $Th(\mathcal{G})$ is finitely axiomatizable in a finite language L. Then there is a finitely axiomatizable abelian structure with constants, $\mathbb{G}$, which is interdefinable with a finitely axiomatizable expansion of $\mathcal{G}$ by finitely many constants.*

Proof. Consider $\mathcal{M}_0 \preceq \mathcal{G}$, the prime model of $Th(\mathcal{G})$, which exists by ω-stability of $Th(\mathcal{G})$. Consider the abelian structure $\mathbb{G}$ described above which is M_0-interdefinable with $\mathcal{G}$. As the language L is finite, we can choose a finite family $H_1, \ldots, H_k$ and a finite sequence $m_0, \ldots, m_n$ of elements from the prime model M_0, such that every symbol from the language L can be defined in the restriction, denoted $\mathbb{G}_f$, of $\mathbb{G}$ to the finite language

$$L_f := \{+, -, 0, (H_i)_{1 \leq i \leq k}, \{m_0, \ldots, m_n\}\}$$

and such that every H_i is definable in $\mathscr{G}$ over $\{m_0, \dots, m_n\}$. Now add $\{m_0, \dots, m_n\}$ as new constants to the language L of $\mathscr{G}$. As M_0 is atomic, the type of the tuple $(m_0, \dots, m_n)$ in L is isolated over $\emptyset$. By Lemma 4.4, the theory of the expansion $\mathscr{G}'$ of $\mathscr{G}$ to $L' := L \cup \{m_0, \dots, m_n\}$ remains finitely axiomatizable. The structures $\mathscr{G}'$ and $\mathbb{G}_f$ are interdefinable (over $\emptyset$). Hence by Proposition 4.3 $Th(\mathbb{G}_f)$ is finitely axiomatizable. ⊣

We will now check, using the fact that the connected component is a definable subgroup of finite index, that an $\aleph_1$-categorical group is finitely axiomatizable if and only if its connected component is. This is a particular case of the transfer of finite axiomatizability by bi-interpretability.

Recall the definition of bi-interpretability from [2], or [20] (as remarked before, we can suppose that functions are given by their graphs and that our languages contain only relations and constants):

Definition 4.8. Let $\mathscr{M} = (M, L_1)$ and $\mathscr{N} = (N, L_2)$. We say that $\mathscr{M}$ is $\emptyset$-*interpretable* in $\mathscr{N}$ if there is an $\emptyset$-definable subset U of N^n, and a surjective map f, from U onto M such that:

- the equivalence relation E_f on $U \times U$ defined, for $(a, b) \in U \times U$, by $f(a) = f(b)$ is $\emptyset$-definable in $\mathscr{N}$,
- for every k-ary relation symbol R in L_1, the subset R_f of U^k, $R_f := \{(a_1, \dots, a_k) \in U^k \,;\, \mathscr{M} \models R(f(a_1), \dots, f(a_k))\}$ is $\emptyset$-definable in $\mathscr{N}$,
- for every constant symbol c in L_1, the subset $c_f := \{a \in U : \mathscr{M} \models f(a) = c\}$ is $\emptyset$-definable in $\mathscr{N}$.

It follows that f induces an isomorphism of L_1-structures between U/E_f (subset of N^{eq}) and $\mathscr{M}$.

If $\mathscr{M}$ is interpretable in $\mathscr{N}$, via the surjective map f from $U \subset N^k$ onto M, and $\mathscr{N}$ is interpretable in $\mathscr{Q} = (Q, L_3)$, via the surjective map g from $V \subset Q^n$ onto N, let $W := \{(q_1, \dots, q_k) \in (Q^n)^k : (g(q_1), \dots, g(q_k)) \in U\}$. We denote by $f \circ g$ the obvious induced map from $W \subset Q^{nk}$ onto M. One checks easily that $f \circ g$ is an interpretation of $\mathscr{M}$ in $\mathscr{Q}$.

If $\mathscr{M}$ is interpretable in $\mathscr{N}$, via f, and $\mathscr{N}$ is interpretable in $\mathscr{M}$ via g, we say that $\mathscr{M}$ and $\mathscr{N}$ are *bi-interpretable* if $f \circ g$ is an $\emptyset$-definable map in $\mathscr{M}$ and $g \circ f$ an $\emptyset$-definable map in $\mathscr{N}$. Interdefinablity corresponds to the trivial case where $f = g = Id$.

Proposition 4.9. *Let $\mathscr{M}$ be an L_1-structure and $\mathscr{N}$ an L_2-structure which are bi-interpretable. Then $Th(\mathscr{M})$ is finitely axiomatizable if and only if $Th(\mathscr{N})$ is.*

Proof. Suppose that $\mathscr{M}$ is interpretable in $\mathscr{N}$, via the surjective map f from $U \subset N^k$ onto M, and $\mathscr{N}$ is interpretable in $\mathscr{M}$, via the surjective map g from $V \subset M^n$ onto N such that $f \circ g$ is an $\emptyset$-definable map in $\mathscr{M}$ (from $g^{-1}(U)$ onto M) and $g \circ f$ an $\emptyset$-definable map in $\mathscr{N}$. Denote by π_g (resp. π_f) the quotient map definable in $\mathscr{M}^{eq}$ (resp. in $\mathscr{N}^{eq}$) from $V \subset M^n$ onto V/E_g (resp. from $U \subset N^k$ onto U/E_f). We denote by $\bar{f}$ the isomorphism of L_1-structures induced by f from U/E_f onto M.

Let $\mathscr{M}_g$ be the two sorted structure (living in $\mathscr{M}^{eq}$) consisting of $(M, V/E_g, L_1, \pi_g)$.

(a) We extend the isomorphism $(\bar{f})^{-1}$ to an isomorphism h from $\mathscr{M}_g$ into $\widetilde{\mathscr{M}_g} := (U/E_f, N, L_1, \widetilde{\pi_g})$, by letting:

– on M, $h = (\bar{f})^{-1} : M \mapsto U/E_f$,
– on V/E_g, $h = \bar{g} : V/E_g \mapsto N$.

Then for $(b_1, \dots, b_n) \in h(V) \subset (U/E_f)^n$, $b_j = h(a_j)$, we let $\widetilde{\pi_g}(b_1, \dots, b_n) := \bar{g}(\pi_g(a_1, \dots, a_n))$.

We now identify, via this isomorphism, $\mathscr{M}_g$ and $\widetilde{\mathscr{M}_g}$ and can hence suppose that the domain of $\mathscr{M}_g$ is equal to $U/E_f \cup N$.

(b) We claim that the structure $\mathscr{M}_g$ is $\emptyset$-definable in the structure $\mathscr{N}_f := (N, U/E_f, L_2, \pi_f)$:

– the sorts are definable, as $M = U/E_f$, and $V/E_g = N$.
– For any symbol s in L_1, using the notation introduced just before Lemma 4.2, interpretability of $\mathscr{M}$ in $\mathscr{N}$ tells us that the set $\{(u_1, \dots, u_m) \in U^m : \mathscr{M} \models \delta_s(f(u_1), \dots, f(u_m))\}$ is definable in $\mathscr{N}$, via a formula $\theta_s(\bar{x}_1, \dots, x_m)$ in L_2. So, having identified M and U/E_f, via h, we see that $\mathscr{M}_g \models \delta_s(b_1, \dots, b_m)$ iff $\mathscr{M} \models \delta_s(b_1, \dots, b_m)$ iff, in $\mathscr{N}_f$, there exist $u_1, \dots u_m$ in U, such that $b_i = \pi_f(u_i)$, $1 \le i \le m$, and $\theta_s(u_1, \dots, u_m)$ holds. This is a formula in $\mathscr{N}_f$.
– There remains only to show that the map π_g is definable in $\mathscr{N}_f$. The domain of π_g is $V \subset (U/E_f)^n$, its image is N. For $(b_1, \dots, b_n) \in V$, we have that $\pi_g(b_1, \dots, b_n) = c$ iff $b_i = \pi_f(u_i)$, $1 \le i \le n$, and $c = (g \circ f)(u_1, \dots, u_n)$, and gof is, by the assumption of bi-interpretability, a definable map.

(c) Similarly, we show that $\mathscr{N}_g$ is $\emptyset$-definable in $\mathscr{M}_f$, using the fact that $f \circ g$ is a definable map in $\mathscr{M}$.

Now, assume that $Th(\mathscr{M})$ is finitely axiomatizable. We can assume that L_1 is finite and $Th(\mathscr{M})$ has a finite axiomatisation in L_1. By Lemma 4.4, the theory of $\mathscr{M}_g = (M, V/E_g, L_1, \pi_g)$ also has a finite axiomatization. By transfer through inter-definability (Proposition 4.3), it follows that the theory of $\mathscr{N}_f = (N, U/E_f, L_2, \pi_f)$ is finitely axiomatizable in a finite sub-language. Again by Lemma 4.4, it follows that $\mathscr{N}$ is finitely axiomatizable. ⊣

We now recall the definition of an induced structure on a definable subset:

Definition 4.10. Let $\mathscr{M}$ be an L-structure and D be an $\emptyset$-definable subset of M^n. The *induced structure* from $\mathscr{M}$ on D is the structure $(D, (P_\phi)_{\{\phi \text{ formula in } L\}})$, where, for $\phi(x_1, \dots, x_k)$, $|x_j| = n$, P_ϕ is a predicate of arity k which is interpreted on D by the set $D^k \cap \phi(M^{nk})$.

Lemma 4.11. *Let $\mathscr{M}$ be an L-structure which is the union of a finite definable partition, that is, $M = M_1 \cup \dots \cup M_n$, where, for $1 \le i < j \le n$, M_i is $\emptyset$-definable and $M_i \cap M_j = \emptyset$. Suppose furthermore that for each $i > 1$ there is a $\emptyset$-definable bijection, f_i from M_1 onto M_i and that L contains n constant symbols $\{c_1, \dots, c_n\}$ which are interpreted in $\mathscr{M}$ by distinct elements of M_1. Consider M_1 together with its induced structure from $\mathscr{M}$, denoted $\mathscr{M}_1$. Then $\mathscr{M}$ and $\mathscr{M}_1$ are bi-interpretable.*

Proof. Let $U := M_1 \times \{c_1, \dots, c_n\} \subset {M_1}^2$ and let $f : U \mapsto M$, $f(x, c_i) = f_i(x)$, where f_1 is the identity on M_1. Then f gives an (injective) interpretation of $\mathscr{M}$ into $\mathscr{M}_1$. Indeed, as f is $\emptyset$-definable in $\mathscr{M}$, for any k-ary predicate symbol R from L, the set $R_f = \{(a_1, \dots, a_k) \in U : \mathscr{M} \models R(f(a_1), \dots, f(a_k))\}$ is a basic predicate in the language of the induced structure, hence certainly definable. Similarly, if d is a constant symbol, for the set $d_f = \{a \in U : \mathscr{M} \models f(a) = d\}$.

Let $g : M_1 \subset M \mapsto M_1$ be the identity. Let $W := \{(x, y) \in U : f(x, y) \in M_1\} = M_1 \times \{c_1\}$. Then $g \circ f : W \mapsto M_1$ is an interpretation of $\mathscr{M}_1$ into itself. For $(x, c_1) \in U$, $g \circ f(x, c_1) = x$ is the first projection and is hence $\emptyset$-definable in $\mathscr{M}_1$.

Let $W' := \{(x, y) \in M_1 \times M_1 : (g(x), g(y)) \in U\}$. Then $W' = U = M_1 \times \{c_1, \dots, c_n\}$, and $f \circ g(x, c_i) = f_i(x)$ is $\emptyset$-definable in $\mathscr{M}$. ⊣

Proposition 4.12. *Let $\mathscr{G} = (G, L)$ be an ω-stable group in a language L. There is an expansion of the language L by a finite number of constants, $\tilde{L}$, such that, if $\widetilde{\mathscr{G}} = (G, \tilde{L})$ and $\widetilde{\mathscr{G}^0}$ is the connected component of G, G^0, with the induced structure from $\widetilde{\mathscr{G}}$, the following are equivalent*:

- *$Th(\mathscr{G})$ is finitely axiomatizable.*
- *$Th(\widetilde{\mathscr{G}})$ is finitely axiomatizable.*
- *$Th(\widetilde{\mathscr{G}^0})$ is finitely axiomatizable.*

Proof. By ω-stability, G^0 is $\emptyset$-definable in $\mathscr{G}$ and has finite index in G. By ω-stability again, there is a prime model G_1, $G_1 \preceq \mathscr{G}$, atomic over $\emptyset$. Choose $a_1, \dots, a_n$ in this prime model such that $G = a_1G^0 \cup \dots \cup a_nG^0$, with $a_1 = 1$. Choose $c_1, \dots c_n$ distinct elements from ${G_1}^0$, the connected component of G_1. Then the type of the tuple $a_1, \dots, a_n, c_1, \dots, c_n$ is isolated over the empty set. Let T denote the complete theory of $\mathscr{G}$ in the language $\tilde{L} = L \cup \{a_1, \dots, a_n, c_1, \dots, c_n\}$. By 4.4, $Th(\mathscr{G})$ (in L) is finitely axiomatizable if and only if T is. Let T_0 be $Th(\widetilde{\mathscr{G}^0})$. We are now in the situation of Lemma 4.11: G is the union of a finite $\emptyset$-definable partition $a_1G^0 \cup \dots \cup a_nG^0$, and for each i, there is an $\emptyset$-definable bijection f_i from $a_1G^0 = G^0$ onto a_iG^0, $f_i(g) = a_ig$. The result then follows by 4.11 and 4.9. ⊣

4.2. The classical examples. Before we start on the description of the two emblematic examples, we would like to draw the reader's attention to the following: if T is a theory in L which is finitely axiomatizable in a finite sub-language L_0 of L, then it is certainly finitely axiomatizable in every finite sub-language L_1 of L containing L_0 (Lemma 4.2), but one should be a little careful. For instance, suppose that T is a complete theory in an infinite language L, which is finitely axiomatizable in a finite sub-language L_0 of L, i.e., such that there is a finite set of axioms, in L_0, for T_{L_0}. Let Σ be an arbitrary infinite set of axioms in L for the complete theory T. By compactness, some finite subset Σ_1 of Σ will axiomatize T_{L_0}. But if L_1 is the finite sub-language of L containing all symbols appearing in Σ_1, there is no reason that $\Sigma_1 \vdash T_{L_1}$, or equivalently there is no reason for Σ_1 to axiomatize a complete theory in the language L_1. This explains the care taken in identifying the right set of axioms in the following proofs.

4.2.1. *The trivial example.* First, recall that for any non trivial group G, the theory T_G which describes G acting semi-regularly (the stabilizer of every element is trivial) on an infinite set, in the language $L_G := \{g : g \in G\}$, where each g is a unary function symbol, is strongly minimal, eliminates quantifiers and has trivial geometry. The theory T_G is ω-categorical if and only if G is finite. If G is infinite, the Cayley graph of G (that is the regular action of G on itself by left multiplication) is a model of T_G. The theory T_G can be axiomatized by the following set of axioms, Σ_G, if G is infinite:

- $\forall x\, 1(x) = x$;
- $\forall x\, g(x) \neq x$, for every $g \neq 1 \in G$;
- $\forall x\, g(h(x)) = r(x)$, for every $g, h, r \in G$ such that $gh = r$.

If G is finite, then T_G can be axiomatized by Σ_G together with the scheme for infinity.

Note that, for any model M of T_G, for any $a \in M$, the definable closure of $\{a\}$ in M in the language L_G is the G-orbit of a.

Now suppose that we have a presentation of G: G is isomorphic to the quotient of the free group on $S = \{s_i; i \in I\}$ by a normal subgroup P. Then the theory T_G is clearly interdefinable with the following theory in the language $L_S = \{s : s \in S\}$, which we denote by Σ_S: let W be the set of words on S,

– $\forall x\, w(x) = x$ for every $w \in P$,
– $\forall x\, w(x) \neq x$ for every $w \in W \setminus P$.

Suppose that G is an infinite finitely presented group G, that is, both the free group S and the normal subgroup P are finitely generated. Furthermore suppose that in G there is a finite number of conjugation classes $C_1, \dots, C_k$ such that every non trivial cyclic subgroup of G intersects one of the C_i's. Then T_G is finitely axiomatizable (in the sense of Definition 4.1). Indeed, choose $F = \{g_1, \dots, g_n\} \subset G$ such that: for every j, $g_j \neq 1$, F generates G, there is a finite set P_0 of words on F which generates P, the presentation of G, F is closed under inverse and for every $g \in G \setminus \{1\}$, there is some $m > 0$ such that g^m is conjugate to one of the g_j's. Let Σ_F be the (complete) set of axioms described above, in the finite language $L_F := \{g_1, \dots, g_n\}$, which is interdefinable with T_G. Consider Σ_0, the following finite subset of Σ_F:

– $\forall x\, g_j(x) \neq x$, for every j, $1 \leq j \leq n$,
– $\forall x\, w(x) = x$, for every $w \in P_0$.

We must check that Σ_0 is an axiomatization for Σ_F. If $w \in P$, then, for all x, $w(x) = x$ as P is the normal subgroup generated by P_0. If g is any word on F, and $g \notin P$, we must check that for all x, $g(x) \neq x$. By assumption, there are $m > 0$, $g_j \in F$ and $h \in G$, such that $g^m = h^{-1} g_j h$. If $g(x) = x$ for some x, then $g^m(x) = x = h^{-1}(g_j(h(x))$, hence $h(x) = g_j(h(x))$. But this contradicts Σ_0.

Conversely, suppose that G is infinite and that the theory T_G is finitely axiomatizable. Let $F \subset G$, be finite such that $T := T_G$ is finitely axiomatizable in the sub-language $L_F := \{f; f \in F\}$, that is, such that any model of T_G is interdefinable with its reduct to L_F, and the (complete) theory $T_G|L_F$ is finitely axiomatizable. Let H be the subgroup of G generated by F. Then $T_G|L_H$ contains the theory T_H, which is complete, hence it is equal to T_H. In the language L_G, G which is a model of T_G, is equal to the definable closure of the identity element 1. Similarly, H is, in L_H the definable closure of 1. By interdefinability of L_G with L_F, G is also equal to the definable closure of 1 in L_F, hence also in L_H. It follows that $G = H$.
So we know that G is finitely generated, hence isomorphic to a quotient of the free group on a finite set of generators S, which we suppose closed under inverse, by some normal subgroup P. Let W be the set of all words on F. Pass to the theory T_S (axiomatized by Σ_S) in the finite language $\{s : s \in S\}$, which is interdefinable with T_G. By finite axiomatizability, there is a finite subset W_0 of W (the set of words on S) such that Σ_S can be axiomatized by Σ_{W_0}:

– $\forall x\, w(x) = x$ for every $w \in W_0 \cap P$,
– $\forall x\, w(x) \neq x$ for every $w \in W_0 \setminus P$.

We can suppose that, for every $s \in S$, $ss^{-1} \in W_0$.

Let N be the normal subgroup generated by $W_0 \cap P$ in D, the free group on S. By construction $N \subset P$. The Cayley graph of D/N, in the language L_S is a model of Σ_{W_0}, hence is interdefinable with a model of Σ_S. It follows that $N = P$ and $G \cong D/N$.

Let $h \in G$, $h \neq 1$, let H denote the subgroup generated by h in G, and G/H the set of left cosets, equipped with an L_G structure by the left action of G. As the action of G is not semi-regular on G/H, G/H is not a model of Σ_S, hence by finite axiomatizability, it is not a model of Σ_{W_0}. So there is some $g \in W_0 \setminus N$ and some coset aH such that $g(aH) = aH$, that is, such that $a^{-1}ga = h^n$ for some integer n. So any non trivial element h has a power which is conjugate to one of the $g's$ in W_0.

4.2.2. *Vector spaces.* Let K be any countable division ring. Let L_K be the usual language for K-vector spaces, $L_K := \{+, -, 0, (k)_{k \in K}\}$, where k is a unary function interpreted as scalar multiplication by the element k. Consider T_K the theory of all infinite K-vector spaces in L_K. The theory T_K is $\aleph_1$-categorical and is totally categorical if and only if K is finite.

Suppose that K is an infinite division ring which is finitely presented as a ring. Then the complete theory of K-vector spaces is finitely axiomatizable in the following way: let F be a finite subset of K, which generates K as a ring and such that there is a finite set of terms in F, P, which generates the presentation of K (a two-sided ideal J, such that K is isomorphic to the quotient of the free ring generated by F by the ideal J). Then T_K is finitely axiomatized in $L_F := \{+, -, 0, 1, (f)_{f \in F}\}$ by

- axioms for abelian groups
- $\forall x\, 1(x) = x$,
- $\forall x \forall y\, f(x + y) = f(x) + f(y)$, for every $f \in F$,
- $\forall x\, w(x) = 0$, for every $w \in P$.

For the converse, we now suppose that the theory of infinite K-vector spaces is finitely axiomatizable. This forces K to be infinite. This follows by the classical results on the non finite axiomatizability of totally categorical theories, but one can also check directly that if A is any finite ring, the theory of infinite A-modules cannot be finitely axiomatized.

Proposition 4.13. *Let K be an infinite division ring. If the theory of K-vector spaces in the language L_K is finitely axiomatizable, then K is finitely presented as a ring.*

Proof. Let T_K be the theory of non trivial K-vector spaces, in the usual language $L_K = \{0, +, -, k : k \in K\}$.

Let X be a finite subset of K such that T_K is finitely axiomatizable in the finite language $L_X := \{0, +, -, k : k \in X\}$.

Claim. K is generated as a skew field by X.

Proof. Let K_0 be the subfield of K generated by X. Then the theory T_{K_0} of infinite K_0-vector spaces is a subset of $T_{|L_{K_0}}$ and since T_{K_0} is complete, they are equal. Now, consider K as a K-vector space. Then $K = \mathrm{dcl}_{L_K}(1_K) = \mathrm{dcl}_{L_X}(1_K) = \mathrm{dcl}_{L_{K_0}}(1_K) = K_0$. $\dashv_{\text{Claim}}$

Denote by S_K the classical axiomatization of T_K:

1. $\exists x\, x \neq 0$;
2. axioms for abelian groups;
3. $\forall x\, 1_K(x) = x$;
4. $\forall x \forall y\, k(x + y) = k(x) + k(y)$, $(k \in K)$;

5. $\forall x\ k(x) + k'(x) = k''(x)$, $(k, k', k'' \in K,\ k'' = k + k')$;
6. $\forall x\ k(k'(x)) = k''(x)$, $(k, k', k'' \in K,\ k'' = kk')$.

If A is a subset of K we will denote by S_A the subset of sentences of S_K in the language $L_A := \{0, +, -, k : k \in A\}$. A priori, S_X does not give an axiomatization of the complete theory T_{L_X}, but since there is some finite axiomatization of T_{L_X} by assumption, there exists by compactness a finite subset Y containing X such that S_Y implies T_{L_X}.

We are going to enlarge Y in order that S_Y implies the complete theory T_{L_Y}. First we define the depth of elements of K relatively to X. We assume that $1_K, 0_K \in X$ and we define by induction a sequence $(W_i)_{i\in\omega}$ of subsets of K such that $W_0 := X$ and

$$W_{i+1} := \{k \in K : k = -k_1 \text{ or } k = k_1^{-1} \text{ or } k = k_1 + k_2 \text{ or } k = k_1k_2 \text{ for } k_1, k_2 \in W_i\}.$$

Then $K = \cup_{i\in\omega} W_i$ since X generates K as a skew field. We define the depth of $k \in K$ as the smallest integer n such that $k \in W_n$. Now by an easy induction, we can enlarge Y so that it remains finite and for each $k \in Y$, if the depth of k is $n + 1$ then there exists $k_1, k_2 \in Y$ of depths at most n such that $k = -k_1$, or $k = k_1^{-1}$, or $k = k_1 + k_2$ or $k = k_1k_2$.

Claim. Then T_{L_Y} is axiomatized by S_Y.

Proof. We choose, by induction on the depth of elements of Y, for each $k \in Y$, a formula $\phi_k(x, y) \in L_X$ such that $S_Y \vdash \forall x\forall y\ (kx = y) \leftrightarrow \phi_k(x, y)$ (we know by assumption that there is such a formula for which $T \vdash \forall x\forall y\ (kx = y) \leftrightarrow \phi_k(x, y)$, but we want one such that the equivalence can be deduced from S_Y). If $k \in Y$ has depth 0 (i.e., $k \in X$), then we let $\phi_k(x, y) := (kx = y)$. Assume that we have chosen a formula ϕ_k for each $k \in Y$ of depth less or equal to n. Let $k \in Y$ have depth $n + 1$. Then there exist $k_1, k_2 \in Y$, of depth at most n, such that at least one of the following cases occur:

- $k = -k_1$; in this case we let $\phi_k(x, y) := \phi_{k_1}(x, -y)$,
- or $k = k_1^{-1}$; in this case we let $\phi_k(x, y) := \phi_{k_1}(y, x)$,
- or $k = k_1 + k_2$; in this case we let
$$\phi_k(x, y) := \exists t_1\exists t_2\ \left(\phi_{k_1}(x, t_1) \wedge \phi_{k_2}(x, t_2) \wedge (y = t_1 + t_2)\right),$$
- or $k = k_1k_2$; in this case we let
$$\phi_k(x, y) := \exists t\ \left(\phi_{k_2}(x, t) \wedge \phi_{k_1}(t, y)\right).$$

Since T_K is finitely axiomatizable in the language L_X and for each $k \in Y$, $T_{L_Y} \vdash \forall x\forall y\ (kx = y) \leftrightarrow \phi_k(x, y)$, the complete theory T_{L_Y} is axiomatized by

$$T_{L_X} \cup \{\forall x\forall y\ (kx = y) \leftrightarrow \phi_k(x, y) : k \in Y\}.$$

It follows that S_Y is an axiomatization of T_{L_Y}. $\dashv_{\text{Claim}}$

Now, we are going to prove that K is isomorphic to the finitely presented ring A, given by the set of generators $\{k : k \in Y\}$ and the presentation:

$$\{1_K - 1\} \cup \{k_1 + k_2 - k_3 : k_1, k_2, k_3 \in Y; k_3 = k_1 + k_2\} \cup \{k_1k_2 - k_3 : k_1, k_2, k_3 \in Y; k_3 = k_1k_2\}.$$

Remark that every non trivial A-module is a model of S_Y as an L_Y-structure, hence any two non trivial A-modules are elementarily equivalent in the language L_Y. Furthermore, any non trivial A-module has a canonical expansion to a K-vector space: by assumption, for each $k \in K$, there is a formula $\theta_k \in L_X \subset L_Y$, such that $T \vdash \forall x \forall y\ (kx = y) \leftrightarrow \theta_k(x, y)$. Define, for $k \in K$, $m, n \in M$, $km = n$ iff $M \models \theta_k(m, n)$.

Let ψ be the canonical morphism from A to K which sends each generator $k \in Y$ to $k \in K$. The morphism ψ is injective: consider the A-module structure on K given via ψ, i.e., define $ax := \psi(a)x$. As A-modules, K and A are elementarily equivalent. In A, if $a \neq 0$, then for some x, $ax \neq 0$, hence this is also true in K, which implies that $\psi(a) \neq 0$.

Hence A has no zero divisors. Again by completeness of the theory of non trivial A-modules, this implies that in all non trivial A-modules, if $a \neq 0 \in A$, if $x \neq 0$, then $ax \neq 0$. This implies that A is a division ring: if $a \in A \setminus \{0\}$ was not left invertible, A/Aa would be a non-trivial A-module satisfying that $ax = 0$ for some $x \neq 0$ ($x = 1 + Aa$). Since $X \subset \psi(A)$, X generates K as a skew field and $\psi(A)$ is a skew field, we obtain that $\psi(A) = K$. ⊣

REMARK. As we have mentioned above, it seems to be an open question whether there exists an infinite division ring which is finitely generated as a ring. It is easily seen, though, that there is no such commutative division ring. Let K be a field which is finitely generated as a ring, and let k denote its prime field ($k = \mathbb{F}_p$ or $k = \mathbb{Q}$). As K is finitely generated as a ring over k, K is contained in k^{alg}, the algebraic closure of k (see for example [14], Chapter IX.1, or argue by modelcompleteness of the theory of algebraically closed fields). If K has characteristic $p > 0$, then $K = \mathbb{F}_p[a_1, \dots, a_n]$ is finite. Otherwise $k = \mathbb{Q}$, and there are $a_1, \dots, a_n \in \mathbb{Q}^{alg}$, such that $K = \mathbb{Z}[a_1, \dots, a_n]$. In that case, for some integer $m > 0$, the a_i's are entire over $A := \mathbb{Z}[1/m]$ and K is finitely generated as an A-module. As A is Noetherian, K is Noetherian as an A-module, and $\mathbb{Q}$, as an A-submodule, must also be finitely generated, which is impossible.

4.3. Finitely axiomatizable strongly minimal abelian structures. We suppose that $\mathbb{G}$ is a strongly minimal abelian structure in a finite language $L_c = L_0 \cup \{c \in C\}$ such that its theory $\mathbb{T}$ is finitely axiomatizable. Recall from corollary 2.4 that $\mathbb{T}$ is axiomatized by the set of sentences $T(\mathbb{G})$ together with the pp-type of the constants. Let $\mathbb{B}$ be a finite axiomatization of $\mathbb{T}$ which consists of a finite subset $\mathbb{A}$ of $T(\mathbb{G})$ together with a finite subset of the pp-type of the constants. Denote by $\mathbb{A}_1$ the following finite subset of $\mathbb{T}_1$ (as defined in Section 2.2):

- the axioms for abelian groups,
- for each original predicate H from L_0, H is a subgroup,
- the equivalence sentences in $\mathbb{A}$,
- $[\phi : \psi] \geq k$ for every k and for every pair of pp-formulas such that in $\mathbb{G}$, $\psi(G) \subset \phi(G) \subset G$, $[\phi(G) : \psi(G)]$ is infinite and the sentence $[\phi : \psi] \geq k$ is in $\mathbb{A}$,
- $[\phi : \psi] = 1$ for every pair of pp-formulas such that in $\mathbb{G}$, $\psi(G) \subset \phi(G) \subset G$ and for some integer k, the sentence $[\phi(G) : \psi(G)] = k$ is in $\mathbb{A}$,
- $\exists x\ x \neq 0$.

LEMMA 4.14. *For every model* $\mathbb{G}$ *of* $\mathbb{T}$ *and every model* $\mathbb{S}$ *of* $\mathbb{A}_1$, *the* L_c*-structure* $\mathbb{G}_S$ $(= \mathbb{G} \oplus \mathbb{S}$ *as in Section* 2.2) *is a model of* $\mathbb{B}$ *and so, of* $\mathbb{T}$. *Moreover,* $G \oplus \{0\}$ *is an elementary submodel of* $\mathbb{G}_S$.

PROOF. The proof that $\mathbb{G}_S$ is a model of $\mathbb{B}$ is exactly similar to the proof of 2.13. One needs to check the dimensions only for the pp-subgroups ϕ, ψ such that a dimension sentence of the form $[\phi : \psi] \geq n$ (if it is infinite) or $[\phi : \psi] = n$ (if it is finite) appears in $\mathbb{A}$. ⊣

We are going to show that $\mathbb{A}_1$ gives an axiomatization for the complete theory $\mathbb{T}_1$. It suffices to show that every model of $\mathbb{A}_1$ is infinite:

LEMMA 4.15. *If all models of* $\mathbb{A}_1$ *are infinite then* $\mathbb{A}_1$ *is a finite axiomatization of* $\mathbb{T}_1$.

PROOF. We show that any two models of $\mathbb{A}_1$ of cardinality $\aleph_1$ are isomorphic, then if $\mathbb{A}_1$ has no finite models, it is complete and hence axiomatizes $\mathbb{T}_1$. Let $\mathbb{G}$ be a countable model of $\mathbb{T}$. (One can choose acl$(\emptyset)$ if it is infinite.) Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two models of $\mathbb{A}_1$ of cardinality $\aleph_1$. Then by strong minimality, as $G \oplus \{0\}$ is algebraically closed in $\mathbb{G}_{S_i}$, there is an isomorphism between $\mathbb{G}_{S_1}$ and $\mathbb{G}_{S_2}$ which is the identity on $G \oplus \{0\}$. From this isomorphism one induces easily an isomorphism between $\mathbb{S}_1$ and $\mathbb{S}_2$. ⊣

PROPOSITION 4.16. *The ring of quasi-endomorphisms of* $\mathbb{G}$ *is infinite.*

PROOF. Suppose not. Let K denote the ring of quasi-endomorphisms, then $K = \mathbb{F}_q$ and G has bounded exponent. In particular every finitely generated subgroup of G is finite. We are going to construct a finitely generated subgroup of G which is a model of $\mathbb{T}$, contradicting the completeness of $\mathbb{T}$.

First, we add the quasi-endomorphisms as predicates to the language: for each $\alpha \in \mathbb{F}_q$, denote by H_α the corresponding quasi-endomorphism, which is a strongly minimal subgroup of G^2, definable over $\emptyset$ by Lemma 2.6, such that its first projection is equal to G. We add to L_c a predicate $\hat{H}_\alpha$ for each $\alpha \in \mathbb{F}_q$. This preserves the finite axiomatizability of $\mathbb{G}$. So we can assume that the language L_c already contains the quasi-endomorphisms as predicates. Now, we also, if necessary, add finitely many new predicates for some pp-definable subgroups which appear in the axiomatization $\mathbb{B}$, so that $\mathbb{T}$ has a finite axiomatization containing only sentences of the following type where X, Y and Z are amongst the predicates $\hat{H}_i$ of L_c:

1. G is a group;
2. the $\hat{H}_i$'s are subgroups;
3. the projection of X on the first $k-1$ coordinates is equal to Y (where X is k-ary);
4. the cartesian product of X and Y is equal to Z;
5. the intersection of X and Y is equal to Z;
6. X is equal to the group Y up to a fixed permutation of coordinates;
7. the index of X in Y is equal to k;
8. the index of X in Y is greater or equal to k;
9. the tuple c is in X;

Remark that the sentences of types 3, 4, 5 and 6 correspond to the equivalence sentences which occur in $\mathbb{A}$.

Note that every subgroup G_0 of G which contains all the constants, satisfies the axioms of types 2, 4, 5, 6, and 9. For each sentence $\Psi_{X,Y,k}$ of types 7 or 8, a subgroup

G_0 satisfies $\Psi_{X,Y,k}$ if and only if it contains at least k elements of Y which are in different cosets modulo X. Thus there exist finitely generated subgroups of G which satisfy the finite set of axioms of types 1, 2, 4, 5, 6, 7, 8 and 9.

To deal with axioms of type 3, we need to find finitely generated subgroups which are also "closed under projection" in the adequate sense. This is done in the two following claims. We say that a subset X of G is *stable under quasi-endomorphisms* if for each $x \in X$ and each $\alpha \in \mathbb{F}_q$, the set $\{y \in G : (x, y) \in H_\alpha\}$ is a subset of X.

Claim. Let X be a definable subgroup of G^k. Then there exists a finite subset D_X of G such that, if G_0 is any subgroup stable under quasi-endomorphisms which contains D_X, if π denotes the projection from G^k onto the first $k-1$ coordinates, then $\pi(X) \cap G_0^{k-1} = \pi(X \cap G_0^k)$.

Proof. Let l be the dimension (algebraic dimension = Morley rank) of X and $(a_1, \dots, a_k)$ a generic point of X, that is a point of dimension l. Then, there are two cases.

Either, a_k is independent of $a_1, \dots, a_{k-1}$. It follows easily in this case that $X = Y \times G$ where $Y = \{(x_1, \dots, x_{k-1}) : (x_1, \dots, x_{k-1}, 0) \in X\}$ and then for every subgroup G_0, $\pi(X) \cap G_0^{k-1} = \pi(X \cap G_0^k)$.

Otherwise, by a permutation of coordinates we can assume that $a_{l+1}, ..., a_k$ are algebraic over $a_1, \dots, a_l$. (Note that then every generic of X satisfies this property.) For each j, $l < j \leq k$, $a_j \in acl(a_1, \dots, a_l)$; so (see Fact 3.1), there exist $\alpha_{j,1}, \dots, \alpha_{j,l} \in \mathbb{F}_q$ and $b_{j,1}, \dots, b_{j,l} \in G$ such that $a'_j = a_j - \sum_{1 \leq i \leq l} b_{j,i} \in \text{acl}(\emptyset)$ and for every $i \leq l$, $(a_i, b_{j,i}) \in H_{\alpha_{j,i}}$.

Let T be the subgroup of G^k of elements $(x_1, \dots, x_k)$ such that there exist $y_{l+1}, \dots, y_k$ with $(x_1, \dots, x_l, y_{l+1}, \dots, y_k) \in X$ and for each j, $l < j \leq k$, there exist $y_{j,1}, \dots, y_{j,l}$ with $x_j = y_j - \sum_i y_{j,i}$ and $i \leq l$, $(x_i, y_{j,i}) \in H_{\alpha_{j,i}}$. Then $(a_1, \dots, a_l, a'_{l+1}, \dots, a'_k) \in T$. Now let $T' := \{(x_{l+1}, \dots, x_k) : (0, \dots, 0, x_{l+1}, \dots, x_k) \in T\}$.

We claim that T' is finite and that $T = G^l \times T'$. Since X is of dimension l, the group $X' := \{(x_{l+1}, \dots, x_k) : (0, \dots, 0, x_{l+1}, \dots, x_k) \in X\}$ is finite. It follows that T' is finite because the cokernels of the quasi-endomorphisms are finite. Let $(x_1, \dots, x_l)$ be a generic of G^l over $(a_1, \dots, a_l)$. By strong minimality, $(x_1, \dots, x_l)$ and $(a_1, \dots, a_l)$ have the same type over $\text{acl}(\emptyset)$. Since $(a'_{l+1}, \dots, a'_k) \in \text{acl}(\emptyset)$, we have $(x_1, \dots, x_l, a'_{l+1}, \dots, a'_k) \in T$ and so $(x_1 - a_1, \dots, x_l - a_l, 0, \dots, 0) \in T$. But $(x_1 - a_1, \dots, x_l - a_l)$ is generic, so $G^l \times \{0\}^{k-l} \subset T$ and thus $T = G^l \times T'$.

Now, let G_0 be any subgroup of G stable under quasi-endomorphisms such that G_0^{k-l} contains T'. Then $\pi(X) \cap G_0^{k-1} = \pi(X \cap G_0^k)$: indeed let $(x_1, \dots, x_l, y_{l+1}, \dots, y_k) \in X$ be such that $x_1, \dots, x_l \in G_0$, we are going to show that $y_k \in G_0$. For each j, $l < j \leq k$, take $y_{j,1}, \dots, y_{j,l}$ such that for each $i \leq l$, $(x_i, y_{j,i}) \in H_{\alpha_{j,i}}$. Then, by stability of G_0 under quasi-endomorphisms, $y_{j,i} \in G_0$ for each j, $l < j \leq k$, and each $i \leq l$. For each j, $l < j \leq k$, let $z_j = y_{j,1} + \dots + y_{j,l}$ and $x_j = y_j - z_j$. Then $(x_1, \dots, x_l, x_{l+1}, \dots, x_k) \in T$ and as $T = G^l \times T'$, $(x_{l+1}, \dots, x_k) \in T' \subset G_0^{k-l}$. So, in particular, $x_k \in G_0$ and thus $y_k = x_k + z_k \in G_0$. $\dashv_{\text{Claim}}$

Claim. For every finite subset $A \subset G$, there is a finite subgroup G_0 of G, containing A, which is stable under quasi-endomorphisms.

Proof. Let A be a finite subset of G. For a subset X of G denote by $\overline{X}$ the set $\cup_{(x,\alpha)\in X\times\mathbb{F}_q}\{y \in G : (x, y) \in H_\alpha\}$. Note that $\overline{X}$ is not necessarily stable under quasi-endomorphisms (i.e., $\overline{\overline{X}}$ is not necessarily equal to $\overline{X}$).

For each $(\alpha, \beta) \in {\mathbb{F}_q}^2$, let $H_\alpha \circ H_\beta$ denote the subgroup of G^2 defined by the formula

$$\exists z\ ((x, z) \in H_\beta \wedge (z, y) \in H_\alpha).$$

The quasi-endomorphism $H_{\alpha\beta}$ is equal to the connected component of $H_\alpha \circ H_\beta$. Let $X_0 := \{y \in G : (0, y) \in H_\alpha \circ H_\beta \text{ for some } (\alpha, \beta) \in {\mathbb{F}_q}^2\}$ and let A_0 be the finite subgroup generated by A and $\overline{X_0}$. Let B be the set $\overline{A_0}$. We prove that B is stable under quasi-endomorphisms: let $\alpha \in \mathbb{F}_q$, $x \in B$ and $y \in G$ be such that $(x, y) \in H_\alpha$. By definition of B, there exists $\beta \in \mathbb{F}_q$ and $z \in A_0$ such that $(z, x) \in H_\beta$. So $(z, y) \in H_\alpha \circ H_\beta$. Let $y' \in G$ be such that $(z, y') \in H_{\alpha\beta}$. Then $y - y' \in X_0$ since $(0, y - y') \in H_\alpha \circ H_\beta$. Remark that if $\alpha = 0$ then $y = 0$ and if $\beta = 0$ then $y \in X_0$. So assume that $\alpha\beta \neq 0$. Let $t \in G$ be such that $(y - y', t) \in H_{(\alpha\beta)^{-1}}$. Then $t \in A_0$ since $y - y' \in X_0$ and $\overline{X_0} \subseteq A_0$. Thus $y \in B$ since $(z + t, y' + (y - y')) \in H_{\alpha\beta}$ and $z + t \in A_0$. Now consider G_0 the subgroup generated by B. Then G_0 is also stable under quasi-endomorphisms since for each $x_1, x_2 \in G$ and each $\alpha \in \mathbb{F}_q$,

$$\{y \in G : (x_1 + x_2, y) \in H_\alpha\} = \{y_1 \in G : (x_1, y_1) \in H_\alpha\} + \{y_2 \in G : (x_2, y_2) \in H_\alpha\}.$$

$\dashv_{\text{Claim}}$

Now by the previous claims we can find a finite subgroup of G, which contains sufficiently many elements in different cosets for the axioms of type 7 or 8 to be satisfied, which is stable under quasi-endomorphisms and contains each D_{H_i}. Such a finite group is a model of $\mathbb{T}$. $\dashv$

Corollary 4.17. *$\mathbb{A}_1$ is a finite axiomatization of the complete theory $\mathbb{T}_1$.*

Proof. By lemma 4.15, it suffices to show that every model of $\mathbb{A}_1$ is infinite. Let $\mathbb{S}$ be a model of $\mathbb{A}_1$. We work in the structure $\mathbb{G}_S$ which is a model of $\mathbb{T}$ by 4.14. For $r \in K$, let ϕ_r denote the corresponding pp-formula (over $\emptyset$). In $\mathbb{G}_S$, the kernel and cokernel of ϕ_r are finite, hence, by Lemma 4.14, they must be contained in $G \oplus \{0\}$ This means that ϕ_r restricted to $\{0\} \oplus S$ is a well-defined map. Let $s \in S \setminus \{0\}$. For each $r \in K$, consider the unique $(x_r, y_r) \in G_S$ such that $((0, s), (x_r, y_r)) \in \phi_r$. Then, if $r \neq r'$, $y_r \neq y_{r'}$: indeed, if for $r \neq r'$, $y_r = y_{r'}$ then $((0, s), (x_r - x_{r'}, 0)) \in (\phi_r - \phi_{r'})$, $((x_r - x_{r'}, 0), (0, s)) \in (\phi_r - \phi_{r'})^{-1}$ and hence $s = 0$. $\dashv$

By Proposition 3.4, $\mathbb{T}_1$ is interdefinable with the theory of non trivial K-vector spaces, where K is the ring of quasi-endomorphisms of $\mathbb{G}$. By Corollary 4.17 and Lemma 4.2, the theory of K-vector spaces is finitely axiomatizable. By Proposition 4.13 we derive immediately:

Corollary 4.18. *The division ring K of quasi-endomorphisms of $\mathbb{G}$ is finitely presented as a ring.*

4.4. Finitely axiomatizable strongly minimal groups.

Theorem 4.19. *Let $\mathcal{G}$ be a strongly minimal group. If $Th(\mathcal{G})$ is finitely axiomatizable in a finite language L, then the ring of quasi-endomorphisms of $\mathcal{G}$ is an infinite division ring which is finitely presented as a ring.*

PROOF. By [10], a finitely axiomatizable strongly minimal group must be locally modular, hence one-based.

CLAIM. If $\mathscr{G}'$ is an expansion of $\mathscr{G}$ by a set C of constants, then $\mathscr{G}$ and $\mathscr{G}'$ have the same ring of quasi-endomorphisms.

PROOF. This uses only the fact that $\mathscr{G}$ is stable one-based. For $A \subset G$, let S be any connected A-definable subgroup of $G \times G$ in $\mathscr{G}'$. Then S is also definable in $\mathscr{G}$, over $A \cup C$. As $\mathscr{G}$ is one-based, by Fact 4.5, S is definable over $acl^{eq}(\emptyset)$ in $\mathscr{G}$. Conversely, any definable connected subgroup H in $\mathscr{G}$ remains connected in $\mathscr{G}'$. $\dashv_{\text{CLAIM}}$

Recall the construction from Corollary 4.7: we add finitely many constants from $\mathscr{M}_0$, the prime model of $Th(\mathscr{G})$. Let $\mathscr{G}'$ denote the expansion of $\mathscr{G}$ to the new language $L' = L \cup \{m_0, \dots, m_n\}$. Then $\mathscr{G}'$ is interdefinable with some finitely axiomatizable abelian structure $\mathbb{G} =< G, +, -, 0, H_1, \dots, H_k, m_0, \dots, m_n >$. It follows that $\mathbb{G}$ and $\mathscr{G}$ have the same quasi-endomorphisms ring, as a quasi-endomorphism is a definable connected subgroup of $G \times G$. (Note that, as in $\mathbb{G}$ every definable connected subgroup is defined over $\emptyset$ (Proposition 2.6), the same is true in $\mathscr{G}'$. Hence in $\mathscr{G}$ every definable connected subgroup was already definable over $\{m_0, \dots, m_n\}$.)

By Corollary 4.18, the division ring of quasi-endomorphisms of $\mathbb{G}$ is finitely presented as a ring. As remarked above, this is also the division ring of quasi-endomorphisms of $\mathscr{G}$. $\dashv$

By Proposition 4.12, we derive immediately:

COROLLARY 4.20. *Let $\mathscr{G}$ be a Morley Rank one group. If $\mathscr{G}$ is finitely axiomatizable then the quasi-endomorphism ring of its connected component is an infinite division ring which is finitely presented as a ring*

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