A measure for perfect PAC fields with pro-cyclic Galois group

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Abstract

In "Pseudo-finite fields and related structures" Hrushovski asks (in a more general context) whether the notion of measure on definable sets in pseudo-finite fields can be extended to perfect PAC fields whose Galois group is bounded but not $\hat{\mathbb{Z}}$. We define a suitable generalization of measure and give an answer to this question: yes if the Galois group is pro-cyclic, no otherwise.

Key words: pseudo-algebraically closed fields, measure, definable sets, bounded Galois group
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1 Introduction

To understand definable sets up to definable bijections, it is helpful to have invariants of these sets. In the case of pseudo-finite fields, one well-known such invariant is the measure defined in [1]. By the "measure" of a definable set X, one should think of the size of the part of highest dimension of X.

In [2], Hrushovski asks (in a more general context) whether this measure can be generalized to cases where the Galois group is not $\hat{\mathbb{Z}}$. More precisely, of the three conditions on pseudo-finite fields—pseudo algebraically closed (PAC), perfect, and the Galois group is $\hat{\mathbb{Z}}$—we only keep the first two, and the third one is weakened to the requirement that the Galois group is bounded, i.e. it has only a finite number of quotients of each cardinality. In such fields, one still has almost-quantifier-elimination as in pseudo-finite fields, so one can hope to control the definable sets. The question is whether one can find an appropriate definition of a "measure" on the definable sets of such fields.

Of course, such a measure can not have all the properties of the measure on pseudo-finite fields. (This was proven in [4].) In particular, one can not hope to have a "Fubini theorem" as in the case of pseudo-finite fields. Fubini states, more or less, that for a surjective map $X \twoheadrightarrow Y$ with constant fiber size, the measure of X is equal to the measure of Y times the measure of a fiber. This is already false, for example, for the map $K^\times \to K^\times, x \mapsto x^2$ if K is algebraically closed.

In this article, instead of requiring the measure to satisfy Fubini, we will just require that it is invariant under definable bijections. The main theorem (Theorem 16) states that if the Galois group is pro-cyclic, then such a measure indeed exists, and is—under one additional hypothesis—even unique. In the case of pseudo-finite fields our measure is just the usual measure of [1]. In the case of algebraically closed fields, it counts the irreducible components of the sets. Our definition of measure can be seen as an attempt to find a common generalization of these two extreme cases.

Note that uniqueness of the measure is a new result even in the case of pseudo-finite fields: up to now, it was known only if one requires the measure to satisfy Fubini.

The main theorem only gives a measure if the Galois group is pro-cyclic. Indeed, if the Galois group is not pro-cyclic but still bounded, then (in general) no measure exists. In Subsection 7.1, we will give a counter-example which works for a lot of different non-pro-cyclic Galois groups.

So for fields, the question of Hrushovski is answered: a measure does exist if and only if the Galois group is cyclic—at least for measures in our sense. See Subsection 7.2 for further weakenings of the notion of measure.

1.1 Overview over the article

In Section 2 we will define all our notation and state some basic facts. In Section 3 we state the main theorem and give an idea of the proof. The proof itself is done in Section 6, after proving some lemmas in the previous section. In particular, we prove a variant of Chebotarevs density theorem on bounded Galois groups in Subsection 4.3. In the last section (Section 7) we give the counter-example when the Galois group is not pro-cyclic and we mention some open problems.

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2 Preliminaries

2.1 Setting and notation

In the whole article, K will be a field, fixed once and for all. Most of the time, K will be perfect and PAC and with bounded or even pro-cyclic absolute Galois group. The only parts where K will be an arbitrary field are some of the following subsections (where we mainly define things) and section 4 where we prove some lemmas which are true in more generality.

We will denote the algebraic closure of K by $\tilde{K}$ and its absolute Galois group by $\mathrm{Gal}_K := \mathrm{Gal}(\tilde{K}/K)$.

We will try to avoid the language of schemes as much as possible. The reader can think of an algebraic set as a set defined by polynomials. Morphisms between algebraic sets will always be considered to be morphisms between the $\tilde{K}$-valued points of the sets. In particular, if we have a morphism $f\colon V \to W$ and a point $w \in W(K)$ over K, then $f^{-1}(w)$ denotes the set $\{v \in V(\tilde{K}) \mid f(v) = w\}$ of all points over $\tilde{K}$ in the fiber, not only the ones over K. (In the language of schemes, this means: we always work with geometric points and we write $W(K)$ for those geometric points which factor through K.)

All our definable sets will be definable in the language of rings and with parameters in K. Also, all our varieties, algebraic sets, and morphisms between such will be defined over K, with one exception: absolutely irreducible components of varieties will (of course) only be defined over $\tilde{K}$.

Sometimes, we will have to consider homomorphisms between groups, in particular from Gal_K to some finite group G. We will always want these homomorphisms to be continuous. Nevertheless, we will only write $\mathrm{Hom}(\mathrm{Gal}_K, G)$

to keep the notation light. One may think of it as working in the category of topological groups, where finite groups have the discrete topology.

We will need a notion of dimension. One possible definition of the dimension of a definable set $X \subset K^n$ is to take for $\dim X$ the (usual algebraic) dimension of the Zariski closure of X in $\tilde{K}^n$ (see [5]). As K is supersimple of rank 1, another equivalent definition is to take the SU-Rank of X.

As customary (see e.g. [6]), we will describe definable sets in terms of "Galois stratifications". In the remainder of this section, we repeat the necessary definitions and basic facts. However, we have to work in a slightly more general context than usual: our Galois group need not be $\hat{\mathbb{Z}}$, so we will need a generalized version of the Artin symbol.

2.2 Galois covers

In this and the two following subsections, K may be an arbitrary field. Everything is defined over K.

We use the following definition of Galois cover (note the slightly unusual requirement that W is absolutely irreducible).

Definition 1 *A* Galois cover *is a morphism $f\colon V \twoheadrightarrow W$ such that V is irreducible, W is absolutely irreducible, f is finite and étale and* $\mathrm{Aut}_W(V)$ *acts regularly on the fibers of f (i.e. for any two elements v_1, v_2 in one fiber, there exists exactly one morphism in* $\mathrm{Aut}_W(V)$ *which maps v_1 to v_2).*

The group $G := \mathrm{Aut}_W(V)^{\mathrm{opp}}$, which acts from the right on V, is called the group *of the Galois cover. (We will avoid to call G "Galois group" to avoid confusion with the Galois group of K.) We will write $f\colon V \overset{G}{\twoheadrightarrow} W$ to say that f is a Galois cover with group G.*

"Étale" intuitively means that locally, V looks like several copies of W. For an exact definition, see e.g. [3].

As right actions are a bit difficult to read, we will always write them with a lower dot, i.e. for example "$v.g$" if g acts on v.

If $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover, then we have canonically $W \cong V/G$.

If follows from the irreducibility of V that $\mathrm{Aut}_W(V)$ acts freely. If V is not irreducible (and $\mathrm{Aut}_W(V)$ maybe does not act freely), then one can get a Galois cover by replacing V by one of its irreducible components.

Definition 2 *For two Galois covers $f\colon V \overset{G}{\twoheadrightarrow} W$ and $f'\colon V' \overset{G'}{\twoheadrightarrow} W$ of the same set W, we say that $V' \overset{G'}{\twoheadrightarrow} W$ is a* refinement *of $V \overset{G}{\twoheadrightarrow} W$ if there is a finite étale map $g\colon V' \twoheadrightarrow V$ such that $f' = f \circ g$.*

If $V' \overset{G'}{\twoheadrightarrow} W$ is a refinement of $V \overset{G}{\twoheadrightarrow} W$, then there is a natural surjective map $G' \twoheadrightarrow G$.

Definition 3 *If $f\colon V \overset{G}{\twoheadrightarrow} W$ is a Galois cover, $W' \subset W$ an absolutely irreducible algebraic subset, and $V' \subset V$ an irreducible component of $f^{-1}(W')$, then $f\restriction_{V'}\colon V' \twoheadrightarrow W'$ is again a Galois cover. We call it a* restriction *of f to W'.*

Remark 4 *Denote the group of the restricted cover $V' \twoheadrightarrow W'$ by G'. If W' is dense in W, then $f^{-1}(W')$ is irreducible and $G' = G$. In general, we only have $G' \hookrightarrow \operatorname{Aut}_{W'}(f^{-1}(W')) \hookleftarrow G$.*

Here are some more well-known facts about Galois covers which we will be using without further mentioning:

Fact 5 *(1) Suppose $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover and $H \subset G$ is a subgroup. If H acts transitively on the components of V, then $V \overset{H}{\twoheadrightarrow} V/H$ is also a Galois cover. If H is normal, then $V/H \overset{G/H}{\twoheadrightarrow} W$ is also a Galois cover.*

(2) If $f \circ g$ is a Galois cover and both f and g are finite and étale, then g is a Galois cover, too.

(3) Given two Galois covers $V_1 \overset{G_1}{\twoheadrightarrow} W$ and $V_2 \overset{G_2}{\twoheadrightarrow} W$ of the same set, there exists a common refinement of both.

(4) If W is absolutely irreducible and $f\colon V \to W$ is a finite étale map, then there exists an algebraic set $\tilde{V}$ and a finite étale map $g\colon \tilde{V} \to V$ such that $f \circ g$ is a Galois cover. (In particular, g is also a Galois cover.) If f is only finite, dominant, and generically étale, then this is still true after replacing W by some suitable dense open subset W' of W and V by $f^{-1}(W')$.

We will need one more technical lemma to construct generically étale maps. It describes how a map can be split into a generically étale and a "purely inseparable" part. For V affine and irreducible, we will write $K[V]$ for the ring of regular functions on V and $K(V)$ for the field of rational functions.

Lemma 6 *Suppose V and W are irreducible and $f\colon V \to W$ is finite and surjective. Then after replacing W by a dense open subset and V by the preimage $f^{-1}(W)$, we can find*

$$V \xrightarrow{g} V' \xrightarrow{h} W$$

where V' is irreducible, $f = h \circ g$, both g and h are finite and surjective, h is generically étale, and the field extension $K(V') \hookrightarrow K(V)$ induced by g

is purely inseparable. In addition, if K is perfect then g induces a bijection $V(K) \stackrel{1:1}{\to} V'(K)$ on the K-valued points.

Proof. If the characteristic of K is zero, then set $V' := V$ and there is nothing to prove.

By shrinking V and W, we can suppose that they are both affine and normal. Let $R := K[W]$, $S := K[V]$, $E := K(W)$, and $F := K(V)$ be the corresponding rings of regular functions resp. fields of rational functions. Define F' to be the separable closure of E in F, S' to be the integral closure of R in F', and $V' := \operatorname{spec} S'$ to be the corresponding variety. The inclusions $R \hookrightarrow S' \hookrightarrow S$ induce maps $g\colon V \to V'$ and $h\colon V' \to W$ which have all the desired properties.

For the bijectivity of g on the K-valued points, we have to check that each ring homomorphism $S' \to K$ can be extended in a unique way to a homomorphism $S \to K$. As K is perfect, this follows from the fact that for each element $s \in S$, there is some k such that $s^{p^k} \in S'$, where p is the characteristic of K. $\square$

2.3 The Artin symbol

Given a Galois cover $V \stackrel{G}{\twoheadrightarrow} W$ and an element $w \in W(K)$, the usual Artin symbol of w is a conjugacy class of elements of G, which one gets as images of a generator of Gal_K under certain maps from Gal_K to G. When Gal_K is not $\hat{\mathbb{Z}}$, it is better to consider directly conjugacy classes of maps from Gal_K to G. By "conjugacy class in $\mathrm{Hom}(\mathrm{Gal}_K, G)$", we will always mean a class under conjugation by elements of G (i.e. ρ and ρ' are conjugate if $\rho' = \rho^g := \mathrm{Int}(g) \circ \rho$ for some $g \in G$).

Note that if the absolute Galois group of K is bounded, then $\mathrm{Hom}(\mathrm{Gal}_K, G)$ is finite: There are only finitely many quotients of Gal_K whose cardinality is less or equal to the cardinality of G and each homomorphism $\rho \in \mathrm{Hom}(\mathrm{Gal}_K, G)$ factors through such a quotient. (And of course, there are only finitely many homomorphisms from that quotient to G.)

Another remark: When G_1 is a subgroup of G_2, we will often identify $\mathrm{Hom}(\mathrm{Gal}_K, G_1)$ with the corresponding subset of $\mathrm{Hom}(\mathrm{Gal}_K, G_2)$.

Definition 7 *Suppose $f\colon V \stackrel{G}{\twoheadrightarrow} W$ is a Galois cover and $w \in W(K)$. Then we have a (left) action of Gal_K on the fiber $f^{-1}(w) \subset V(\tilde{K})$. Suppose $v \in f^{-1}(w)$ lies in that fiber. By the regularity of the action of G on $f^{-1}(w)$, there is a unique map $\rho\colon \mathrm{Gal}_K \to G$ such that $\sigma v = v.\rho(\sigma)$ for all $\sigma \in \mathrm{Gal}_K$. We call this map the* Frobenius symbol *of v and denote it by* $\mathrm{Fr}(v)$.

For $w \in W(K)$*, we call the set* $\{\mathrm{Fr}(v) \mid v \in f^{-1}(w)\}$ *the* Artin symbol *of* w *and denote it by* $\mathrm{Ar}(w)$*.*

Remark 8 *It is easy to check that* $\mathrm{Fr}(v)$ *is a continuous group homomorphism from* Gal_K *to* G *and that* $\mathrm{Ar}(w)$ *is exactly one conjugacy class in* $\mathrm{Hom}(\mathrm{Gal}_K, G)$ *(under conjugation by* G*).*

Definition 9 *Given a Galois cover* $f\colon V \overset{G}{\twoheadrightarrow} W$ *and a union* C *of conjugacy classes in* $\mathrm{Hom}(\mathrm{Gal}_K, G)$*, we define the subset of* $W(K)$ *consisting of those elements whose Artin symbol lies in* C*:*

$$X(V \overset{G}{\twoheadrightarrow} W, C) := \{w \in W(K) \mid \mathrm{Ar}(w) \subset C\} \ .$$

Remark 10 *Suppose that* $f\colon V \overset{G}{\twoheadrightarrow} W$ *and* $f'\colon V' \overset{G'}{\twoheadrightarrow} W'$ *are two Galois covers and that* $C \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$ *is closed under conjugation.*

- *If* f' *is a refinement of* f*, then* $X(V \overset{G}{\twoheadrightarrow} W, C) = X(V' \overset{G'}{\twoheadrightarrow} W', C')$*, where* $C' = \{\rho \in \mathrm{Hom}(\mathrm{Gal}_K, G') \mid \pi \circ \rho \in C\}$ *and* π *is the natural map* $G' \twoheadrightarrow G$*.*
- *If* f' *is a restriction of* f *to* W'*, then* $X(V \overset{G}{\twoheadrightarrow} W, C) \cap W' = X(V' \overset{G'}{\twoheadrightarrow} W', C')$*, where* $C' = C \cap G'$ *(here, both* G *and* G' *are identified with the corresponding subgroup of* $\mathrm{Aut}_{W'}(f^{-1}(W'))$*).*

2.4 Galois stratifications

When our field K is perfect and PAC and has bounded Galois group, then the main idea to get hold of definable sets is that given such a set $X \subset K^n$, we can cut K^n into locally closed subsets such that on each of these subsets, X has the form $X(V \overset{G}{\twoheadrightarrow} W, C)$. To make this precise, we define Galois stratifications. In the remainder of this subsection, K may still be arbitrary.

Definition 11 *A* Galois stratification $\mathcal{A}$ *of a variety* W *consists of:*

- *a partition of* W *into finitely many absolutely irreducible locally closed subsets* $W_i \subset W$ $(i \in I)$*,*
- *for each* $i \in I$*, a Galois cover* $f_i\colon V_i \overset{G_i}{\twoheadrightarrow} W_i$*,*
- *for each* $i \in I$*, a union* C_i *of conjugacy classes of* $\mathrm{Hom}(\mathrm{Gal}_K, G_i)$*.*

We shall say that such a Galois stratification defines *the following subset of* $W(K)$*:*

$$\bigcup_{i \in I} X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)$$

The data of a Galois stratification denoted by $\mathcal{A}$ will always be denoted by V_i, W_i, G_i, C_i, and analogously with primes for $\mathcal{A}'$, $\mathcal{A}''$, etc. This will not always be explicitely mentioned.

Definition 12 *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are two Galois stratifications. We say that $\mathcal{A}'$ is a* refinement *of $\mathcal{A}$, if:*

- *Each W_i is a union $\bigcup_{j\in J_i} W'_j$ for some $J_i \subset I$.*
- *For each $i \in I$ and each $j \in J_i$, the Galois cover $V'_j \stackrel{G'_j}{\twoheadrightarrow} W'_j$ is a refinement of a restriction of the Galois cover $V_i \stackrel{G_i}{\twoheadrightarrow} W_i$ to the set W'_j.*
- *C'_j is constructed out of C_i as described in Remark 10, such that $X(V'_j \stackrel{G'_j}{\twoheadrightarrow} W'_j, C'_j) = X(V_i \stackrel{G_i}{\twoheadrightarrow} W_i, C_i) \cap W'_j$.*

By the third condition, $\mathcal{A}$ and $\mathcal{A}'$ define the same set.

One important reason for Galois stratifications being handy to use is the following well-known lemma:

Lemma 13 *If $\mathcal{A}$ and $\mathcal{A}'$ are two Galois stratifications, then there exist refinements $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ of $\mathcal{A}$ resp. $\mathcal{A}'$ which differ only in the sets $\tilde{C}_i$ (resp. $\tilde{C}'_i$).*

2.5 Definable sets and Galois stratifications

Now suppose K is perfect and PAC and its Galois group Gal_K is bounded. The important lemma about definable sets is:

Lemma 14 *For each definable set $X \subset K^n$, there exists a Galois stratification $\mathcal{A}$ of $\mathbb{A}^n$ which defines X.*

This is well known if K is a pseudo-finite field. In our case, the same proof works. We give a short sketch of it.

The following lemma was proved in [2] in a very general context (Corollary 1.16).

Lemma 15 *Each definable subset X of K^n can be defined by a formula of the form $\exists \overline{y}\, \phi(\overline{x}, \overline{y})$, where ϕ is quantifier-free and such that for each $\overline{a} \in K^n$, there are only finitely many $\overline{b} \in K^m$ such that $\phi(\overline{a}, \overline{b})$ holds.*

Sketch of a proof of Lemma 14. One can easily eliminate negations in ϕ by replacing $p(\overline{x}, \overline{y}) \neq 0$ by $p(\overline{x}, \overline{y}) \cdot z = 1$. So we can suppose that ϕ defines an algebraic subset of $\mathbb{A}^{n+m}$, which we will denote by V.

As we are only interested in K-valued points, we can suppose that V is the Zariski closure of $V(K)$. (Otherwise replace V by that set.) In particular, irreducible components of V are already absolutely irreducible. It is enough to find Galois stratifications for the projections of those components: By Lemma 13, we can suppose that these stratifications differ only in the sets C_i. Therefore we can take the union of those C_i and get a stratification defining the whole X. So suppose now V is absolutely irreducible.

Let W be the Zariski closure of the image of V in $\mathbb{A}^n$ and write f for the map $V \to W$. We proceed by induction on the dimension of W. This means that we are allowed to replace W by a dense open subset W' and V by $V' := f^{-1}(W')$. (This yields a Galois stratification for $X \cap W'$. By induction, we get a Galois stratification for $X \cap (W \setminus W') = f(V \setminus V')$; putting both together, we get the desired result.)

As K is PAC, the points $W(K)$ are dense in $W(\tilde{K})$. On $W(K)$, f has finite fibers, so it follows that $\dim V = \dim W$. (By a theorem of Chevalley, W contains a dense open subset on which the dimension of the fibers is equal to $\dim V - \dim W$.) From this we get that there is a dense open subset of W on which f is finite. We replace W by that subset.

Next, we would like to suppose that f is generically étale. By Lemma 6, we can shrink V and W such that we get maps $V \twoheadrightarrow V' \twoheadrightarrow W$ where $V' \twoheadrightarrow W$ is generically étale and $V \to V'$ induces a bijection on $V(K) \to V'(K)$. We replace V by V'.

After replacing W again by a dense open subset, we get a Galois cover $\tilde{f} = f \circ g \colon \tilde{V} \overset{G}{\twoheadrightarrow} W$ where $g \colon \tilde{V} \overset{H}{\twoheadrightarrow} V$ is also a Galois cover, with group $H \subset G$. We have

$$\begin{aligned} X = f(V(K)) &= \tilde{f}(\{\tilde{v} \in \tilde{V}(\tilde{K}) \mid \sigma(g(\tilde{v})) = g(\tilde{v}) \ \forall \sigma \in \mathrm{Gal}_K\}) \\ &= \tilde{f}(\{\tilde{v} \in \tilde{V}(\tilde{K}) \mid \operatorname{im} \mathrm{Fr}(g) \subset H\}) = X(\tilde{V} \overset{G}{\twoheadrightarrow} W, C)\,, \end{aligned}$$

where $C := \mathrm{Hom}(\mathrm{Gal}_K, H)^G$. $\square$

Note that the converse of Lemma 14 is also true: Any set defined by a Galois stratification is definable in the usual sense. Indeed, in order to speak about the Artin symbol of an element $w \in W(K)$ with respect to a given Galois cover $V \overset{G}{\twoheadrightarrow} W$, it is enough to work in a finite extension of K (this uses the boundedness of the Galois group), and finite extensions of K are interpretable in K.

3 The main theorem

3.1 The statement

The main result of this article is the following:

Theorem 16 *Let K be a perfect PAC field, with pro-cyclic Galois group. Then there is exactly one function μ (a "measure") from the definable sets over K to $\mathbb{Q}$ satisfying:*

(1) $\mu(K) = 1$.
(2) If there is a definable bijection between X_1 and X_2, then $\mu(X_1) = \mu(X_2)$.
(3) Suppose X_1 and X_2 are disjoint definable sets. If $\dim X_1 = \dim X_2$, then $\mu(X_1 \cup X_2) = \mu(X_1) + \mu(X_2)$; if $\dim X_1 > \dim X_2$, then $\mu(X_1 \cup X_2) = \mu(X_1)$.
(4) Suppose $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover where V is absolutely irreducible and suppose $C \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$ is a conjugacy class. Then the measure of $X(V \overset{G}{\twoheadrightarrow} W, C)$ only depends on C and on G as an abstract group (and not on V, W, the map $V \to W$ and the action of G on V).

Moreover, this measure satisfies:

(5) $\mu(X) \geq 0$ *for any X, and* $\mu(X) = 0$ *if and only if X is empty.*
(6) $\mu(X_1 \times X_2) = \mu(X_1) \cdot \mu(X_2)$.

Remark 17 *Condition (4) is the "additional hypothesis" needed to get uniqueness mentioned in the introduction. It can be seen in the following way:*

To get uniqueness, one has to fix the measure of absolutely irreducible varieties. This is done by Condition (4) applied to $X(\mathrm{id}\colon W \overset{1}{\twoheadrightarrow} W, 1)$ for any absolutely irreducible W (together with Condition (1)). In this sense, Condition (4) is a generalization of fixing the measure of absolutely irreducible varieties.

We give now the main steps of the proof. Details will be done later.

3.2 Sketch of the proof of existence

Condition (4) suggests to define a "measure" which associates a value $\mu_G(C)$ to each finite group G and each each conjugacy class $C \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$, and then to define $\mu(V \overset{G}{\twoheadrightarrow} W, C) := \mu_G(C)$. Indeed, to prove the existence, we will state a theorem similar to Theorem 16 about such a measure $\mu_G(C)$ on the groups (Theorem 22). In this manner, the whole proof is divided into one part in which the real work of finding a measure is done, but which is only group theoretical (Section 5), and one part which consists in transfering the result to definable sets (Subsection 6.1).

There is one technical complication to this: Condition (4) only treats the case when V is absolutely irreducible. If V is not absolutely irreducible, then one has to take into account the action of the Galois group on the absolutely irreducible components of V. This makes the measure on the groups somewhat uglier.

The definition of the measure we will give can be interpreted as follows: In the case of pseudo-finite fields, a measure exists. If the Galois group of K is a subgroup of $\hat{\mathbb{Z}}$, then in a certain sense the lanugage is a "simplification" of the language of pseudo-finite fields: when defining sets using Galois covers, then there are fewer possible Artin symbols. Using an embedding $\mathrm{Gal}_K \hookrightarrow \hat{\mathbb{Z}}$, we will therefore be able to "pull back" the well-known measure on pseudo-finite fields to K.

3.3 Sketch of the proof of uniqueness

One idea would be to use the same approach as for the existence: Prove the uniqueness of a measure on groups and then transfer this result to definable sets. However, there are a some technical reasons which would make this proof unnecessarily complicated. In particular we would need a version of Condition (4) which also treats non-irreducible V. To avoid this, we will prove the uniqueness directly. However, while doing this, we will still have the above idea in mind.

4 Some useful lemmas

Before we really start with the proof of the main theorem, we need some lemmas. In Subsections 4.1 and 4.2, K may be an arbitrary field; in subsection 4.3, we require K to be PAC and perfect, but we have no restriction on the Galois group of K.

Lemma 18 *Suppose $f\colon V \overset{G}{\twoheadrightarrow} W$ is a Galois cover and $w \in W(K)$ has Artin symbol $C := \mathrm{Ar}(w) \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$. Choose a $\rho \in C$. Then the number*

$$\#\left\{v \in V(\tilde{K}) \mid f(v) = w \wedge \mathrm{Fr}(v) = \rho\right\}$$

of elements in the fiber of w with Frobenius symbol ρ is $\frac{|G|}{|C|}$.

Proof. For $\rho \in C$, write $F(\rho) := \{v \in V(\tilde{K}) \mid f(v) = w \wedge \mathrm{Fr}(v) = \rho\}$ for the part of the fiber with ρ as Frobenius symbol. We will show that for any two $\rho, \rho' \in C$, $F(\rho)$ and $F(\rho')$ have the same cardinality. As the whole preimage of w consists of $|G|$ elements, the lemma follows.

Given $\rho, \rho' \in C$, there exists a $g \in G$ such that $\rho' = \rho^g$. Suppose $v \in F(\rho)$, i.e. $\sigma v = v.\rho(\sigma)$ for all $\sigma \in \mathrm{Gal}_K$. Then $\sigma v.g^{-1} = v.\rho(\sigma)g^{-1} = v.g^{-1}\rho(\sigma)^g = v.g^{-1}\rho'(\sigma)$, i.e. $v.g^{-1} \in F(\rho')$. So g^{-1} maps $F(\rho)$ to $F(\rho')$. By the same argument, g maps $F(\rho')$ to $F(\rho)$, so we have a bijection. $\square$

Lemma 19 *Suppose we have the following diagram, where the maps $f_1\colon V \to W_1$ and $f_2\colon V \to W_2$ are Galois covers with groups G_1 resp. G_2. Note that we have naturally $G_1 \subset G_2$ and $\mathrm{Hom}(\mathrm{Gal}_K, G_1) \subset \mathrm{Hom}(\mathrm{Gal}_K, G_2)$.*

$$\begin{tikzcd} V \arrow[d, two heads, "f_1"'] \arrow[dr, two heads, "f_2"] & \\ W_1 \arrow[r, two heads, "\phi"'] & W_2 \end{tikzcd}$$

Suppose additionally that $C_1 \subset \mathrm{Hom}(\mathrm{Gal}_K, G_1)$ is a conjugacy class and set $C_2 := C_1^{G_2} \subset \mathrm{Hom}(\mathrm{Gal}_K, G_2)$. Then the image under ϕ of $X_1 := X(V \overset{G_1}{\twoheadrightarrow} W_1, C_1)$ is $X_2 := X(V \overset{G_2}{\twoheadrightarrow} W_2, C_2)$. In addition, ϕ restricts to a bijection $X_1 \to X_2$ if and only if $\frac{|G_1|}{|C_1|} = \frac{|G_2|}{|C_2|}$.

Proof. "$\phi(X_1) \subset X_2$": Suppose $w_1 \in X_1$ and suppose $v \in f_1^{-1}(w_1)$ is a preimage. Then $\mathrm{Fr}(v) \in C_1$. But v is also a preimage of $\phi(w_1)$, so $\mathrm{Ar}(\phi(w_1))$ contains $\mathrm{Fr}(v)$ and is therefore equal to $C_1^{G_2}$.

"$\phi(X_1) \supset X_2$": Suppose $w_2 \in X_2$. As $\mathrm{Ar}(w_2)$ contains C_1, we may choose some preimage $v \in V$ of w_2 with $\mathrm{Fr}(v) \in C_1$. In particular, $\mathrm{im}\,\mathrm{Fr}(v) \subset G_1$, which means that Gal_K fixes $v.G_1 \in V/G_1 \cong W_1$. So the image $f_1(v)$ lies in $W_1(K)$. As $\mathrm{Ar}(f_1(v))$ contains $\mathrm{Fr}(v)$, we have $\mathrm{Ar}(f_1(v)) = C_1$, so $f_1(v)$ is a preimage of w_2.

It remains to check that the fibers of $\phi\restriction_{X_1}$ have size one if and only if $\frac{|G_1|}{|C_1|} = \frac{|G_2|}{|C_2|}$. Indeed, we prove that the size of the fibers is $\frac{|G_2|\cdot|C_1|}{|C_2|\cdot|G_1|}$: Fix any element $\rho \in C_1$. For any $w_2 \in X_2$, consider the set $F_2 := \{v \in f_2^{-1}(w_2) \mid \mathrm{Fr}(v) = \rho\}$. By Lemma 18, this set has $\frac{|G_2|}{|C_2|}$ elements.

For any $v \in F_2$, $f_1(v)$ lies in $X_1 \cap \phi^{-1}(w_2)$ (lying in X_1 holds by the same argument as in "$\phi(X_1) \supset X_2$"). So the sets $F_1(w_1) := \{v \in f_1^{-1}(w_1) \mid \mathrm{Fr}(v) = \rho\}$, $w_1 \in X_1 \cap \phi^{-1}(w_2)$, form a partition of F_2. By Lemma 18, $F_1(w_1)$ has $\frac{|G_1|}{|C_1|}$ elements, so w_2 has $\frac{|G_2|\cdot|C_1|}{|C_2|\cdot|G_1|}$ preimages in X_1. □

4.2 A closer look on Galois covers

A Galois cover comes with some additional data which we will need several times. We now fix notation for that data once and for all. So suppose $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover.

Suppose $g \in G$ fixes some absolutely irreducible component V_0 of V. Any other component V_0' can be written as $V_0' = \sigma V_0$ for some $\sigma \in \mathrm{Gal}_K$, so $V_0'.g = \sigma V_0.g = \sigma V_0 = V_0'$. That is, if g fixes any component, then it fixes all components. One also deduces that the set of $g \in G$ fixing the components form a normal subgroup. We will write S for this subgroup, $T := G/S$ for the quotient, and $\tau\colon G \twoheadrightarrow T$ for the canonical homomorphism. Note that T acts regularly on the set of components of V.

Now choose one absolutely irreducible component V_0 of V. This yields a map $\eta\colon \mathrm{Gal}_K \to T$ defined by $V_0.\eta(\sigma) = \sigma V_0$. Using the fact that the actions of T and of Gal_K on the set of absolutely irreducible componenents of V commute, we get that η is a (continuous) group homomorphism. As Gal_K acts transitively on the set of components, η is surjective.

$$\begin{tikzcd} & \mathrm{Gal}_K \arrow[dr, "\eta"] & \\ S \arrow[r, hook] & G \arrow[r, two heads, "\tau"] & T \end{tikzcd}$$

Note that another choice of V_0 would yield a map $\mathrm{Gal}_K \to T$ which is conjugate to η by some element of T.

In the remainder of this article, whenever we have a Galois cover denoted by $V \overset{G}{\twoheadrightarrow} W$, then S, T, τ, η will always denote the objects described here, and analogously for $V' \overset{G'}{\twoheadrightarrow} W'$, $V_i \overset{G_i}{\twoheadrightarrow} W_i$, etc. with primes resp. indices. This will not always be explicitely mentioned. (We will of course have to pay attention to the fact that η depends on the choice of V_0.)

4.3 Which of the $X(V \overset{G}{\twoheadrightarrow} W, C)$ are empty?

Let $f\colon V \overset{G}{\twoheadrightarrow} W$ be a Galois cover. To construct the measure, we need to know precisely for which conjugacy classes $C \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$ the sets $X(V \overset{G}{\twoheadrightarrow} W, C)$ are empty and for which they are not. This is what the following lemma states. It can be seen as a variant of Chebotarev's density theorem. In fact, the idea of the proof was taken from the proof of Chebotarev's density theorem in [6].

The second assertion of the lemma will be needed in the proof of the uniqueness of the measure. It is part of the same lemma because the proof is the same.

Lemma 20 *Suppose the field K we work over is perfect and PAC, and suppose further that $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover and $C \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$ is a conjugacy class. Define S, T, τ, η as in section 4.2. Then $X(V \overset{G}{\twoheadrightarrow} W, C)$ is non-empty if and only if there is a $\rho \in C$ such that $\tau \circ \rho = \eta$. In that case, $\dim X(V \overset{G}{\twoheadrightarrow} W, C) = \dim W$.*

If the conjugacy class C consists of a single element ρ and $\tau \circ \rho = \eta$, then there exists another Galois cover $\hat{V} \overset{S}{\twoheadrightarrow} W$ where $\hat{V}$ is absolutely irreducible and S is as above, and such that $X(V \overset{G}{\twoheadrightarrow} W, C) = X(\hat{V} \overset{S}{\twoheadrightarrow} W, \{1\})$.

Remark 21 *Concerning the choice involved in η, note that the condition that C contains an element ρ satisfying $\tau \circ \rho = \eta$ does not change if η is replaced by an element conjugate to it.*

Proof. We start with the easy direction of the first statement: Suppose $X(f\colon V \overset{G}{\twoheadrightarrow} W, C)$ is not empty. Choose $w \in X(V \overset{G}{\twoheadrightarrow} W, C)$ and choose a preimage $v \in f^{-1}(w) \cap V_0$, where V_0 is the absolutely irreducible component of V which gave rise to η as in Subsection 4.2. The corresponding Frobenius symbol $\rho := \mathrm{Fr}(v)$ is an element of C. This ρ satisfies $\tau \circ \rho = \eta$: For any $\sigma \in \mathrm{Gal}_K$, look at the component of V containing the left hand side resp. the right hand side of $v.\rho(\sigma) = \sigma v$. The one on the left is $V_0.\rho(\sigma) = V_0.\tau(\rho(\sigma))$, the one on the right is $\sigma V_0 = V_0.\eta(\sigma)$. As T acts freely on the set of components, it follows that $\tau(\rho(\sigma)) = \eta(\sigma)$.

For the remainder of the lemma, the goal is to construct a surjective map $\hat{f}\colon \hat{V} \twoheadrightarrow W$ such that $\hat{V}$ is absolutely irreducible and $\hat{f}(\hat{V}(K)) \subset X(V \overset{G}{\twoheadrightarrow} W, C)$. As K is PAC, it follows that $X(V \overset{G}{\twoheadrightarrow} W, C)$ is not empty and has the right dimension. If $|C| = 1$, then this $\hat{f}$ will be a Galois cover and we will have an equality $X(\hat{V} \overset{S}{\twoheadrightarrow} W, \{1\}) = \hat{f}(\hat{V}(K)) = X(V \overset{G}{\twoheadrightarrow} W, C)$.

Here is a sketch of the construction of $\hat{V}$ and $\hat{f}$:

$$\begin{tikzcd}
& V' \arrow[dl, two heads, "H"'] \arrow[dd, two heads, "G'"'] \arrow[drr, two heads] & \subset & V \times U \arrow[d, two heads] \arrow[ddll, two heads] \\
V'/H =: \hat{V} \arrow[dr, two heads, "\hat{f}"'] & & & V \arrow[dll, two heads, "G"] \\
& W & &
\end{tikzcd}$$

(1) Define some Galois cover $U \overset{Q}{\twoheadrightarrow} \{\mathrm{Pt}\}$ and consider the product $V \times U$.
(2) Restrict the map $V \times U \twoheadrightarrow W$ to an irreducible component V' of $V \times U$. This yields a Galois cover with group $G' \subset G \times Q$.
(3) Find a set $C' \subset \mathrm{Hom}(\mathrm{Gal}_K, G')$ such that $X(V \overset{G'}{\twoheadrightarrow} W, C') = X(V \overset{G}{\twoheadrightarrow} W, C)$.
(4) Choose a subgroup $H \subset G'$ such that $\hat{V} := V'/H$ is absolutely irreducible. $V' \overset{H}{\twoheadrightarrow} \hat{V}$ is a Galois cover. Choose a subset $C'' \subset \mathrm{Hom}(\mathrm{Gal}_K, H)$ such that $\hat{f}(X(V' \overset{H}{\twoheadrightarrow} \hat{V}, C'')) \subset X(V' \overset{G'}{\twoheadrightarrow} W, C')$, with equality if $|C| = 1$.
(5) Check that $X(V' \overset{H}{\twoheadrightarrow} \hat{V}, C'') = \hat{V}(K)$.
(6) Check that, if $|C| = 1$, then $\hat{f} \colon \hat{V} \to W$ is a Galois cover with group S.

Now let's get to work:

Step 1: First choose $\rho_0 \in C$ such that $\tau \circ \rho_0 = \eta$. (This exists by the hypothesis.)

Denote the image $\operatorname{im} \rho_0 \subset G$ by Q and write π_0 for the induced map $\mathrm{Gal}_K \twoheadrightarrow Q$.

$$\begin{tikzcd}
\mathrm{Gal}(L/K) \arrow[dr, "\overline{\pi}_0"', "\wr"] & & \mathrm{Gal}_K \arrow[ll, two heads] \arrow[dl, two heads, "\pi_0"'] \arrow[dd, "\rho_0"] \arrow[ddr, two heads, "\eta"] & \\
& Q \arrow[dr, phantom, "\subset"] & & \\
& S \arrow[r, hook] & G \arrow[r, two heads, "\tau"] & T
\end{tikzcd}$$

Let L be the extension of K corresponding to the quotient Q of Gal_K. In particular, π_0 factors over $\mathrm{Gal}(L/K)$ and the induced map $\overline{\pi}_0 \colon \mathrm{Gal}(L/K) \to Q$ is an isomorphism.

As K is perfect, L can be generated by a single element: $L = K(u_0)$. Let f be the minimal polynomial of u_0 over K and define $U := V(f)$ to be the set of its zeros (as an algebraic set defined over K). One easily checks that the group of automorphisms of U is isomorphic to Q^{opp} and that the isomorphism can be choosen such that for any $\sigma \in \mathrm{Gal}_K$, we have $\sigma u_0 = u_0.\pi_0(\sigma)$, where the right action of $\pi_0(\sigma) \in Q$ on U is the one induced by the isomorphism $Q \to \mathrm{Aut}(U)^{\mathrm{opp}}$. It is also easy to check that this yields a Galois cover $U \overset{Q}{\twoheadrightarrow} \{\mathrm{Pt}\}$.

In other words, $U := \operatorname{spec} L$ as a scheme over K. As K is perfect, $\mathrm{Aut}(U)^{\mathrm{opp}} = \mathrm{Gal}(L/K) = Q$ and $U \overset{Q}{\twoheadrightarrow} \{\mathrm{Pt}\}$ is a Galois cover. Let u_0 be the point of U defined by the inclusion $L \hookrightarrow \tilde{K}$. Then we have $\sigma u_0 = u_0.\pi_0(\sigma)$ for any $\sigma \in \mathrm{Gal}_K$.

Step 2: Let V_0 be the absolutely irreducible component of V as in Subsection 4.2. Then for any $\sigma \in \mathrm{Gal}_K$ we have

$$\sigma V_0 = V_0.\eta(\sigma) = V_0.\rho_0(\sigma) = V_0.\pi_0(\sigma)\,. \tag{1}$$

Let V' be the irreducible component of $V \times U$ containing the absolutely irreducible component $V_0 \times \{u_0\}$. We have:

$$\begin{aligned} V' &= \bigcup_{\sigma\in\mathrm{Gal}_K} \sigma(V_0 \times \{u_0\}) = \bigcup_{\sigma\in\mathrm{Gal}_K} \sigma V_0 \times \sigma\{u_0\} \\ &\overset{(1)}{=} \bigcup_{\sigma\in\mathrm{Gal}_K} V_0.\pi_0(\sigma) \times \{u_0\}.\pi_0(\sigma) = \bigcup_{q\in Q} V_0.q \times \{u_0\}.q\,. \end{aligned} \tag{2}$$

In particular, the absolutely irreducible components of V' are in bijection with $U(\tilde{K})$ and the diagonal action of Q on these components is regular.

The action of G on V and the action of Q on U induce an action of $G \times Q$ on $V \times U$. This gives rise to a Galois cover $V' \overset{G'}{\twoheadrightarrow} W$, where G' is the subgroup of $G \times Q$ fixing V'. We want to understand a bit more precisely how G' looks like.

An element $(g, q) \in G\times Q$ fixes V' if and only if it maps $V_0\times\{u_0\}$ to some other absolutely irreducible component of V'. By (2) these absolutely irreducible components are of the form $V_0.q' \times \{u_0\}.q'$ for some $q' \in Q$. In other words, $(g, q) \in G'$ is equivalent to: there exists a $q' \in Q$ such that $V_0.g = V_0.q'$ and $u_0.q = u_0.q'$. As Q acts regularly on $U(\tilde{K})$, it follows that $q = q'$, so we finally get:

$$\begin{aligned} G' &= \{(g, q) \in G \times Q \mid V_0.g = V_0.q\} \\ &= \{(g, q) \in G \times Q \mid \tau(g) = \tau(q)\} \end{aligned} \tag{3}$$

Step 3: We now want to find a union of conjugacy classes $C' \subset \mathrm{Hom}(\mathrm{Gal}_K, G')$

such that $X(V \overset{G}{\twoheadrightarrow} W, C) = X(V' \overset{G'}{\twoheadrightarrow} W, C')$.

We have canonically

$$\mathrm{Hom}(\mathrm{Gal}_K, G') \subset \mathrm{Hom}(\mathrm{Gal}_K, G \times Q) \cong \mathrm{Hom}(\mathrm{Gal}_K, G) \times \mathrm{Hom}(\mathrm{Gal}_K, Q)\,.$$

Using these identifications, define $C' := (C \times \pi_0^Q) \cap \mathrm{Hom}(\mathrm{Gal}_K, G')$. ($C'$ might consist of several conjugacy classes.)

Claim: $X(V \overset{G}{\twoheadrightarrow} W, C) = X(V' \overset{G'}{\twoheadrightarrow} W, C')$.

Proof of the claim: For any $w \in W(K)$, we can choose a preimage $v \in V_0(\tilde{K})$. This also yields a preimage $(v, u_0) \in V'(\tilde{K})$.

w is an element of the left hand side of the claim if and only if there exists a $\rho \in C$ such that $v.\rho(\sigma) = \sigma v$ for all $\sigma \in \mathrm{Gal}_K$. w is an element of the right hand side of the claim if and only if there exists a $(\rho, \pi) \in C'$ such that $v.\rho(\sigma) = \sigma v$ and $u_0.\pi(\sigma) = \sigma u_0$ for all $\sigma \in \mathrm{Gal}_K$.

Now "$\supset$" is clear (using the same ρ).

"$\subset$": We use the same ρ and we set $\pi := \pi_0$. Then we have $v.\rho(\sigma) = \sigma v$ and $u_0.\pi(\sigma) = \sigma u_0$, and it remains to check that (ρ, π_0) is indeed an element of C'. The only thing which is not clear here is that it is an element of $\mathrm{Hom}(\mathrm{Gal}_K, G')$, i.e. that the image of (ρ, π_0) lies in G'. But $(v, u_0).(\rho, \pi_0)(\sigma) = \sigma(v, u_0)$, so $(\rho, \pi_0)(\sigma)$ maps the irreducible component V' of $V \times U$ to itself, and therefore $(\rho, \pi_0)(\sigma) \in G'$.

So the claim is proven.

Step 4: Define $H := \{(q, q) \in G \times Q \mid q \in Q\}$ to be the diagonal embedding of Q in $G \times Q$. By (2), H is a subgroup of G' (as it fixes V') and acts regularly on the absolutely irreducible components of V'. It follows that $\hat{V} := V'/H$ is absolutely irreducible. Denote the map from $\hat{V}$ to W by $\hat{f}$.

Obviously, $\mathrm{im}(\rho_0, \pi_0) \subset H$, so we can set $C'' := (\rho_0, \pi_0)^H \subset \mathrm{Hom}(\mathrm{Gal}_K, H)$. By Lemma 19, the image of $X(V' \overset{H}{\twoheadrightarrow} \hat{V}, C'')$ under $\hat{f}$ is $X(V' \overset{G'}{\twoheadrightarrow} W, (\rho_0, \pi_0)^{G'})$, which is included in $X(V' \overset{G'}{\twoheadrightarrow} W, C')$.

If $|C| = 1$, then the conjugation action of G on ρ_0 is trivial, i.e. the image of ρ_0 lies in the center of G. In particular, Q is abelian. It follows that $C' = (C \times \pi_0^Q) \cap \mathrm{Hom}(\mathrm{Gal}_K, G') = \{(\rho_0, \pi_0)\}$, so in particular, $\hat{f}(X(V' \overset{H}{\twoheadrightarrow} \hat{V}, C'')) = X(V' \overset{G'}{\twoheadrightarrow} W, (\rho_0, \pi_0)^{G'}) = X(V' \overset{G'}{\twoheadrightarrow} W, C')$.

Step 5: We want to check that $X(V' \overset{H}{\twoheadrightarrow} \hat{V}, C'')$ is the whole of $\hat{V}(K)$. Suppose

$w \in \hat{V}(K)$. The preimage of w in $V'(\tilde{K})$ has exactly one element (v, u) for each second coordinate $u \in U(\tilde{K})$, as the composition $H \hookrightarrow G' \twoheadrightarrow Q$ is an isomorphism and Q acts regularly on U. Suppose (v, u_0) is the preimage with second coordinate u_0. We have to check that $\sigma(v, u_0) = (v, u_0).(\rho_0, \pi_0)(\sigma)$. Both the left and the right hand side are preimages of w, so it is enough to compare their second coordinates. But indeed, $\sigma(u_0) = u_0.\pi_0(\sigma)$.

Step 6: Now suppose $|C| = 1$. Then Q is abelian and $H \subset \mathrm{Z}(G) \times Q$, so H is a normal subgroup of $G \times Q$ and in particular of G'. Therefore, $\hat{V} \to W$ is a Galois cover with group G'/H. It remains to check that this group is isomorphic to S.

$$\begin{tikzcd}
(g,q) \arrow[rr, mapsto] & & gq^{-1} \\
G \times Q \arrow[rr, two heads, "\phi"'] & & G \\
\cup & & \cup \\
H \subset G' \arrow[r, two heads] & G'/H \arrow[r, dashed] & S
\end{tikzcd}$$

Consider the map $\phi\colon G' \to G$, $(g, q) \mapsto gq^{-1}$. The kernel of ϕ is exactly H (by definition of H), so we get an injective map from G'/H to G. By (3), $\tau(gq^{-1}) = 1$, so the image of ϕ lies in the kernel of τ, which is S. On the other hand, given an element $g \in S$, we get $(g, 1) \in G'$ (again by (3)) and $\phi(g, 1) = g$; so the image of ϕ is the whole S. Therefore, the map $G'/H \to S$ is an isomorphism. $\square$

5 The measure on the groups

As mentioned in the sketch of the proof of the main theorem, to get the existence of the measure, we first define a measure on the groups. In order to do that, we first have to clarify what exactly we want to associate a measure to. Then we will state a theorem about the existence of such a measure, and finally, we will prove it.

In this section, Gal_K will just be an abstract profinite group for us. However, we still write Gal_K to avoid unnecessary confusion.

5.1 The theorem for groups

Fix a subgroup Gal_K of $\hat{\mathbb{Z}}$. Suppose we are given the following data:

- A finite group G.
- A quotient T of G. We will write τ for the canonical map $G \twoheadrightarrow T$ and S for the kernel of τ.
- A continuous surjective map $\eta\colon \operatorname{Gal}_K \twoheadrightarrow T$.
- A subset $C \subset \operatorname{Hom}(\operatorname{Gal}_K, G)$ which is closed under conjugation by G.

It is to these data that we want to associate a measure. We will denote that measure by $\mu_{G,\eta}(C)$.

Theorem 22 *Suppose Gal_K is a subgroup of $\hat{\mathbb{Z}}$. Then, there exists a map μ which associates to each G, T, η, C as above a rational number $\mu_{G,\eta}(C) \in \mathbb{Q}$ satisfying the following conditions:*

(1) If G, T and η are trivial and $C = \operatorname{Hom}(\operatorname{Gal}_K, G)$ is the only existing morphism, then $\mu_{G,\eta}(C) = 1$.
(2) If $C_1, C_2 \subset \operatorname{Hom}(\operatorname{Gal}_K, G)$ are disjoint, then $\mu_{G,\eta}(C_1 \cup C_2) = \mu_{G,\eta}(C_1) + \mu_{G,\eta}(C_2)$.
(3) Suppose we have the following commutative diagram of groups, where S resp. S' is the kernel of τ resp. τ':

$$\begin{tikzcd}
S \arrow[d, two heads] & \subset & G \arrow[r, two heads, "\tau"] \arrow[d, two heads, "\pi"] & T \arrow[d, two heads] & \\
& & & & \operatorname{Gal}_K \arrow[ul, two heads, "\eta"'] \arrow[dl, two heads, "\eta'"] \\
S' & \subset & G' \arrow[r, two heads, "\tau'"] & T' &
\end{tikzcd}$$

Suppose further that $C' \subset \operatorname{Hom}(\operatorname{Gal}_K, G')$ is a conjugacy class and that $C := \{\rho \in \operatorname{Hom}(\operatorname{Gal}_K, G) \mid \pi \circ \rho \in C'\}$ is its preimage. Then $\mu_{G,\eta}(C) = \mu_{G',\eta'}(C')$.
(4) Suppose we have the following commutative diagram of groups, where S_i is the kernel of τ_i:

$$\begin{tikzcd}
S_1 \subset G_1 \arrow[dr, two heads, "\tau_1"] & & \\
\cap \quad\ \cap & T & \operatorname{Gal}_K \arrow[l, two heads, "\eta"'] \\
S_2 \subset G_2 \arrow[ur, two heads, "\tau_2"'] & &
\end{tikzcd}$$

Suppose further that we have a single conjugacy class $C_1 \subset \operatorname{Hom}(\operatorname{Gal}_K, G_1)$, that $C_2 := C_1^{G_2}$ is the induced conjugacy class in $\operatorname{Hom}(\operatorname{Gal}_K, G_2)$, and that these classes satisfy $\frac{|G_1|}{|C_1|} = \frac{|G_2|}{|C_2|}$. Then $\mu_{G_1,\eta}(C_1) = \mu_{G_2,\eta}(C_2)$.
(5) $\mu_{G,\eta}(C) \geq 0$ for any G, η and C, and $\mu_{G,\eta}(C) > 0$ if and only if C contains an element ρ which is compatible with η, i.e. such that $\tau \circ \rho = \eta$.
(6) $\mu_{G_1 \times G_2, (\eta_1, \eta_2)}(C_1 \times C_2) = \mu_{G_1,\eta_1}(C_1) \cdot \mu_{G_2,\eta_2}(C_2)$.

Proof of Theorem 22. Gal_K is isomorphic to $\prod_{p\in P}\mathbb{Z}_p$ for some subset P of the primes and $\hat{\mathbb{Z}}$ can be written as $\prod_{p\in P}\mathbb{Z}_p \times \prod_{p'\notin P}\mathbb{Z}_{p'}$. Fix an injection $\iota\colon \mathrm{Gal}_K \hookrightarrow \hat{\mathbb{Z}}$ which maps Gal_K to the corresponding factors of $\hat{\mathbb{Z}}$. For any finite group G, composition with ι defines a map from $\mathrm{Hom}(\hat{\mathbb{Z}}, G)$ to $\mathrm{Hom}(\mathrm{Gal}_K, G)$. Denote this map by ϕ.

Suppose G, S, T, τ, η are given as in Subsection 5.1. Then first note that there is exactly one continuous map $\hat{\eta} \in \mathrm{Hom}(\hat{\mathbb{Z}}, T)$ such that $\hat{\eta}\circ\iota = \eta$. Indeed, as T is a quotient of Gal_K, the prime factors of $|T|$ lie in P. Therefore, the only map from $\mathbb{Z}_{p'}$ to T for any $p' \notin P$ is the trivial one.

It follows that for any $\hat{\rho} \in \mathrm{Hom}(\hat{\mathbb{Z}}, G)$, we have $\tau\circ\hat{\rho} = \hat{\eta}$ if and only if $\tau\circ\phi(\hat{\rho}) = \eta$.

$$\begin{tikzcd}
\mathrm{Gal}_K \arrow[r, hook, "\iota"] \arrow[dr, "\eta"'] & \hat{\mathbb{Z}} \arrow[d, two heads, "\hat{\eta}"] \\
G \arrow[r, two heads, "\tau"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}(\mathrm{Gal}_K, G) \arrow[d, phantom, "\cup"] & \mathrm{Hom}(\hat{\mathbb{Z}}, G) \arrow[l, two heads, "\phi"'] \arrow[d, phantom, "\cup"] \\
\{\rho \mid \tau\circ\rho = \eta\} & \{\hat{\rho} \mid \tau\circ\hat{\rho} = \hat{\eta}\} \arrow[l, two heads, "\phi"']
\end{tikzcd}$$

Here is a definition of the measure:

$$\mu_{G,\eta}(C) := \frac{\#\{\hat{\rho}\in\phi^{-1}(C) \mid \tau\circ\hat{\rho} = \hat{\eta}\}}{|S|} = \frac{|\phi^{-1}(\{\rho\in C \mid \tau\circ\rho = \eta\})|}{|S|}.$$

We check all the conditions required by the theorem.

Conditions (1) and (2) are clear.

Condition (3): We have the following commutative diagram:

$$\begin{tikzcd}
S \subset G \arrow[d, two heads] \arrow[r, two heads, "\tau"] & T \arrow[d, two heads] & & \\
& & \mathrm{Gal}_K \arrow[ul, two heads, "\eta"'] \arrow[dl, two heads, "\eta'"] \arrow[r, hook] & \hat{\mathbb{Z}} \arrow[ull, two heads, "\hat{\eta}"'] \arrow[dll, two heads, "\hat{\eta}'"] \\
S' \subset G' \arrow[r, two heads, "\tau'"] & T' & &
\end{tikzcd}$$

with the vertical map $G \twoheadrightarrow G'$ labelled π.

We also have a conjugacy class $C' \subset \mathrm{Hom}(\mathrm{Gal}_K, G')$ and its preimage $C := \{\rho\in\mathrm{Hom}(\mathrm{Gal}_K, G) \mid \pi\circ\rho\in C'\}$. Fix a generator σ_0 of $\hat{\mathbb{Z}}$. Then maps from $\hat{\mathbb{Z}}$ to a finite group are in bijection to the elements of that finite group by taking the image of σ_0.

Define $D := \{\hat{\rho}(\sigma_0) \in G \mid \hat{\rho} \in \phi^{-1}(C)\}$ and $D' := \{\hat{\rho}'(\sigma_0) \in G' \mid \hat{\rho}' \in \phi^{-1}(C')\}$. Using this, what we have to check translates to

$$\frac{\#\{g \in D \mid \tau(g) = \hat{\eta}(\sigma_0)\}}{|S|} = \frac{\#\{g' \in D' \mid \tau(g') = \hat{\eta}'(\sigma_0)\}}{|S'|}. \tag{4}$$

Choose $g_0 \in G$ such that $\tau(g_0) = \hat{\eta}(\sigma_0)$. Then the sets in the numerators of (4) can be rewritten as $D \cap g_0 S$ resp. $D' \cap \pi(g_0)S'$. The homomorphism π restricts to a map $g_0 S \twoheadrightarrow \pi(g_0)S'$ with fiber size $\frac{|S|}{|S'|}$. So we are finished as soon as we have checked that D is the preimage of D' under π. But this follows from the hypothesis that C is the preimage of C': Applying ϕ^{-1} to C yields

$$\phi^{-1}(C) = \{\hat{\rho} \in \mathrm{Hom}(\hat{\mathbb{Z}}, G) \mid \pi \circ \hat{\rho} \circ \iota \in C'\} = \{\hat{\rho} \in \mathrm{Hom}(\hat{\mathbb{Z}}, G) \mid \pi \circ \hat{\rho} \in \phi^{-1}(C')\}.$$

From this, we get $D = \{g \in G \mid \pi(g) \in D'\}$.

Condition (4): We have the following commutative diagram of groups:

$$\begin{tikzcd}
S_1 \subset G_1 \arrow[dr, two heads] & & \mathrm{Gal}_K \arrow[dl, two heads, "\eta"'] \arrow[dd, hook, "\iota"] \\
\cap \qquad \cap & T & \\
S_1 \subset G_2 \arrow[ur, two heads] & & \hat{\mathbb{Z}} \arrow[ul, two heads, "\hat{\eta}"]
\end{tikzcd}$$

We also have a single conjugacy class $C_1 \subset \mathrm{Hom}(\mathrm{Gal}_K, G_1)$ and the induced class $C_2 := C_1^{G_2} \subset \mathrm{Hom}(\mathrm{Gal}_K, G_2)$, and these conjugacy classes satisfy $\frac{|G_1|}{|C_1|} = \frac{|G_2|}{|C_2|}$. We want to check that $\mu_{G_1,\eta}(C_1) = \mu_{G_2,\eta}(C_2)$.

To avoid confusion, we will write ϕ_i (instead of just ϕ) for the composition with ι from $\mathrm{Hom}(\hat{\mathbb{Z}}, G_i)$ to $\mathrm{Hom}(\mathrm{Gal}_K, G_i)$.

$$\begin{array}{ccccc}
C_2 & \subset & \mathrm{Hom}(\mathrm{Gal}_K, G_2) & \overset{\phi_2}{\twoheadleftarrow} & \mathrm{Hom}(\hat{\mathbb{Z}}, G_2) \\
\cup & & \cup & & \cup \\
C_1 & \subset & \mathrm{Hom}(\mathrm{Gal}_K, G_1) & \overset{\phi_1}{\twoheadleftarrow} & \mathrm{Hom}(\hat{\mathbb{Z}}, G_1)
\end{array}$$

We will check

$$\frac{\#\{\rho \in C_1 \mid \tau_1 \circ \rho = \eta\}}{|C_1|} = \frac{\#\{\rho \in C_2 \mid \tau_2 \circ \rho = \eta\}}{|C_2|} \tag{5}$$

and

$$\frac{\left|\phi_1^{-1}(\{\rho \in C_1 \mid \tau_1 \circ \rho = \eta\})\right|}{\#\{\rho \in C_1 \mid \tau_1 \circ \rho = \eta\}} = \frac{\left|\phi_2^{-1}(\{\rho \in C_2 \mid \tau_2 \circ \rho = \eta\})\right|}{\#\{\rho \in C_2 \mid \tau_2 \circ \rho = \eta\}}. \tag{6}$$

Together with $\frac{|S_2|}{|S_1|} = \frac{|G_2|}{|G_1|} = \frac{|C_2|}{|C_1|}$, this implies

$$\frac{\left|\phi_1^{-1}(\{\rho \in C_1 \mid \tau_1 \circ \rho = \eta\})\right|}{|S_1|} = \frac{\left|\phi_2^{-1}(\{\rho \in C_2 \mid \tau_2 \circ \rho = \eta\})\right|}{|S_2|}, \tag{7}$$

which is what we have to show. (In (6), it is possible that a denominator is zero. However by (5) then both of them have to be zero and it is clear that in that case both numerators are zero, too, so (7) still holds.)

For the proof of (5), consider the image of C_i in $\operatorname{Hom}(\mathrm{Gal}_K, T)$ under composition with τ_i. It is exactly one conjugacy class, and it does not depend on i, as $(\tau_1\circ)(C_1) \subset (\tau_2\circ)(C_2)$. If this conjugacy class does not contain η, then both sides of (5) are zero, and there is nothing more to prove. So suppose now that $(\tau_1\circ)(C_1) = (\tau_2\circ)(C_2) = \eta^T$.

The fibers of the restricted composition map $(\tau_i\circ)\restriction_{C_i}$ all have the same size, as they are conjugate. The set $\{\rho \in C_i \mid \tau_i \circ \rho = \eta\}$ is one such fiber, so $|C_i| = \#\{\rho \in C_i \mid \tau_i \circ \rho = \eta\} \cdot |\eta^T|$. As this is true for both i, (5) follows.

For the proof of (6), consider the map ϕ_2 restricted to $\phi_2^{-1}(C_2) \to C_2$. The fibers are conjugate by elements of G_2 and therefore all have the same size, which we denote by k. So the right hand side of (6) is k.

Now consider $\phi_2^{-1}(C_1)$. A priori, this is a set of maps from $\hat{\mathbb{Z}}$ to G_2. However, we will check that the image of these maps lies in G_1. This means that $\phi_2^{-1}(C_1) = \phi_1^{-1}(C_1)$ and it follows that the left hand side of (6) is k, too.

So suppose $\hat{\rho} \in \operatorname{Hom}(\hat{\mathbb{Z}}, G_2)$ is such that $\rho := \phi_2(\hat{\rho}) \in C_1$. As $|\,\mathrm{C}_{G_1}(\operatorname{im}\rho)| = \frac{|G_1|}{|C_1|} = \frac{|G_2|}{|C_2|} = |\,\mathrm{C}_{G_2}(\operatorname{im}\rho)|$ and $\mathrm{C}_{G_1}(\operatorname{im}\rho) \subset \mathrm{C}_{G_2}(\operatorname{im}\rho)$, we have $\mathrm{C}_{G_1}(\operatorname{im}\rho) = \mathrm{C}_{G_2}(\operatorname{im}\rho)$, so in particular $\mathrm{C}_{G_2}(\operatorname{im}\rho) \subset G_1$. Now $\operatorname{im}\hat{\rho}$ is abelian (as $\hat{\mathbb{Z}}$ is abelian) and contains $\operatorname{im}\rho$, so $\operatorname{im}\hat{\rho} \subset \mathrm{C}_{G_2}(\operatorname{im}\rho) \subset G_1$.

Condition (5) is clear.

Condition (6) is straight forward to check. □

6 Proof of the main theorem

6.1 Existence

We know that there exists a measure on the groups as described by Theorem 22. Now we have to define a measure on the definable sets. We do this by defining the measure of a Galois stratification and then showing that this measure only depends on the set the Galois stratification defines.

Let $\mathcal{A} = (f_i \colon V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)_{i\in I}$ be a Galois stratification of K^n for some n, and let T_i, τ_i, η_i be as usual. Let d be the maximal dimension of those W_i where $\mu_{G_i,\eta_i}(C_i) \neq 0$, or $d := 0$ if all $\mu_{G_i,\eta_i}(C_i) = 0$. Then we define the measure of $\mathcal{A}$ to be

$$\mu(\mathcal{A}) := \sum_{\{i\in I \mid \dim W_i = d\}} \mu_{G_i,\eta_i}(C_i)\,.$$

Remember that η_i is only well defined up to conjugacy (Subsection 4.2), so for this definition to make sense, we have to check that $\mu_{G_i,\eta_i}(C_i)$ does only depend on the conjugacy class of η_i. Indeed, if $\eta_i' = \eta_i^t$ for some $t \in T$, then choose a preimage $g \in \tau_i^{-1}(t)$ and apply Condition (3) of Theorem 22 with $G = G'$ and $\pi := \operatorname{Int} g$ the corresponding internal automorphism.

Remark 23 *By Condition (5) of Theorem 22, $\mu_{G_i,\eta_i}(C_i) \neq 0$ if and only if there is a $\rho \in C_i$ such that $\tau_i \circ \rho = \eta_i$. By Lemma 20, this is equivalent to $X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)$ being non-empty. As in the latter case* $\dim X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i) = \dim W_i$ *(also by Lemma 20), it follows that d is the dimension of the set defined by $\mathcal{A}$.*

Lemma 24 *If two Galois stratifications $\mathcal{A}$ and $\mathcal{A}'$ define the same set, then their measures (as defined above) are equal.*

Proof. By Lemma 13, there exist refinements $\tilde{\mathcal{A}}$ resp. $\tilde{\mathcal{A}}'$ of $\mathcal{A}$ resp. $\mathcal{A}'$ which only differ in the sets $\tilde{C}_i$ resp. $\tilde{C}_i'$, and the refinements define the same sets as the originals. So it is enough to check that the measure of two stratifications $\mathcal{A}$ and $\mathcal{A}'$ is equal in two cases: if $\mathcal{A}'$ is a refinement of $\mathcal{A}$ and if $\mathcal{A}$ and $\mathcal{A}'$ differ only in the C_i.

Suppose first that $\mathcal{A} = (f_i \colon V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)_{i\in I}$ and $\mathcal{A}' = (f_i \colon V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i')_{i\in I}$ differ only in the sets C_i and that both define the same set. We show that for all $i \in I$, $\mu_{G_i,\eta_i}(C_i) = \mu_{G_i,\eta_i}(C_i')$. For this, it is enough to check that for any conjugacy class C in $\operatorname{Hom}(\mathrm{Gal}_K, G_i)$ with $\mu_{G_i,\eta_i}(C) \neq 0$, either both or none of C_i and C_i' contain C.

If $X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C)$ is not empty, then indeed both or none of C_i and C_i' contain

C, as by assumtion, we have $X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i) = X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i')$. If on the other hand $X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C)$ is empty, then by Lemma 20, C does not contain any element ρ such that $\tau_i \circ \rho = \eta_i$. By Condition (5) of Theorem 22 it follows that $\mu_{G_i,\eta_i}(C) = 0$.

Now suppose $\mathcal{A}'$ is a refinement of $\mathcal{A}$. We have to check that

$$\sum_{\{i\in I \mid \dim W_i = d\}} \mu_{G_i,\eta_i}(C_i) = \sum_{\{i\in I \mid \dim W_i' = d'\}} \mu_{G_i',\eta_i'}(C_i')\,,$$

where d resp. d' is the dimension of the set defined by $\mathcal{A}$ resp. $\mathcal{A}'$. As $\mathcal{A}$ and $\mathcal{A}'$ define the same set, we have $d = d'$.

For $i \in I$, write J_i of the set of $j \in I'$ such that $W_j' \subset W_i$ and set

$$\mu_i := \sum_{\{j\in J_i \mid \dim W_j' = d\}} \mu_{G_j',\eta_j'}(C_j')\,.$$

The assertion follows from the following claim: If $\dim W_i < d$, then $\mu_i = 0$. Otherwise, $\mu_i = \mu_{G_i,\eta_i}(C_i)$.

If $\dim W_i < d$, then $\dim W_j' < d$ for all $j \in J_i$, so the sum is empty and the claim is true. If $\dim W_i = d$, then (as W_i is irreducible), there exists exactly one W_j' ($j \in J_i$) such that $\dim W_j' = d$. So it remains to check that $\mu_{G_j',\eta_j'}(C_j') = \mu_{G_i,\eta_i}(C_i)$ for this specific j.

By definition of refinement of a Galois stratification, the Galois cover $V_j' \overset{G_j'}{\twoheadrightarrow} W_j'$ is a refinement of a restriction of $V_i \overset{G_i}{\twoheadrightarrow} W_i$ to W_j'. As W_j' is dense in W_i, this restriction is $f_i{\restriction_V}\colon V \overset{G_i}{\twoheadrightarrow} W_j'$ where $V := f_i^{-1}(W_j') \subset V_i$. This means we have the following diagram:

$$\begin{tikzcd}
 & V_j' \arrow[dl, two heads] \arrow[ddl, two heads, "G_j'"] \\
V \arrow[d, two heads, "G_i"'] & \\
W_j' &
\end{tikzcd}$$

On the level of groups, we get:

$$\begin{tikzcd}
S_j' \subset G_j' \arrow[r, two heads, "\tau_j'"] \arrow[d, two heads, "\pi"] & T_j' \arrow[d, two heads] & \\
 & & \mathrm{Gal}_K \arrow[ul, two heads, "\eta_j'"'] \arrow[dl, two heads, "\eta_i"] \\
S_i \subset G_i \arrow[r, two heads, "\tau_i"] & T_i &
\end{tikzcd}$$

Using $C'_j = \{\rho \in \mathrm{Hom}(\mathrm{Gal}_K, G'_j) \mid \pi \circ \rho \in C_i\}$ (also by definition of refinement of a Galois stratification) we get $\mu_{G'_j,\eta'_j}(C'_j) = \mu_{G_i,\eta_i}(C_i)$ by Condition (3) of Theorem 22. $\square$

We proved that the measure of a Galois stratification only depends on the set it defines, so we can define: For any definable set X, choose a Galois stratification $\mathcal{A}$ defining X and set $\mu(X) := \mu(\mathcal{A})$.

Now we have to check that this measures satisfies all the conditions stated in Theorem 16. Before we start, note that if a definable set X has can be defined by a single Galois cover, i.e. $X = X(V \overset{G}{\twoheadrightarrow} W, C)$, then its measure is $\mu_{G,\eta}(C)$ (as expected). Indeed, this is clear if we take a stratification which has W as one of its W_i.

Conditions (1), (3) and (6) are straight forward. (For Condition (3), one chooses Galois stratifications defining the sets X, X' and $X'' := X \mathbin{\dot\cup} X'$ which only differ in the sets C_i and such that $C''_i = C_i \mathbin{\dot\cup} C'_i$.)

Condition (2): Suppose we have two definable sets $X_1 \subset K^{n_1}$ and $X_2 \subset K^{n_2}$ and a definable bijection ϕ between them. We have to check that X_1 and X_2 have the same measure. We will do this by simplifying the situation more and more until we can apply Lemma 19 and Condition (4) of Theorem 22.

The projections $K^{n_1+n_2} \twoheadrightarrow K^{n_i}$ induce definable bijections between the graph of ϕ and X_i. So it is enough to check that the measure is invariant under definable bijections $X \to X'$ which are induced by the projection $\pi\colon K^n \twoheadrightarrow K^{n'}$, where $X \subset K^n, X' \subset K^{n'}$ and $n \geq n'$. (In fact, we are not so much interested in the fact that π is a projection; we only need that it is an algebraic map.)

Now choose a Galois stratification $\mathcal{A}$ defining X. Then $\pi\restriction_X$ decomposes into definable bijections $X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i) \to \pi(X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i))$ $(i \in I)$. So it is enough to check the condition when X is of the form $X(f\colon V \overset{G}{\twoheadrightarrow} W, C)$. (Note that we already checked that the measure is compatible with disjoint union.)

Now choose a Galois stratification $\mathcal{A}'$ defining X'. Again, we can decompose $\pi\restriction_X$, this time into $X \cap \pi^{-1}(W'_i) \to X(V'_i \overset{G'_i}{\twoheadrightarrow} W'_i, C'_i)$. Indeed, an element of $X \cap \pi^{-1}(W'_i)$ is mapped to $X' \cap W'_i$, which is precisely $X(V'_i \overset{G'_i}{\twoheadrightarrow} W'_i, C'_i)$. For any $i \in I'$, the set $X \cap \pi^{-1}(W'_i)$ is defined by a Galois cover which is a restriction of $X(f\colon V \overset{G}{\twoheadrightarrow} W, C)$ to $W \cap \pi^{-1}(W'_i)$. So it is enough to check the condition when X and X' both are defined by single Galois covers: $X = X(f\colon V \overset{G}{\twoheadrightarrow} W, C)$ and $X' = X(f'\colon V' \overset{G'}{\twoheadrightarrow} W', C')$. Write ϕ for the map π restricted to $W \to W'$.

Next, we want to modify the situation such that ϕ becomes finite, surjective, and generically étale. For this, first note that we are allowed to replace W' by a dense open subset and V', W, and V by the corresponding preimages, as removing parts of lower dimension doesn't change the measure.

X' is contained in $\operatorname{im}\phi$ and is Zariski dense in W', so ϕ is dominant. As $\dim W = \dim X = \dim X' = \dim W'$, ϕ is finite on a dense open subset of W'. Shrinking W' appropriately makes ϕ finite and surjective.

To make ϕ generically étale we use Lemma 6: After further shrinking the sets, we get maps $W \xrightarrow{\psi} W'' \xrightarrow{\phi''} W'$ where ψ is "purely inseparable" and ϕ'' generically étale. We have to check that we can transform the Galois cover defining X into a Galois cover defining $X'' := \psi(X)$ (in a way which ensures $\mu(X'') = \mu(X)$). Afterwards, we can replace X by X'' and ϕ by ϕ''.

By shrinking the sets again, we can suppose that V, W, and W'' are normal. We give names to the rings of regular functions and to the fields of fractions: $R := K[W]$, $R'' := K[W'']$, $S := K[V]$, $E := K(W)$, $E'' := K(W'')$, and $F := K(V)$. Let F'' be the separable closure of E'' in F, S'' be the integral closure of R'' in F'' and $V'' := \operatorname{spec} R''$ be the corresponding variety. We have:

$$
\begin{array}{ccc}
F \overset{\text{insep.}}{\supset} F'' & S \supset S'' & V \rightarrow V'' \\
\cup \qquad \cup & \cup \quad \cup & \downarrow \qquad \downarrow \\
E \overset{\text{insep.}}{\supset} E'' & R \supset R'' & W \rightarrow W''
\end{array}
$$

We claim that $V'' \to W''$ is a Galois cover with group G and that $X'' = X(V'' \overset{G}{\twoheadrightarrow} W'', C)$. Indeed, this follows easily using $\operatorname{Aut}_W(V) = \operatorname{Gal}(F/E) = \operatorname{Gal}(F''/E'') = \operatorname{Aut}_{W''}(V'')$ and the the bijectivity of $V \to V''$ and $W \to W''$.

Now that ϕ is finite, surjective, and generically étale, we claim that we can suppose $V = V'$. Indeed: After shrinking W', we can extend the composition $\phi \circ f$ to a Galois cover $\hat{V} \twoheadrightarrow W'$.

$$
\begin{tikzcd}
 & \tilde{V} \arrow[dl] \arrow[dddl, two heads] \arrow[ddr, two heads] \arrow[dddr, two heads] & \\
\hat{V} \arrow[d, two heads] \arrow[ddrr, two heads] & & \\
V \arrow[d, two heads, "f"'] & & V' \arrow[d, two heads, "f'"] \\
W \arrow[rr, "\phi"'] & & W'
\end{tikzcd}
$$

Now we take a common refinement $\tilde{V} \twoheadrightarrow W'$ of the Galois covers $\hat{V} \twoheadrightarrow W'$ and $V' \overset{G'}{\twoheadrightarrow} W'$. As $\tilde{V} \twoheadrightarrow W'$ is a Galois cover, so is $\tilde{V} \twoheadrightarrow W$. Replace V and

V' by $\tilde{V}$. (X and X' can be defined using the Galois covers $\tilde{V} \twoheadrightarrow W$ and $\tilde{V} \twoheadrightarrow W'$, as these are refinements of $V \overset{G}{\twoheadrightarrow} W$ resp. $V' \overset{G'}{\twoheadrightarrow} W'$.) So we can suppose $X = X(V \overset{G}{\twoheadrightarrow} W, C)$ and $X' = X(V \overset{G'}{\twoheadrightarrow} W', C')$ and $G \subset G'$.

We decompose $\phi\restriction_X$ one last time: into $\phi\restriction_{X(\tilde{V} \overset{\tilde{G}}{\twoheadrightarrow} W, D)}$ where $D \subset C$ ranges over the conjugacy classes in C. By Lemma 19, the image $\phi(X(V \overset{G}{\twoheadrightarrow} W, D))$ is $X(V \overset{G'}{\twoheadrightarrow} W', D^{G'})$. So we can suppose that C consists of a single conjugacy class and that $C' = C^{G'}$ is the induced class.

As $\phi\restriction_X$ is a bijection, we can apply the second statement of Lemma 19 and get $\frac{|G|}{|C|} = \frac{|G'|}{|C'|}$. One esily verifies that on the level of groups, we have the diagram of Condition (4) of Theorem 22. ($T = T'$ follows from the fact that W is absolutely irreducible.) Therefore, we can apply that condition and get $\mu(X) = \mu_{G,\eta}(C) = \mu_{G',\eta'}(C') = \mu(X')$.

Condition (4): As already noted, a set of the form $X(V \overset{G}{\twoheadrightarrow} W, C)$ has measure $\mu_{G,\eta}(C)$. If V is absolutely irreducible, then T and η are trivial, so $\mu(X(V \overset{G}{\twoheadrightarrow} W, C)) = \mu_{G,1}(C)$ only depends on G and C.

Condition (5): $\mu(X) \geq 0$ is clear and $\mu(\emptyset) = 0$, too. Now suppose $\mu(X) = 0$. Choose any Galois stratification $\mathcal{A}$ defining X. All $\mu_{G_i,\eta_i}(C_i)$ have to be zero. By Condition (5) of Theorem 22, this implies that C_i contains no ρ such that $\tau_i \circ \rho = \eta_i$. By Lemma 20, it follows that $X(V_i \overset{G_i}{\twoheadrightarrow} W_i, C_i)$ is empty; so X is empty.

6.2 Uniqueness

Suppose μ is any function satifying the conditions of the theorem.

First note that by Lemma 20, given a fixed Galois cover $V \overset{G}{\twoheadrightarrow} W$, all sets $X(V \overset{G}{\twoheadrightarrow} W, C)$ (for some $C \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$ closed under conjugation) which are not empty have the same dimension. It follows that if C_1 and C_2 are disjoint, then $\mu(X(V \overset{G}{\twoheadrightarrow} W, C_1 \cup C_2)) = \mu(X(V \overset{G}{\twoheadrightarrow} W, C_1)) + \mu(X(V \overset{G}{\twoheadrightarrow} W, C_2))$.

Each definable set can be written as disjoint union of sets of the form $X(f\colon V \overset{G}{\twoheadrightarrow} W, C)$. We can further decompose these sets so that each C is a single conjugacy class. Using Condition (3) of the theorem, it is therefore enough to prove the uniqueness for such sets. We do this by induction on the size of the group G.

Suppose first $|G| = 1$. Then $C = \mathrm{Hom}(\mathrm{Gal}_K, G)$ is the only existing conjugacy class and $f\colon V \to W$ is an isomorphism. In particular, V is absolutely irreducible (as W is), so (4) applies: $\mu(X(V \overset{G}{\twoheadrightarrow} W, C))$ is equal to $\mu(X(\mathrm{id}\colon K \overset{1}{\twoheadrightarrow} K, 1))$. But $X(\mathrm{id}\colon K \overset{1}{\twoheadrightarrow} K, 1) = K$, so its measure is 1 by Condition (1).

Now suppose $|G| > 1$. We first treat the case where $|C| > 1$. Choose $\rho \in C$ and let $G' := \mathrm{C}_G(\mathrm{im}\,\rho)$ be the centralizer of ρ in G. Define $W' := V/G'$. Then, $V \overset{G'}{\twoheadrightarrow} W'$ is a Galois cover and we have the following diagram:

$$\begin{tikzcd} V \arrow[d, two heads, "G'"'] \arrow[dr, two heads, "G"] & \\ W' \arrow[r, two heads, "\phi"'] & W \end{tikzcd}$$

As Gal_K is cyclic, so is $\mathrm{im}\,\rho$. Therefore, $\mathrm{C}_G(\mathrm{im}\,\rho)$ contains $\mathrm{im}\,\rho$. So we have $\rho \in \mathrm{Hom}(\mathrm{Gal}_K, G')$, and $C' := \{\rho\}$ is a conjugacy class in $\mathrm{Hom}(\mathrm{Gal}_K, G')$. In addition, $\frac{|G|}{|C|} = |\,\mathrm{C}_G(\mathrm{im}\,\rho)| = \frac{|G'|}{|C'|}$, so we are precisely in the situation of Lemma 19. It follows that ϕ restricts to a bijection $X(V \overset{G'}{\twoheadrightarrow} W', C') \xrightarrow{1:1} X(V \overset{G}{\twoheadrightarrow} W, C)$.

By the assumption $|C| > 1$, we have $|G'| < |G|$, so we can apply the induction hypothesis: We know the measure of $X(V \overset{G'}{\twoheadrightarrow} W', C')$. By Condition (2) of Theorem 16, $X(V \overset{G}{\twoheadrightarrow} W, C)$ has the same measure.

We are left with the case $|C| = 1$. Here, we can apply the second part of Lemma 20. We get another Galois cover $\hat{V} \overset{S}{\twoheadrightarrow} W$ such that $X(V \overset{G}{\twoheadrightarrow} W, C) = X(\hat{V} \overset{S}{\twoheadrightarrow} W, \{1\})$. If $S \subsetneq G$, we know the measure of $X(\hat{V} \overset{S}{\twoheadrightarrow} W, \{1\})$ by induction hypothesis, so we can suppose $S = G$.

Let $Z \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$ be the union of all conjugacy classes of size one ($Z = \mathrm{Hom}(\mathrm{Gal}_K, \mathrm{Z}(G))$). We already know the measure of $X(V \overset{G}{\twoheadrightarrow} W, \mathrm{Hom}(\mathrm{Gal}_K, G) \setminus Z)$. We also know $\mu(X(V \overset{G}{\twoheadrightarrow} W, \mathrm{Hom}(\mathrm{Gal}_K, G))) = \mu(W(K)) = \mu(X(W \overset{1}{\twoheadrightarrow} W, 1)) = 1$. Therefore, we know $X(V \overset{G}{\twoheadrightarrow} W, Z)$.

Now, for any conjugacy class $C \subset Z$, using Lemma 20 and Condition (4) ($S = G$ implies V absolutely irreducible) we get $\mu(X(V \overset{G}{\twoheadrightarrow} W, C)) = \mu(X(\hat{V} \overset{S}{\twoheadrightarrow} W, \{1\})) = \mu(X(V \overset{G}{\twoheadrightarrow} W, \{1\}))$. In other words, $\mu(X(V \overset{G}{\twoheadrightarrow} W, C))$ is the same for any conjugacy class $C \subset Z$, so we get $\mu(X(V \overset{G}{\twoheadrightarrow} W, C)) = \frac{1}{|Z|}\mu(X(V \overset{G}{\twoheadrightarrow} W, Z))$.

7 Generalizations

7.1 If the Galois group is not pro-cyclic

The main theorem proves the existence of a measure if the Galois group is pro-cyclic. In this section, we show that one cannot hope for much more. We give an example which shows that no measure exists for many non-pro-cyclic bounded Galois groups.

Theorem 25 *Suppose K is a perfect PAC field and suppose* $\mathrm{Gal}_K = P_1 * \cdots * P_m$ *is a free product of finitely many (copies of) subgroups P_i of $\hat{\mathbb{Z}}$. Suppose further that there exists a prime p such that at least two of the groups P_i contain $\mathbb{Z}_p$ as a subgroup. Then there does not exist any function μ from the definable sets over K to $\mathbb{Q}$ satisfying Conditions (1) to (4) of theorem 16.*

Remark 26 *The prerequisites are in particular satisfied if* Gal_K *is a free product of more that one copy of $\hat{\mathbb{Z}}$.*

Proof of Theorem 25. Let the groups P_i and the prime p be as in the theorem. Suppose G is any finite p-group and consider the maps from P_i (for some fixed i) to G. If P_i does contain $\mathbb{Z}_p$ as a subgroup, then these maps are in bijection to the elements of G (by taking the image of some generator of P_i); otherwise, there is only the trivial map from P_i to G. In the following all our finite groups will be p-groups, so we will fix some generators of the groups P_i and identify $\mathrm{Hom}(\mathrm{Gal}_K, G)$ with G^n, where n is the number of indices i such that P_i contains $\mathbb{Z}_p$.

Suppose a measure μ does exist. We will construct some definable sets which provide a contradiction.

We will have to deal with Galois covers $V \overset{G}{\twoheadrightarrow} W$ where V is absolutely irreducible and $G \cong (\mathbb{Z}/p\mathbb{Z})^2$. So suppose we have such a Galois cover and suppose $\rho \in \mathrm{Hom}(\mathrm{Gal}_K, G)$ is any homomorphism. As G is abelian, the arguments from the proof of the uniqueness of the main theorem concerning one-element conjugacy classes yield

$$\mu(X(V \overset{G}{\twoheadrightarrow} W, \{\rho\})) = \frac{1}{|\mathrm{Hom}(\mathrm{Gal}_K, G)|} = p^{-2n} =: \mu_0 \,,$$

which in particular is greater than zero.

We will now construct definable sets $X = X_1 \,\dot{\cup}\, \ldots \,\dot{\cup}\, X_{p^{n-1}}$ with

$$\mu(X) = \mu(X_1) = \cdots = \mu(X_{p^{n-1}}) = \mu_0 \,,$$

which (as $p^{n-1} > 1$) will be a contradiction to Condition (3).

Let G be the group defined by the following generators and relations:

$$G = \langle g_1, g_2, z \mid g_1^p = g_2^p = z^p = 1, g_1 z = z g_1, g_2 z = z g_2, g_1 g_2 = g_2 g_1 z \rangle \ .$$

Its center $Z := \mathrm{Z}(G)$ is $\langle z \rangle \cong \mathbb{Z}/p\mathbb{Z}$, and the quotient $\hat{G} := G/Z$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$. Write π for the canonical map $G \twoheadrightarrow \hat{G}$. The centralizer of g_1 is $G' := \mathrm{C}_G(g_1) = \langle g_1, z \rangle \cong (\mathbb{Z}/p\mathbb{Z})^2$.

Now suppose $V \overset{G}{\twoheadrightarrow} W$ is a Galois cover with this group and that V is absolutely irreducible. (It is well known how to construct such a Galois cover for any finite group G.) Define $W' := V/G'$ and $\hat{V} := V/Z$. We get Galois covers $V \overset{G'}{\twoheadrightarrow} W'$ and $\hat{V} \overset{\hat{G}}{\twoheadrightarrow} W$.

$$\begin{tikzcd}
V \arrow[r, two heads] \arrow[d, two heads, "(\mathbb{Z}/p\mathbb{Z})^2 \cong G'"'] \arrow[dr, two heads, "G"] & \hat{V} \arrow[d, two heads, "\hat{G} \cong (\mathbb{Z}/p\mathbb{Z})^2"] \\
W' \arrow[r, two heads] & W
\end{tikzcd}$$

Choose $\hat{\rho} := (\pi(g_1), 1, 1, \ldots, 1) \in \hat{G}^n \cong \mathrm{Hom}(\mathrm{Gal}_K, \hat{G})$ and consider the set $X := X(\hat{V} \overset{\hat{G}}{\twoheadrightarrow} W, \{\hat{\rho}\})$. It has measure μ_0.

This set can also be described using the Galois cover $V \overset{G}{\twoheadrightarrow} W$: $X := X(V \overset{G}{\twoheadrightarrow} W, C)$, where $C = \{\rho \in G^n \mid \pi(\rho) = \hat{\rho}\} = (g_1 Z) \times Z \times \cdots \times Z$. The G-conjucacy classes in C are of the form $(g_1 Z) \times \{(z_2, \ldots z_n)\}$ for any $z_2, \ldots, z_n \in Z$. Denote the elements of the form $(g_1, z_2, \ldots z_n)$ by ρ_i ($i \in I := Z^{n-1}$) and the induced conjugacy classes by $C_i := \rho_i^G$. X is the disjoint union of the sets $X_i := X(V \overset{G}{\twoheadrightarrow} W, C_i)$.

As $\rho_i \in G'^n$, we may consider the sets $X'_i := X(V \overset{G'}{\twoheadrightarrow} W', \{\rho_i\})$. Because of $\frac{|G|}{|C_i|} = p^2 = \frac{|G'|}{|\{\rho_i\}|}$, Lemma 19 applies and we get a bijection $X'_i \to X_i$. So $\mu(X_i) = \mu(X'_i) = \mu_0$. This terminates the contradiction. $\square$

Note that this counter-example does not depend very much on the target ring of the measure. In particular, it still works when $\mathbb{Q}$ is replaced by any ring of characteristic zero.

7.2 Some open questions

- Is Condition (4) of the main theorem really necessary for uniqueness? It would be nice to replace it by a condition which only requires the measure of absolutely irreducible varieties to be 1. Indeed, the only place where this condition is really used is in the proof that for $\rho \in \mathrm{Hom}(\mathrm{Gal}_K, \mathrm{Z}(G)) \subset \mathrm{Hom}(\mathrm{Gal}_K, G)$, the measure of $X(V \overset{G}{\twoheadrightarrow} W, \{\rho\})$ does not depend on ρ.

- Is Condition (4) of the main theorem necessary to construct the counter-example of Subsection 7.1? To provide a counter-example without using that condition, it would be necessary to construct some Galois covers more explicitely.
- We proved that for non-procyclic Galois groups, no measure in the sense of the main theorem can exist. However, one might still hope to be able to define some kind of invariant of definable sets which is not additive.

  Note however the following: If one still describes definable sets as unions of sets of the form $X(V \overset{G}{\twoheadrightarrow} W, C)$, one will need some kind of relation between the measure of a union and the components. However, the counter-example of Subsection 7.1 easily adapts to prove that no sensible invariant exists if there is *any* function f such that $\mu(X_1 \mathbin{\dot\cup} X_2) = f(\mu(X_1), \mu(X_2))$ for any definable sets X_1 and X_2 of the same dimension.

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