Superrigidity and the classification problem for the torsion-free abelian groups of finite rank

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Definition 1 *$G \leqslant \mathbb{Q}^n$ has rank n iff G contains n linearly independent elements.*

Some History:

1937 Baer gave a satisfactory classification of the rank 1 groups.

1938 Kurosh and Malcev independently gave an unsatisfactory classification of the higher rank groups.

Suppose that G is a torsion-free abelian group and that $0 \neq x \in G$. For each prime $p \in \mathbb{P}$, the *p-height* of x is defined to be the supremum

$$h_x(p) \in \mathbb{N} \cup \{\infty\}$$

of the natural numbers n such that

$$(\exists y \in G) p^n y = x$$

and the *characteristic* $\chi(x)$ of x is defined to be the function

$$\langle h_x(p) \mid p \in \mathbb{P} \rangle \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}.$$

Two functions $\chi_1,\ \chi_2 \in (\mathbb{N} \cup \{\infty\})^{\mathbb{P}}$ are said to be *similar* or to *belong to the same type*, written $\chi_1 \equiv \chi_2$, iff

(a) $\chi_1(p) = \chi_2(p)$ for almost all primes p; and

(b) if $\chi_1(p) \neq \chi_2(p)$, then both $\chi_1(p)$ and $\chi_2(p)$ are finite.

The *type* $\tau(x)$ of x is the $\equiv$-class containing $\chi(x)$.

If $0 \neq G \leqslant \mathbb{Q}$, then $\tau(G) = \tau(x)$, where x is an arbitrary nonzero element of G.

Theorem 2 (Baer 1937) *If $0 \neq G, H \leqslant \mathbb{Q}$, then $G \cong H$ iff $\tau(G) = \tau(H)$.*

Fix $n \geq 2$. Kurosh and Malcev independently found *unsatisfactory* complete invariants of the form

$$\langle M_p \mid p \in \mathbb{P} \rangle \qquad M_p \in GL_n(\mathbb{Q}_p)$$

Why unsatisfactory? Because the problem of deciding whether two sequences are equivalent is as difficult as deciding whether the corresponding groups are isomorphic!

Definition 3 *$R(\mathbb{Q}^n)$ is the set of all subgroups $G \leqslant \mathbb{Q}^n$ of rank n.*

The classification problem for $R(\mathbb{Q}^n)$ can be reduced to that for $R(\mathbb{Q}^{n+1})$ by the explicit map

$$R(\mathbb{Q}^n) \to R(\mathbb{Q}^{n+1})$$
$$A \mapsto A \oplus \mathbb{Q}$$

Question 4 *Does there exist an* explicit *map $f : R(\mathbb{Q}^{n+1}) \to R(\mathbb{Q}^n)$ such that*

$$A \cong B \quad \textit{iff} \quad f(A) \cong f(B)?$$

Question 5 *Which functions* $f : \mathbb{R} \to \mathbb{R}$ *are explicit?*

An analogue of Church's Thesis:

$$\text{EXPLICIT} \subseteq \text{BOREL}$$

BOREL = MEASURABLE modulo null

X is a standard Borel space; i.e. a complete separable metric space equipped with its σ-algebra of Borel subsets.

e.g. $\mathbb{R}$, $[0,1]$, $\mathbb{Q}_p$, ...

Some less obvious examples:

- $\mathcal{P}(\mathbb{Q}^n) = 2^{\mathbb{Q}^n}$
- The space $R(\mathbb{Q}^n)$ of rank n groups
- The space of finitely generated groups
- The space of countable groups

Theorem 6 *There exists a unique uncountable standard Borel space up to isomorphism.*

Definition 7 *Let E, F be equivalence relations on X, Y.*

(a) $E \leq_B F$ iff there exists a Borel function $f : X \to Y$ such that

$$xEy \iff f(x)Ff(y).$$

f is called a Borel reduction from E to F.

(b) $E \sim_B F$ iff both $E \leq_B F$ and $F \leq_B E$.

(c) $E <_B F$ iff both $E \leq_B F$ and $E \nsim_B F$.

Let $\cong_n$ denote the isomorphism relation on $R(\mathbb{Q}^n)$. Then:

$$(\cong_1) \leq_B \underbrace{(\cong_2) \leq_B (\cong_3) \leq_B (\cong_4) \leq_B \cdots}_{\text{Hard ?}}$$

Using the theory of Borel equivalence relations

- $\cong_1$ is the $<_B$-least nontrivial classification problem.

- Let E_∞ denote the isomorphism relation on the space of finitely generated groups. Then $(\cong_n) \leq_B E_\infty$ for all $n \geq 1$.

Conjecture 8 (Hjorth-Kechris) $(\cong_2) \sim_B (E_\infty)$

Note that if A, $B \in R(\mathbb{Q}^n)$, then

$$A \cong B \quad \text{iff} \quad \exists g \in GL_n(\mathbb{Q}) \quad g(A) = B.$$

We want to show that there doesn't exist a Borel map

$$f : R(\mathbb{Q}^{n+1}) \to R(\mathbb{Q}^n)$$

A, B lie in the same $GL_{n+1}(\mathbb{Q})$-orbit iff $f(A)$, $f(B)$ lie in the same $GL_n(\mathbb{Q})$-orbit.

Lattices in Lie Groups:

Let G be a connected simple Lie group with $\mathbb{R}$-rank$(G) \geq 2$.

e.g. $SL_m(\mathbb{R})$ for $m \geq 3$.

Let ν be a fixed Haar measure on G. Then $\Delta \leqslant G$ is a *lattice* iff

- Δ is a discrete subgroup
- $\nu(G/\Delta) < \infty$.

e.g. $SL_m(\mathbb{Z})$ is a lattice in $SL_m(\mathbb{R})$.

A suggestive non-example:
Let $G = \mathbb{R} \oplus \mathbb{R}$ and $\Delta = \mathbb{Z} \oplus \mathbb{Z}$. Then G/Δ is the torus.

The Measure Context (Zimmer):
Let G be a connected centreless simple Lie group with $\mathbb{R}$-rank$(G) \geq 2$ and let Δ be a lattice in G. Suppose that

- X is a standard Borel Δ-space.
- Δ acts freely on X.
- μ is a Δ-invariant ergodic probability measure on X.
- E_{Δ}^{X} is the orbit equivalence relation on X.

Question 9 *To what extent does the data*

$$(X, E_{\Delta}^{X}, \mu)$$

"remember" the group Δ and its action on X?

Definition 10 (X_0, Δ_0, μ_0) *and* (X_1, Δ_1, μ_1) *are* orbit equivalent *iff there exist*

- Δ_i*-invariant Borel subsets* $Y_i \subseteq X_i$ *with* $\mu_i(Y_i) = 1$

- *a measure preserving Borel bijection* $f : Y_0 \to Y_1$

such that for all a, $b \in Y_0$,

$$\Delta_0 \cdot a = \Delta_0 \cdot b \iff \Delta_1 \cdot f(a) = \Delta_1 \cdot f(b).$$

Superrigidity for optimists:
Does orbit equivalence imply isomorphism?

Unfortunately, not ...

Theorem 11 (Connes-Feldman-Weiss) *If H_0, H_1 are amenable, then (X_0, H_0, μ_0) and (X_1, H_1, μ_1) are orbit equivalent.*

However ...

Theorem 12 (Zimmer Superrigidity) *With the above hypotheses, if (X_0, Δ_0, μ_0) and (X_1, Δ_1, μ_1) are orbit equivalent, then $G_0 \cong G_1$.*

A reminder ...

We want to show that there doesn't exist a Borel map

$$f : R(\mathbb{Q}^{n+1}) \to R(\mathbb{Q}^n)$$

A, B lie in the same $GL_{n+1}(\mathbb{Q})$-orbit iff $f(A)$, $f(B)$ lie in the same $GL_n(\mathbb{Q})$-orbit.

Unfortunately ...

- $GL_n(\mathbb{Q})$ is not a lattice.

- There does not exist a $GL_n(\mathbb{Q})$-invariant probability measure on $R(\mathbb{Q}^n)$.

- $GL_n(\mathbb{Q})$ does not act freely on $R(\mathbb{Q}^n)$.

But we can recover ...

Suppose that $f : R(\mathbb{Q}^{n+1}) \to R(\mathbb{Q}^n)$ is such a Borel map.

- Consider the action of $SL_{n+1}(\mathbb{Z})$ on $R(\mathbb{Q}^{n+1})$.

- There exists an ergodic $SL_{n+1}(\mathbb{Z})$-invariant probability measure μ on $R(\mathbb{Q}^{n+1})$. In fact, there exists a class of p-local groups $R^*(\mathbb{Q}^{n+1}) \subseteq R(\mathbb{Q}^{n+1})$ such that
$$(R^*(\mathbb{Q}^{n+1}), GL_{n+1}(\mathbb{Q}))$$
is essentially the same as
$$(PG(n, \mathbb{Q}_p), GL_{n+1}(\mathbb{Q})).$$

- Slightly oversimplifying ... reduce to the case when the stabiliser of each $f(A)$ is a *fixed* group L. Then we obtain a free action of $N(L)/L$.

Borel analogue of Zimmer superrigidity:

Suppose that there is a Borel reduction

$$f : R(\mathbb{Q}^{n+1}) \to R(\mathbb{Q}^n).$$

Then $PSL_{n+1}(\mathbb{R})$ is involved in $PSL_n(\mathbb{R})$; i.e. there exist Lie subgroups

$$N \trianglelefteq H \leqslant PSL_n(\mathbb{R})$$

such that

$$H/N \cong PSL_{n+1}(\mathbb{R}),$$

which is a contradiction!

Theorem 13 (S.T. 2000) *The complexity of the classification problem for the torsion-free abelian groups of rank n increases strictly with n.*

Actually we also need ...

Theorem 14 (Hjorth 1998) *Rank 1 groups are strictly easier than rank 2 groups.*

The essential point ...

$GL_1(\mathbb{Q}) = \mathbb{Q}^*$ is amenable.

$GL_2(\mathbb{Q})$ is nonamenable.

Question 15 *Is the classification problem for $R(\mathbb{Q}^2)$ "genuinely difficult"?*

Question 16 *Does there exist a Borel equivalence relation E such that*

$$(\cong_1) <_B E <_B (\cong_2)?$$

Definition 17 *An abelian group A is said to be p-local iff A is q-divisible for every prime $q \neq p$.*

Definition 18 *$R^{(p)}(\mathbb{Q}^n)$ is the standard Borel space of all p-local subgroups $G \leqslant \mathbb{Q}^n$ of rank n.*

Definition 19 *The isomorphism relation on $R^{(p)}(\mathbb{Q}^n)$ is denoted by $\cong_n^{(p)}$.*

Conjecture 20 (S.T. 2000) *If $p \neq q$ are distinct primes, then $\cong_2^{(p)}$ and $\cong_2^{(p)}$ are incomparable with respect to Borel reducibility. Hence*

$$(\cong_1) <_B (\cong_2^{(p)}) <_B (\cong_2).$$

Theorem 21 *For each prime p, let E^p be the orbit equivalence relation arising from the natural action of $GL_2(\mathbb{Q})$ on the projective line $\mathbb{Q}_p \cup \{\infty\}$.*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

Then $\cong_2^{(p)} \sim_B E^p$.

Theorem 22 (S.T. 2002) *Fix some $n \geq 3$. If $p \neq q$ are distinct primes, then the classification problems for p-local and q-local torsion-free abelian groups of rank n are incomparable with respect to Borel reducibility.*

Makes use of

- Zimmer superrigidity for $SL_n(\mathbb{R})$.

- The Ratner measure classification theorem for real Lie groups.

- The Kazhdan property for $SL_n(\mathbb{Z})$.

Theorem 23 (Hjorth-S.T. 2004) *If $p \neq q$ are distinct primes, then $\cong_2^{(p)}$ and $\cong_2^{(p)}$ are incomparable with respect to Borel reducibility. Hence*

$$(\cong_1) <_B (\cong_2^{(p)}) <_B (\cong_2).$$

Makes use of

- Zimmer superrigidity for

$$SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p).$$

- The Margulis-Tomanov measure classification theorem for products of real and p-adic Lie groups.

- Property (τ) for $SL_2(\mathbb{Z}[1/p])$.