Complexity of the conjugacy problem in amalgamated products of free groups

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Applications

Let G be a group, $g \in G$.

The Conjugator Search Problem for g:

CSP: given g^y, find x: $\quad g^x = g^y$

Let

$$G = A *_C B$$

is the free product of free groups $A = F(X)$, $B = F(Y)$ of finite ranks amalgamated over a subgroup C of finite rank.

Chuck Miller III: examples with undecidable conjugacy problem

Let A, B, C be groups given by presentations

$$A = \langle X \mid R_A \rangle, \quad B = \langle Y \mid R_B \rangle, \quad C = \langle Z \mid R_C \rangle$$

and

$$\phi : C \longrightarrow A \text{ and } \psi : C \longrightarrow B$$

be monomorphisms.

$$G = A *_C B$$

has presentation

$$G = \langle X \cup Y \mid R_A \cup R_B \cup \{z^\phi = z^\psi \mid z \in Z\} \rangle.$$

If the groups A and B are finitely presented and C is finitely generated then the group G is finitely presented.

Denote

$$z^{\phi} = u_z(x), \quad z^{\psi} = v_z(y)$$

then G has presentation

$$G = \langle X \cup Y \mid R_A = 1, R_B = 1, u_z(x) = v_z(y),\ z \in Z \rangle.$$

The *factors* A and B can be identified with the subgroups in G generated by X and Y.

$$C = A \cap B = \langle u_z \rangle_{z \in Z} = \langle v_z \rangle_{z \in Z} \leqslant G.$$

Canonical forms of elements

For an element $g \in (A \cup B) \setminus C$ we define $F(g) = A$ if $g \in A$ and $F(g) = B$ if $g \in B$.

S and T fixed systems of right coset representatives of C in A and B.

An arbitrary g in $G = A *_C B$ can be uniquely written

$$g = cg_1g_2 \cdots g_n,$$

where $c \in C$, $g_i \in T \cup S \setminus \{1\}$, and $F(g_i) \neq F(g_{i+1})$.

$l(g) = n$ is a length function $l : G \longrightarrow \mathbb{N}$.

A reduced form of g:

$$g = cg_1g_2\cdots g_n$$

is called *cyclically reduced* if one of the following conditions is satisfied:

(a) $l(g) = 0$;

(b) $l(g) = 1$ and g is not conjugate to an element in C;

(c) $l(g \geqslant 2$ and $l(g^2) = 2l(g)$.

Every g is conjugate to a cyclically reduced element g_0.

g_0 is not uniquely defined, but its length $l(g_0)$ is uniquely determined by g.

Reduced length function $l_0 : G \to \mathbb{N}, \quad l_0(g) = l(g_0)$.

Notice that $l_0(g) = 0$ amounts to saying

> "the element g is conjugate to some element in C",

while $l_0(g) = 1$ amounts to saying

> " the element g is conjugate to some element in A or B but not conjugate to an element in C."

Conjugacy criterion

(iii) Let $l_0(g) = r \geqslant 2$ and $g = g_1 \cdots g_r$ be a cyclically reduced form of g. Assume that g is conjugate to a cyclically reduced element $h = h_1 \cdots h_s$ in G.

Then $r = s$ and h can be obtained from g by a cyclic permutation of the elements $g_1, \ldots, g_r$ followed by a conjugation by an element from C.

Generalised normaliser

- If C is a finitely generated subgroup of a free group A then the generalised normaliser

$$N_A^*(C) = \{\, a \in A \mid C \cap C^a \neq 1 \,\}$$

  is a regular set and an union of finitely many cosets of C.

- If $|A : C| = \infty$ then $N_A^*(C)$ is sparse.

Regular and singular elements

$$G = A *_C B$$

- $g \in G$ is **singular** if it is conjugate to an element in

$$N_G^*(C)$$

- **regular** otherwise

Theorem: for regular elements $g \in G$, **CSP** is decidable.

“Black hole” (in Miller’s groups)

$$B = \{g \in C \mid \quad \textbf{GSP} \quad \text{is undecidable}\}$$

is sparse in C.

Random elements in $A *_C B$:

- Alternating words in $F(X \cup Y)$

  $$w = a_1 b_2 \cdots a_{n-1} a_n$$

  where syllables a_i, b_{i+1} are independent and taken from $F(X)$, $F(Y)$.

- The number n of syllables is distributed exponentially with the mean N.

- The lengths of sylables a_i, b_j are distributed exponentially with the mean L.

There are concepts of "thick", "sparse" sets, etc, similar to these for $F(X)$.

- There is a subset R in $F(X \cup Y)$ with "sparse" complement which is mapped to regular elements in $A *_C B$.
- **A very crude estimate:**

  There is a subset Q in $F(X \cup Y)$ with "sparse" complement which is mapped to a set of elements in $A *_C B$ such that **CSP** is solvable in quadratic time on average (in terms of input lengths $l_{X \cup Y}(g)$, $l_{X \cup Y}(g^y)$).