NOTES ON THE BAIZHANOV'S THEOREM ON AN EXPANSION OF A WEAKLY O-MINIMAL THEORY

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ABSTRACT. A totally ordered structure is called *weakly o-minimal*, if any definable with parameters subset is a finite union of convex sets. A theory is weakly o-minimal, if all its models are.

In this notes I give an explanation of the proof of the Baizhanov's theorem: an expansion of a model with a weakly o-minimal theory by an unary convex predicate has a weakly o-minimal theory. [B0, B2]

1. INTRODUCTION

Throughout the notes M will be a model with a weakly o-minimal theory, N an elementary $|M|^+$-saturated extension of M. All considered sets A, B, C, ... will be such that N is $|A \cup B \cup C \cup \dots|^+$-saturated. We shall consider an expansion of M by a unary convex predicate.

We write $A < B$ iff for any $a \in A$ and for any $b \in B$, $a < b$. Let C be a subset of M. Then

$$C_M^+ \triangleq \{a \in M : \text{for any } c \in C \text{ we have } M \models c < a\}$$
$$C_M^- \triangleq \{a \in M : \text{for any } c \in C \text{ we have } M \models a < c\}$$

Note, if $C = \emptyset$, then both C_M^+ and C_M^- is equal to M. Observe that if C is definable, then both C_M^+ and C_M^- are definable. Usually, we shall apply these notations to subsets of N and write just C^+, C^-.

Definition 1.1. Subset C of an ordered structure M is called a *cut*, if it is an initial segment, i.e. if $(C_M^+)_M^- = C$.

Claim that given a cut C we can define a type $s \in S_\varphi(M)$ for $\varphi(x; y) = x < y$ by $x > d \in s$ iff $d \in C$. By the same way any φ-type defines a cut. So, sometimes we call a φ-type by a cut. Note that any cut has at most two extensions to complete types over M, whose realizations are always convex [MMS, K]. If $p \in S_1(M)$, we write C_p for a unique cut, which is consistent with p, i.e. for the φ-part of type p.

Given a convex set I we shall say that the left border of I is definable, if I^- is a definable set. Similarly, the right border of I is definable, if I^+ is a definable set.

Definition 1.2. [B0] Non-isolated one-type p over a set $A \subset N$ is called *irrational*, if both borders of $p(N)$ are not definable, otherwise we call a type *quasirational*. Moreover, a quasirational type p is called *quasirational to the right*, if the right border of $p(N)$ is definable, otherwise it is called *quasirational to the left*.

In particular, a cut is called *irrational*, if it is not definable, or that is the same, if the corresponding φ-type is not definable; otherwise it is called *quasirational*

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Notion of a quasirational type is a generalization of the notion of a rational type, introduced by Lascar–Poizat in 1979. In an o-minimal model any quasirational one-type over an algebraically closed set is rational. Note that if a border of $p(N)$ is definable for $p \in S_1(A)$, then this border is definable over A. Claim that if $p \in S_1(M)$ is irrational, then $\sup(C_p)$ is not in model M. Observe, that a one-type over a model is definable iff the corresponding cut is definable.

To prove the theorem it is sufficient by [B0] to prove that we can eliminate the quantification 'there exists in model M':

Theorem 1.3. *For any $\bar{\alpha} \in N \setminus M$ there exists $\bar{\gamma} \in N \setminus M$ with the following property: for any $L(M \cup \bar{\alpha})$-formula $\phi(x, \bar{z}, \bar{\alpha})$ there exists an $L(M \cup \bar{\alpha} \cup \bar{\gamma})$-formula $K_\phi(\bar{z}, \bar{\alpha}, \bar{\gamma})$ such that the following holds:*

$$\forall \bar{a} \in M\ [\exists b \in M, N \models \phi(b, \bar{a}, \bar{\alpha}) \Longleftrightarrow N \models K_\phi(\bar{a}, \bar{\alpha}, \bar{\gamma})].$$

Below we sketch the proof of Theorem 1.3. So, given an $(M \cup \bar{\alpha})$-definable formula ϕ, we have to investigate the question if the property that $\phi(N) \cap M = \emptyset$ is definable. By weak o-minimality, $\phi(N, \bar{a}, \bar{\alpha})$ is a finite union of convex sets, which are definable over the same parameters as ϕ. Denote them by $\phi_i(N, \bar{a}, \bar{\alpha})$, for $i < n$. So, if $\phi(N, \bar{a}, \bar{\alpha}) \cap M = \emptyset$, then $\phi_i(N, \bar{a}, \bar{\alpha}) \subset s_{i,\bar{a}}(N)$ for some cut $s_{i,\bar{a}}$ in $S_\varphi(M)$ for $\varphi(x; y) \triangleq x < y$.

Case 1. Suppose that for a given $\bar{a} \in M$ and $i < n$, cut $s_{i,\bar{a}}$ is irrational, i.e. there is a unique extension of $s_{i,\bar{a}}$ to a complete type $q_{i,\bar{a}}$ over M. Then by compactness there are $\delta^1_{i,\bar{a}}, \delta^2_{i,\bar{a}} \in q_{i,\bar{a}}(N)$ such that $\delta^1_{i,\bar{a}} < \phi_i(N, \bar{a}, \bar{\alpha}) < \delta^2_{i,\bar{a}}$. Since N is $|M|^+$-saturated, there exists $\tau_{i,\bar{a}} \in \mathrm{Aut}(N/M)$ such that $\tau_{i,\bar{a}}(\delta^1_{i,\bar{a}}) = \delta^2_{i,\bar{a}}$. Then

$$\delta^1_{i,\bar{a}} < \phi_i(M, \bar{a}, \bar{\alpha}) < \delta^2_{i,\bar{a}} < \phi_i(M, \bar{a}, \tau_{i,\bar{a}}(\bar{\alpha})), \text{ so}$$

$$\phi_i(M, \bar{a}, \bar{\alpha}) \cap \phi_i(M, \bar{a}, \tau_{i,\bar{a}}(\bar{\alpha})) = \emptyset$$

Let $\mathbb{X} \triangleq \{\bar{\alpha}\} \cup \{\tau_{i,\bar{a}}(\bar{\alpha}) : q_{i,\bar{a}} \text{ is irrational}, \bar{a} \in M,\ i < n\}$. Since we take only one automorphism for each $i < n$ and $\bar{a}$ in the definition of $\mathbb{X}$, $|\mathbb{X}| = |M|$. Clearly,

$$\phi'(x, \bar{y}, \mathbb{X}) \triangleq \forall \bar{z}(\bar{z} \in \mathbb{X} \to \phi(x, \bar{y}, \bar{z}))$$

has no intersection with irrational 1-types over M, although at the moment we do not know if $\mathbb{X}$ is definable, even with parameters in N.

Case 2. Assume that cut $s_{i,\bar{a}}$ is quasirational. Then there are two types extending this cut, say, $q_{i,\bar{a},r}$, which is quasirational to the right, and $q_{i,\bar{a},l}$, which is quasirational to the left. Let this cut be definable over M by some formula U. We have that $\phi_i(N, \bar{a}, \alpha) \subseteq q_{i,\bar{a},l}(N) \cup q_{i,\bar{a},r}(N)$. Consider $q_{i,\bar{a},r}$ and suppose that $\phi_i(N, \bar{a}, \alpha) \cap q_{i,\bar{a},r}(N) \neq \emptyset$. Since N is sufficiently saturated and $q_{i,\bar{a},r}$ is quasirational to the right, there exists $\delta_1 \in q_{i,\bar{a},r}(N)$ such that $\delta_1 < \phi_i(N, \bar{a}, \alpha)$.

Take an arbitrary $\delta_2 \in \phi_i(N, \bar{a}, \alpha) \cap q_{i,\bar{a},r}(N)$. There exists $\tau \in \mathrm{Aut}(N/M)$ such that $\tau(\delta_1) = \delta_2$. Then $\delta_2 < \phi_i(N, \bar{a}, \tau(\bar{\alpha}))$. Let

$$\begin{aligned}\mathbb{Y}_{i,\bar{a},r} \quad = \quad & \{\tau(\bar{\alpha}) : \tau \in \mathrm{Aut}(N/M) \text{ and for some } \delta_2 \in \phi_i(N, \bar{a}, \alpha) \cap q_{i,\bar{a},r}(N), \\ & \tau(\delta_1) = \delta_2\}\end{aligned}$$

So, it is obvious that $\emptyset = q_{i,\bar{a},r}(N) \cap \bigcap\{\phi_i(N, \bar{a}, \bar{z}) : \bar{z} \in \mathbb{Y}_{i,\bar{a},r} \text{ or } \bar{z} = \bar{\alpha}\}$. Similarly, we can define $\mathbb{Y}_{i,\bar{a},l}$. Let

$$\mathbb{Y} \quad = \quad \bigcup\{\mathbb{Y}_{i,\bar{a},t} : s_{i,\bar{a}} \text{ is quasirational}, \bar{a} \in M,\ i < n,\ t = l, r\} \cup \{\bar{\alpha}\}$$

Then $\phi'(x,\bar a,\mathbb{Y})$ has no intersection with quasirational types. Thus if $\phi(N,\bar a,\bar\alpha)\cap M=\emptyset$, then $\phi'(N,\bar a,\mathbb{X}\cup\mathbb{Y})=\emptyset$. If some element of M satisfies $\phi(x,\bar a,\bar\alpha)$, then it satisfies $\phi'(x,\bar a,\mathbb{X}\cup\mathbb{Y})$, since all elements of $\mathbb{X}\cup\mathbb{Y}$ have the same type over M. Thus we have reduced Theorem 1.3 to

Lemma 1.4. *Assume that for any $\bar\alpha\in N$ there exists $\bar\gamma\in N$, such that for any formula $\phi(x,\bar y,\bar\alpha)$ there exists $(M\cup\bar\alpha\cup\bar\gamma)$-definable formula $F(\bar z)$ with $l(\bar z)=l(\bar\alpha)$ such that $\mathbb{X}\cup\mathbb{Y}\subseteq F(N)\subseteq p_\phi(N)$ for some $\mathbb{X}$ and $\mathbb{Y}$, where p_ϕ is a ϕ-type of $\bar\alpha$ over M. Then*

$$K_\phi(\bar y,\bar\alpha,\bar\gamma)\triangleq\exists x\forall\bar z(F(\bar z)\rightarrow\phi(x,\bar y,\bar z))$$

satisfies the assertion of Theorem 1.3.

In fact, we shall not work directly with $\mathbb{X}$ and $\mathbb{Y}$. Nevertheless, these sets give some intuition and are closed to our main notion "big enough", which will be introduced in the next section. We shall often write that F satisfies the assertions of Lemma 1.4, meaning that the corresponding formula $K_\phi(\bar y,\bar\alpha,\bar\gamma)\triangleq\exists x\forall\bar z(F(\bar z)\rightarrow\phi(x,\bar y,\bar z))$ satisfies the assertion of Theorem 1.3.

2. $\bar\alpha$ IS A SINGLETON

In this section we show existence of F from Lemma 1.4, when α is a singleton.
Case 1. Let $\alpha\models p\in S_1(M)$. Suppose that for all $\bar a\in M$ and $i<n$ types $q_{i,\bar a}$ are irrational. Since $|\mathbb{X}|=|M|$, by compactness there are γ_1 and $\gamma_2\in p(N)$, such that $\gamma_1<\mathbb{X}<\gamma_2$. Let

$$\begin{aligned}F_1(z,\gamma_1,\gamma_2)&\triangleq\gamma_1<z<\gamma_2\\K_\phi(\bar y,\gamma_1,\gamma_2)&\triangleq\exists x\forall z(F_1(z,\gamma_1,\gamma_2)\rightarrow\phi(x,\bar y,z))\end{aligned}$$

Then K_ϕ satisfies the assertions of Theorem 1.3.

To work independently of the choice of ϕ and to clarify where γ_i are taken from, Baizhanov in [B0] suggested the following definition.

Definition 2.1. Let $r\in S_1(A)$, A, $B\subset N$ such that N is $|B\cup A|^+$-saturated. Then *a neighbourhood of set B in type r* is the following set:

$$\begin{aligned}V_r(B)\ \triangleq\ &\{\gamma\in N:\gamma\in H(N)\text{ for some }L(A\cup B)\text{-formula }H(x)\text{ such that}\\&\gamma_1<H(N)<\gamma_2\text{ for some }\gamma_1,\gamma_2\in r(N)\}\end{aligned}$$

Let $\bar\alpha=\langle\alpha_1,\alpha_2,\ldots,\alpha_k\rangle$ then $V_r(\bar\alpha)\triangleq V_r(\{\alpha_1,\alpha_2,\ldots,\alpha_k\})$.

Claim that if p is irrational, then $tp(\gamma/M\cup\alpha)$ is a coheir of p, whenever $\gamma\in p(N)\setminus V_p(\alpha)$. So, there are two coheirs, the first one is less than $V_p(\alpha)$, the second one is greater. The next lemma is obvious.

Lemma 2.2. *If δ_1, $\delta_2\in p(N)$, and both are either less than $V_p(\alpha)$, or greater, then $tp(\delta_1/M\cup\alpha)=tp(\delta_2/M\cup\alpha)$. (The same is true for a quasirational type p.)*

Proof. Assume the contrary. We can assume that $\delta_1<\delta_2<V_p(\alpha)$. Then for some $(M\cup\alpha)$-definable formula $H(x)$ we have $N\models\neg H(\delta_1)\wedge H(\delta_2)$. There is no cost to suppose that $H(x)\wedge x<\alpha$ defines a convex set. This set contains δ_2 and bounded by δ_1. Then $\delta_2\in V_p(\alpha)$, it is a contradiction. $\square$

Lemma 2.3. *Let $p\in S_1(A)$ be irrational. And let $\alpha,\beta\in p(N)$ be such that $V_p(\alpha)\cap V_p(\beta)=\emptyset$. Then for any $r\in S_1(A)$, $V_r(\alpha)\cap V_r(\beta)=\emptyset$.*

Proof. Assume the contrary, there are A-definable formulae $\phi_1(x,\alpha)$ and $\phi_2(x,\beta)$, such that $\phi_1(N,\alpha) \subset r(N)$, $\phi_2(N,\beta) \subset r(N)$, and $\phi_1(N,\alpha) \cap \phi_2(N,\beta) \neq \emptyset$. We can assume that $\phi_1(N,\alpha)$ and $\phi_2(x,\beta)$ are convex and $\alpha < \beta$.

Consider the case $\phi_1(N,\alpha) \subseteq \phi_2(N,\beta)$. Let $\psi(x,\beta) \triangleq (\emptyset \neq \phi_1(N,x) \subseteq \phi_2(N,\beta))$, and $\psi_i(N,\beta)$ a convex part of $\psi(N,\beta)$ containing α. Since $V_p(\alpha) \cap V_p(\beta) = \emptyset$, so $\alpha \notin V_p(\beta)$. And since p is irrational, there is a convex formula $\theta_1 \in p$ such that $\theta_1(x) \wedge x < \beta$ implies $\psi_i(N,\beta)$. Taking into consideration irrationality of p, one can see that there is a convex formula $\theta_2 \in p$ whose left border are greater than the left border of $\theta_1(N)$. Then $\theta(x) \triangleq \theta_1(x) \wedge x < \theta_2(N)$ implies $\psi_i(x,\beta)$.

Consider $(\exists x)\,\theta(x) \wedge \phi_1(y,x)$. This is an A-definable formula, which implies $\phi_2(y,\beta)$, and the latest implies $r(y)$—a type over A. So, we obtain a contradiction with the definition of a type.

Up to symmetry it is sufficient to consider the case when both intersections $\phi_1(N,\alpha) \cap \phi_2(N,\beta)$ and $\neg\phi_1(N,\alpha) \cap \phi_2(N,\beta)$ are not empty. Let $\psi(x,\beta)$ say that some border of $\phi_1(N,x)$ is inside $\phi_2(N,\beta)$. Without loss of generality we may assume that this border is right. By the similar to the previous case reason there exists $\theta(x) \in p(x)$ such that some border of $(\exists x)\,\theta(x) \wedge \phi_1(N,x)$ is inside $\phi_2(N,\beta) \subset r(N)$. Since there are δ_1, $\delta_2 \in r(N)$ such that $\delta_1 < \phi_2(N) < \delta_2$, we obtain that A-definable formula divides a type over A, that is a contradiction. $\square$

So, it is not difficult to show that to satisfy the assertions of Lemma 1.4 and Theorem 1.3 it is sufficient to pose that $\gamma_1 < V_p(\alpha) < \gamma_2$.

We shall investigate the properties of a neighbourhood in Section 3. One of the most interesting is the following:

Lemma 2.4. [B2, B3] *Let $\alpha \models p \in S_1(M)$. Then $E(x,y) \triangleq x \in V_p(y)$ is an equivalence relation on set $p(N)$ (not necessary definable).*

Case 2. Assume that for some $\bar{a} \in M$ and for some $i < n$, we have that cut $s_{i,\bar{a}}$ is definable. In Section 3 we shall show

Lemma 2.5. *If at least one cut $s_{i,\bar{a}}$ or type p is definable, then all cuts $s_{i,\bar{a}}$ are definable as well as type p.*

We shall show in Section 3 that there exists $\gamma \in p(N)$ such that all $\delta \in \mathbb{Y}$ are between γ and the definable border of p. Suppose that the right border of p is definable, i.e. $p(N)^+ = U(N)$. Let

$$\begin{aligned} F_1(z,\gamma) &\triangleq \gamma < z \wedge \neg U(z) \\ K_\phi(\bar{y},\gamma) &\triangleq \exists x \forall z (F_1(z,\gamma) \to \phi(x,\bar{y},z)) \end{aligned}$$

Then K_ϕ satisfies the assertions of the theorem, because $\mathbb{Y} \subset F_1(N,\gamma_1,\gamma_2)$ and $\mathbb{X} = \emptyset$ by Lemma 2.5.

As in Case 1, to work independently of the choice of ϕ and to clarify where γ is taken from, Baizhanov in [B0] suggested the following definition.

Definition 2.6. Let $r \in S_1(A)$, A, $B \subset N$ such that N is $|B \cup A|^+$-saturated. Then *a quasineighbourhood of set B in type r* is the following set:

$$\begin{aligned} QV_r(B) \;\triangleq\; &\{\gamma \in N : \gamma \in H(N) \text{ for some } L(A \cup B)\text{-formula } H(x) \text{ such that} \\ &H(N) \subset r(N)\} \end{aligned}$$

Let $\bar{\alpha} = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$ then $QV_r(\bar{\alpha}) \triangleq QV_r(\{\alpha_1, \alpha_2, \ldots, \alpha_k\})$.

Claim, for a quasirational type p, $tp(\delta/M \cup \alpha)$ is a coheir of p, provided that $\delta \in p(N) \setminus QV_p(\alpha)$. If $\delta \in QV_p(\alpha) \setminus V_p(\alpha)$, then $tp(\delta/M \cup \alpha)$ is a heir of p.

We shall show later that to satisfy the assertions of Theorem 1.3 it is sufficient to pose that $\gamma \notin QV_p(\alpha)$. We shall investigate the properties of a quasineighbourhood in Section 3 and give one property right now.

Lemma 2.7. *Let p be an irrational type. Then for any B, $QV_p(B) = V_p(B)$.*

Proof. The both borders of $p(N)$ are not definable. Then for any B-definable formula ψ, if $\psi(N) \subseteq p(N)$, then the following set of formulae

$$\{\theta(x_1) \wedge \theta(x_2) \wedge (x_1 < \psi(N) < x_2) \ : \ \theta \in p, \ \psi(N) \subset p(N), \ \psi \text{ is } B\text{-definable}\}$$

is consistent and realized in N. □

In the general case as in the case when $\bar{\alpha}$ is a singleton, we shall work with condition that a formula contains quasineighbourhood of $\bar{\alpha}$. We express this in the following definition.

Definition 2.8. We say that a formula $\Phi(z)$ is *big enough in a one-type r for $\bar{\beta}$*, if the following hold:

(1) If r is irrational, then for some δ_1, $\delta_2 \in r(N)$ such that $\delta_1 < V_r(\bar{\beta}) < \delta_2$ we have $(\delta_1, \delta_2) \subset \Phi(N)$.
(2) If r is quasirational, then for some $\delta \in r(N) \setminus QV_r(\bar{\beta})$, set $\Phi(N)$ contains all elements between δ and the definable border of $r(N)$.

One-type q is said to be *big enough in a one-type r for $\bar{\beta}$*, if all formulae of q are.

3. Non-orthogonality of 1-types over a set

In this section we show existence of γ for Case 2 in Section 2 and prove Lemma 2.5. First we claim that all considered types p and $q_{i,\bar{a}}$ are non weakly orthogonal following the next definition:

Definition 3.1. Let $p \in S_n(A)$ and $q \in S_k(A)$. We say that p and q are *weakly orthogonal* and write $p \perp^w q$, if $p(\bar{x}) \cup q(\bar{y})$ defines a complete type in $S_{n+k}(A)$.

In the opposite case we say that they are *non weakly orthogonal* and write $p \not\perp^w q$.

We say that a tuple $\bar{\alpha}$ is *weakly orthogonal* to $q \in S(A)$ and write $\bar{\alpha} \perp^w q$, if $tp(\bar{\alpha}/A) \perp^w q$.

Suppose that $p \not\perp^w q$ for some p, $q \in S_1(A)$, where $A \subset N$ and N is $|A|^+$-saturated. Then by the definition there are $\alpha \in p(N)$ and β_1, $\beta_2 \in q(N)$ and some A-definable formula $H(x, y)$ such that $N \models H(\beta_1, \alpha) \wedge \neg H(\beta_2, \alpha)$. For definiteness we can suppose that $\beta_1 < \beta_2$. Also we may assume that $H(N, \alpha)$ is convex and moreover that it is a cut, i.e. if $a \in H(N, \alpha)$ and $b < a$, then $b \in H(N, \alpha)$. Indeed, we can take $H'(x, y) \triangleq \exists z(x < z \wedge H(z, y))$ as $H(x, y)$. Thus we can consider H as a function from $p(N)$ to a set of cuts in N. Since a set of cuts is ordered, we can speak of monotonicity of H.

Definition 3.2. Formula $H(x, y)$ is *increasing* in y on a set B, if for any $b_1 < b_2 \in B$ we have that $H(N, b_1) < H(N, b_2)^+$.

Formula $H(x, y)$ is *decreasing* in y on a set B, if for any $b_1 < b_2 \in B$ we have that $H(N, b_2) < H(N, b_1)^+$.

$H(x, y)$ is monotone on B if it is either increasing, or decreasing.

Note, that in this definition increasing of a formula is not strict.

Theorem 3.3. *Let p, $q \in S_1(A)$ be non weakly orthogonal. Let N be $|A|^+$-saturated. Then there is an A-definable formula $H(x,y)$ such that*

1. *Whenever $\alpha \in p(N)$, $H(N,\alpha)$ is a cut such that both $H(N,\alpha)$ and $\neg H(N,\alpha)$ are consistent with $q(N)$;*
2. *$H(x,y)$ is monotone in y on $p(N)$;*
3. *If $H(x,y)$ is increasing in y on $p(N)$, then whenever $\beta \in q(N)$, $H(\beta,N)$ is a cut such that both $H(\beta,N)$ and $\neg H(\beta,N)$ are consistent with $p(N)$, and $H(x,y)$ is increasing in x on $q(N)$;*
4. *If $H(x,y)$ is decreasing in y on $p(N)$, then whenever $\beta \in q(N)$, $\neg H(\beta,N)$ is a cut such that both $H(\beta,N)$ and $\neg H(\beta,N)$ are consistent with $p(N)$, and $\neg H(x,y)$ is decreasing in x on $q(N)$;*

Proof. The first item we have discussed above. Monotonicity follows from [V]. For the class of non-uniformly weakly o-minimal structures of a finite depth it has been shown in [BTV].

(2) can be done directly as in [B0], nevertheless we shall use results of [MMS] and [V]. By [MMS] for any formula $H(x,y)$ there exists a finite definable partition of N into X, I_1, $\ldots$, I_n, such that X is finite, and on each I_j formula H is tidy, i.e. for any $a \in I_j$ there exists an open interval $K \subseteq I_j$ containing x, on which H is monotone in y, and the kinds of monotonicity are the same for all elements of I_j. Consider an equivalence relation $E_1(u,v)$ defined as: u and v are in the same E_1-class iff $H(x,y)$ is monotone in y on $[u,v] \cup [v,u]$.

If all elements of $p(N)$ are in the same E_1-class, then $H(x,y)$ is monotone in y on $p(N)$. If not, consider an ordered structure $N_1 = (N/E_1, <, G_1)$, where

$$
\begin{aligned}
H_1(x,y) &\triangleq \exists z(E_1(y,z) \wedge H(x,z)),\\
G_1(y_1,y_2) &\triangleq H_1(N,y_1) < H_1(N,y_2)^+
\end{aligned}
$$

Model N_1 has a weakly o-minimal theory, so we can apply the result of [MMS] again.

If H_1 is not monotone in y on $p(N)$, we shall consider H_2, $\ldots$, H_n. Since the theory of N is weakly o-minimal, this chain has a bound. So, $H_n(x,y)$ is monotone in y on $p(N)$ for some $n < \omega$.

The last two items are simple corollaries of the properties of monotone functions. □

Corollary 3.4. *Let p and $q \in S_1(A)$ be non weakly orthogonal, and $H(x,y)$ an A-definable formula satisfying the assertions of Theorem 3.3. Let*

$$
\begin{aligned}
E_p(y_1,y_2) &\triangleq H(x,y) \textit{ is constant on } [y_1,y_2] \cup [y_2,y_1] \textit{ in } y\\
E_q(x_1,x_2) &\triangleq H(x,y) \textit{ is constant on } [x_1,x_2] \cup [x_2,x_1] \textit{ in } x
\end{aligned}
$$

Then H forms a bijection τ between definable cuts in $p(N)/E_p$ and $q(N)/E_q$, such that for any $(A \cup \bar\beta)$-definable cut C in $p(N)/E_p$, $\tau(C)$ are also $(A \cup \bar\beta)$-definable.

Moreover, if H is increasing, then τ is order preserving, i.e. for any cuts C_1, C_2 in $p(N)/E_p$, we have $C_1 < C_2$ iff $\tau(C_1) < \tau(C_2)$.

If H is decreasing, then τ is order inverting.

Proof. Let C be a cut in $p(N)/E_p$. We define τ as

$$\tau(C) \triangleq \begin{cases} \exists x' \exists y (y \in C \wedge H(x', y) \wedge E_q(x, x')), & \text{if } H \text{ is increasing;} \\ \exists x' \forall y (y \in C \wedge H(x', y) \wedge E_q(x, x')), & \text{if } H \text{ is decreasing.} \end{cases}$$

Since H is A-definable and monotone on $p(N)$, the rest is obvious. □

The next two corollaries easily follow from Theorem 3.3 and Corollary 3.4.

Corollary 3.5. *Let p, $q \in S_1(A)$ be non weakly orthogonal. Let $H(x, y)$ be an A-definable formula satisfying Theorem 3.3. Then*

(1) *If H is increasing, then the right (left) border of $p(N)$ is definable iff the right (left) border of $q(N)$ is definable;*
(2) *If H is decreasing, then the left (right) border of $p(N)$ is definable iff the right (left) border of $q(N)$ is definable;*
(3) *Thus, p is definable iff q is definable (Lemma 2.5);*
(4) *$\not\perp^w$ is an equivalence relation on $S_1(A)$.*

Proof. (1) Since the property for $H(x, y)$ be monotone is definable, it is monotone in y on some formula $\theta(y) \in p(y)$. Let $U(y)$ defines the right border of $p(N)$. Then $\exists y(\theta(y) \wedge U(y) \rightarrow H(x, y))$ defines the right border of $q(N)$.

(2) is similar. (3) and (4) are obvious. □

Corollary 3.6. *Let $p \in S_1(A)$. Let B and C be some sets. If $\emptyset \neq QV_p(C) \subseteq QV_p(B)$, then for any $q \in S_1(A)$, which is not weakly orthogonal to p, we have that $QV_q(C) \subseteq QV_q(B)$.*

Proof. It is obvious by Corollary 3.4. □

Let us return to Section 2, Case 2. Let $\alpha \models p \in S_1(A)$, where p is definable, and say that the right border of $p(N)$ is definable by an A-formula $U(y)$. Then there exists $\gamma \in p(N)$ which is less than $QV_p(\alpha)$. Obviously, $QV_p(\alpha) \subset QV_p(\gamma)$, because all elements between γ and the definable border of $p(N)$ are in $QV_p(\gamma)$. So, we obtain that

$$\emptyset = \bigcap \{QV_p(\delta) : \gamma < \delta \text{ and } \delta \in p(N)\}$$

By Corollary 3.6, we have that for any $\bar{a} \in M$, $i < n$, $t \in \{r, l\}$, and $q = q_{i,\bar{a},t}$

$$\emptyset = \bigcap \{QV_q(\delta) : \gamma < \delta \text{ and } \delta \in p(N)\}$$

that concludes the proof of the main theorem for the case when α is a single element.

Another useful property is the following:

Lemma 3.7. *Assume that for some $p \in S_1(A)$ and $r \in S_1(A \cup C)$, type r is big enough in p for some set B. Then for any $q \in S_1(A)$ with $q \not\perp^w p$ and for any $r' \in S_1(A \cup C)$, if $QV_q(B) \cap r'(N) \neq \emptyset$, then r' is big enough in q for B. In particular, $QV_q(B) \subset r'(N)$, and r' extends q.*

Proof. It follows from Corollary 3.4 and the definition of a type. □

4. General case. Locally isolated types

Notation. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$;

$$\begin{aligned}
\bar{\alpha}_m &\triangleq (\alpha_1, \ldots, \alpha_m);\\
\bar{\alpha}^{m+1} &\triangleq (\alpha^{m+1}, \ldots, \alpha_n);\\
p_m &\triangleq tp(\alpha_m / M \cup \bar{\alpha}_{m-1});\\
\mathbb{O}_m &\triangleq \{r \in S_1(M) : QV_r(\bar{\alpha}_{m-1}) = \emptyset \text{ and } QV_r(\bar{\alpha}_m) \neq \emptyset\}
\end{aligned}$$

First, we prove some properties of types in $\mathbb{O}_m(\bar{\alpha})$.

Lemma 4.1. *If $r \in \mathbb{O}_m$, then at most two types in $S_1(M \cup \bar{\alpha}_{m-1})$ extend r. Moreover, if r has exactly two extensions, then r is irrational. In particular, these extensions are non-isolated.*

Proof. If $r \perp^w \bar{\alpha}_{m-1}$, then r has a unique extension to a type over $M \cup \bar{\alpha}_{m-1}$ by the definition of weak orthogonality. So, let $r \not\perp^w \bar{\alpha}_{m-1}$, then there exists an $(M \cup \bar{\alpha}_{m-1})$-definable formula $\psi(x)$ such that both $r(x) \cup \psi(x)$ and $r(x) \cup \neg\psi(x)$ are consistent. Since $r \in \mathbb{O}_m$, so $\psi(N) \not\subseteq r(N)$. Thus there is no cost to assume that $\psi(N)$ is convex and defines a cut. Assume that another $(M \cup \bar{\alpha}_{m-1})$-definable formula $\psi'(x)$ satisfies the same property as $\psi(x)$. Then either $\psi(N) \subseteq \psi'(N)$, or $\psi'(N) \subseteq \psi(N)$. Say, the first inclusion holds. Then $\psi'(N) \setminus \psi(N) \subseteq r(N)$ and definable over $M \cup \bar{\alpha}_{m-1}$, consequently, it is empty and $\psi(N) = \psi'(N)$.

Now, assume that r has two extensions, and assume that r is quasirational, say, to the right. Then there is an $(M \cup \bar{\alpha}_{m-1})$-definable formula $U(x)$, which divides $r(N)$. There is no cost to suppose that $U(N)$ is a cut in N. Let $U_1(x)$ say that x is greater than $U(N)$ and less then the right border of $r(N)$. Formula U_1 is $(M \cup \bar{\alpha}_{m-1})$-definable and convex, consequently, $\emptyset \neq U_1(N) \subseteq QV_r(\bar{\alpha}_{m-1})$, a contradiction. $\square$

Suppose, $\phi_i(N, \bar{a}, \bar{\alpha}) \cap M = \emptyset$. As we claimed this before, in this case $\phi_i(N, \bar{a}, \bar{\alpha}) \subset r_1(N) \cup r_2(N)$, for some $r_1 \in S_1(M)$, and intersection of ϕ_i with at least one of $r_i(N)$ is not empty. Then $r_i \in \mathbb{O}_m$ for some m.

We shall construct a formula $F(\bar{z}, \bar{\gamma})$ containing $\mathbb{Z} \triangleq \mathbb{X} \cup \mathbb{Y}$, and $\bar{\gamma}$ by back induction on n. We shall have that

$$\emptyset = \bar{\gamma}_{n+1} \subseteq \bar{\gamma}_n \subseteq \ldots \subseteq \bar{\gamma}_1 = \bar{\gamma}$$

So, first we construct $F_n(\bar{\alpha}_{n-1}, z_n, \bar{\gamma}_n)$, which defines a set containing $\mathbb{Z}(\bar{\alpha}_{n-1}) \triangleq \{\bar{\delta}^n : \bar{\alpha}_{n-1}\bar{\delta}^n \in \mathbb{Z}\}$, then $F_{n-1}(\bar{\alpha}_{n-2}, z_{n-1}, \bar{\gamma}_{n-1})$ such that $F_n(\bar{\alpha}_{n-2}, z_{n-1}, z_n, \bar{\gamma}_n) \wedge F_{n-1}(\bar{\alpha}_{n-2}, z_{n-1}, \bar{\gamma}_{n-1})$ contains $\mathbb{Z}(\bar{\alpha}_{n-2}) \triangleq \{\bar{\delta}^{n-1} : \bar{\alpha}_{n-2}\bar{\delta}^{n-1} \in \mathbb{Z}\}$, etc. Let

$$\begin{aligned}
F^m(\bar{\alpha}_{m-1}, z_m, \ldots, z_n, \bar{\gamma}_m) &\triangleq \bigwedge_{i=m}^{n} F_i(\bar{\alpha}_{m-1}, z_m, \ldots, z_i, \bar{\gamma}_i)\\
F(z_1, \ldots, z_n, \bar{\gamma}) &\triangleq F^1(z_1, \ldots, z_n, \bar{\gamma}_1)\\
\phi^{n+1}(x, \bar{a}, \bar{\alpha}, \bar{\gamma}_{n+1}) &\triangleq \phi(x, \bar{a}, \bar{\alpha})\\
\phi^m(x, \bar{a}, \bar{\alpha}_{m-1}, \bar{\gamma}_m) &\triangleq \forall \bar{z}^m \left(F^m(\bar{\alpha}_{m-1}, \bar{z}^m, \bar{\gamma}_m) \rightarrow \phi(x, \bar{a}, \bar{\alpha}_{m-1}, \bar{z}^m)\right)
\end{aligned}$$

We shall prove by back induction on m that if $\phi(N, \bar{a}, \bar{\alpha})$ has an empty intersection with M, then $\phi^m(N, \bar{a}, \bar{\alpha}_{m-1}, \bar{\gamma}_m)$ has an empty intersection with $r(N)$, for any $r \in (\mathbb{O}_m \cup \ldots \cup \mathbb{O}_n)$. Then, obviously, $K_\phi(x, \bar{a}, \bar{\gamma}) \triangleq \exists x\ \phi^1(x, \bar{a}, \bar{\gamma})$ satisfies the assertion of Theorem 1.3, because $\mathbb{O}_1, \ldots, \mathbb{O}_n$ is a partition of the set of all types, which define cuts, containing a convex part of $\phi(N, \bar{a}, \bar{\alpha})$.

Lemma 4.2. *Assume that $\bar{\gamma}_{m+1} = \emptyset$ and that formula $F_m(\bar{\alpha}_{m-1}, z_m, \bar{\gamma}_m)$ is big enough in $p_m(N)$ for $\bar{\alpha}^m$. Assume also that for all $i \geq m$ formulae $F_i(\bar{\alpha}_{i-1}, z_i, \bar{\gamma}_i)$ isolate $\phi^{i+1}(x, \bar{y}, \bar{\alpha}_{i-1}, z_i)$-parts of types $p_i(z_i)$. Then $\phi^m(x, \bar{a}, \bar{\alpha}_{m-1}, \bar{\gamma}_m)$ is not consistent with any $r \in \mathbb{O}_m \cup \ldots \cup \mathbb{O}_n$, provided that $\phi(N, \bar{a}, \bar{\alpha}) \cap M = \emptyset$.*

Proof. Denote $\phi^{i+1}(x, \bar{y}, \bar{\alpha}_{i-1}, z_i)$-part of type $p_i(z_i)$ by $p_{i,\phi}$. Clearly, for any $i > m$

$$p_i(N) \subseteq F_i(N, \bar{\alpha}_{i-1}) \subseteq p_{i,\phi}(N)$$

Since N is $|M|^+$-saturated, each F_i is big enough in $p_i(N)$ for $\bar{\alpha}^i$ for $i > m$.

We prove the lemma by induction on m. Since the proof for the basic step and the induction step are the same, we prove the lemma just for m.

Assume the contrary, that ϕ^m is consistent with some $r \in \mathbb{O}_m$. In the virtue of Lemma 4.1, r has at most two extensions in $S_1(M \cup \bar{\alpha}_{m-1})$, say r_1 and r_2. We have that some convex part of $\phi^m(N)$ is inside $r(N)$, and then has non-empty intersection with at least one of $r_i(N)$, say $r_1(N)$. Thus $p_m \not\perp^w r_1$.

Since r_1 is non-isolated by Lemma 4.1, so p_m is non-isolated.

If p_m is irrational, then by Corollary 3.5, r_1 is irrational, and $r_1 = r_2$. Further, $F_m(N, \bar{\alpha}_{m-1}, \bar{\gamma}_m)$ contains two elements β_1 and β_2 such that $V_{p_m}(\beta_1) \cap V_{p_m}(\beta_2) = \emptyset$. By Lemma 2.3 we have that $V_{r_1}(\beta_1) \cap V_{r_1}(\beta_2) = \emptyset$. Note, that $\phi^{m+1}(x, \bar{a}, \bar{\alpha}_m)$ is $(M \cup \bar{\alpha}_m)$-definable, so some convex part of $\phi^{m+1}(N, \bar{a}, \bar{\alpha}_m)$ must be in $V_{r_1}(\alpha_m)$. By Corollary 3.4 we have that $\phi^{m+1}(N, \bar{a}, \bar{\alpha}_{m-1}, \beta_1) \cap \phi^{m+1}(N, \bar{a}, \bar{\alpha}_{m-1}, \beta_2) = \emptyset$.

So, let p_m be quasirational, say, to the right. As in the case when $\bar{\alpha}$ was a singleton, we have that

$$\emptyset = \bigcap \{QV_{p_m}(\beta) : \delta < \beta \text{ and } \beta \in p_m(N)\}$$

By Corollary 3.6, for any type r_1, which is non weakly orthogonal with p_m, we have that

$$\emptyset = \bigcap \{QV_{r_1}(\beta) : \delta < \beta \text{ and } \beta \in p_m(N)\}$$

that concludes the proof of the lemma. ☐

Corollary 4.3. *If $\bar{\gamma}_2 = \emptyset$ and $F_m(z_m, \bar{\alpha}_{m-1})$ isolates $\phi^{m+1}(x, \bar{y}, \bar{\alpha}_{m-1}, z_m)$-part of type $p_m(z_m)$ for each $m > 1$, and F_1 is chosen as in Section 2, then F satisfies the assertion of Lemma 1.4 and consequently K_ϕ satisfies the assertion of Theorem 1.3.*

In the following definition we describe the case when $\bar{\gamma}_{m+1}$ can be empty.

Definition 4.4. Let $r \in S(A)$. Then we say that $r(\bar{x})$ is *locally isolated*, if for any A-definable formula $\phi(\bar{x}, \bar{y})$ there exists $\Theta_\phi(\bar{x}) \in r$, which isolates $\phi(\bar{x}, \bar{y})$-part of type $r(\bar{x})$.

Since we have proved Theorem 1.3 provided that all p_m, for $m > 1$, are locally isolated, the last case is that some type p_m is not locally isolated. We shall show in Section 5 that in this case there is a formula $\psi(x, \bar{y}, \bar{\alpha}_{m-1})$, which converges to type p_m, i.e. roughly speaking, for any formula $\theta \in p_m$ there exists $\bar{a} \in M$ such that $\psi(N, \bar{a}, \bar{\alpha}_{m-1}) \subseteq \theta(N)$. We shall show that we can find $\bar{\gamma}_m \in N$ such that $\psi(z_m, \bar{\gamma}_m, \bar{\alpha}_{m-1})$ isolates ϕ^{m+1}-part of p_m, and any $(M \cup \bar{\alpha}_{m-1} \cup \bar{\gamma}_m)$-definable formula, which isolates ϕ^{m+1}-part of p_m, is big enough in p_m for $\bar{\alpha}^m$. In this case we put $F_i(z_m, \bar{\alpha}_{i-m}, \bar{\gamma}_m) \triangleq \psi(z_m, \bar{\gamma}_m, \bar{\alpha}_{m-1})$.

In o-minimal case the proof of Theorem 1.3 is much more simple. By [P] we can omit in our consideration all quasirational types. So, let all types p_m be irrational and let $\alpha'_m \in p_m(N) \setminus V_{p_m}(\bar{\alpha}^m)$. Let $\bar{\gamma}_m \in p(N)$ be such that $\bar{\gamma}_{m-1} = \bar{\alpha}_{m-1}$ and $\gamma_m = \alpha'_m$. We put $F(\bar{z}) \triangleq \bar{z} = \bar{\alpha} \vee \bigvee \bar{z} = \bar{\gamma}_m$. Then F satisfies the assertions of Lemma 1.4.

In weakly o-minimal case we cannot apply the result of [P], as it was shown in [B4]. As we saw above, if type p_m is quasirational, we have to consider infinitely many copies of α_m to provide that the intersection of quasineighbourhoods of these elements is empty. So, the choice of F is not so easy. The main problem is to provide that all elements in $F(N)$ realize the same ϕ-type over M.

5. Existence of $\bar{\gamma}$. Convergence of a formula to a type

The aim of this section is to find some conditions on $\bar{\gamma}$, which allow to prove Lemma 4.2 for non-empty $\bar{\gamma}_m$. To do this, the first thing we can assume is that $F(\bar{\alpha}_{m-1}, z_m, \bar{\gamma}_m)$ is big enough in p_m for $\bar{\alpha}$. Since Theorem 1.3 says that the choice of $\bar{\gamma}$ depends only on $\bar{\alpha}$, we have to suppose that any $(M \cup \bar{\alpha}_{m-1}, \bar{\gamma}_m)$-definable convex formula, containing α_m, is big enough in p_m for $\bar{\alpha}$. We express these conditions in the following definition:

Definition 5.1. Let p_m be a non locally isolated one-type over $A \cup \bar{\alpha}_{m-1}$, and $l(\bar{\alpha}) = n \geq m$. Then $\bar{\gamma}_m$ is called *suitable for* $(p_m, \bar{\alpha})$, if the following hold:

1. For any $(A \cup \bar{\alpha}_{m-1})$-definable formula $\phi(z_m, \bar{u})$ there is an $(A \cup \bar{\alpha}_{m-1} \cup \bar{\gamma}_m)$-definable formula $F_m(z_m)$, which isolates $\phi(z_m, \bar{u})$-part of type $p_m(z_m)$.
2. Let $q_{j,m} = tp(\alpha_j / M \cup \bar{\alpha}_{j-1} \cup \bar{\gamma}_m)$. For any $j \leq m$ the following hold:
   (a) If $j < m$ and p_j is isolated by an $(A \cup \bar{\alpha}_{j-1})$-definable formula $F_j(z_j)$, then this formula $F_j(z_j)$ also isolates type $q_{j,m}$.
   (b) Type $q_{j,m}$ is big enough in p_j for $\bar{\alpha}$.

Further in this section we show existence of $\bar{\gamma}$ and formula F_m for the case, when p_m is not locally isolated. This section is almost coincides with Section 5 in [B0].

We say that a formula $\phi(\bar{x}, \bar{a})$ divides a set B, if both intersections $\phi(N, \bar{a}) \cap B$ and $\neg\phi(N, \bar{a}) \cap B$ are not empty.

Definition 5.2. [B0] We say that a formula $\phi(\bar{z}, \bar{u})$ *weakly converges* to a type $q(\bar{z}) \in S(A)$ and write $WEC(\phi(\bar{z}, \bar{u}), q)$ for this, if for any $\theta \in q$ there exists $\bar{a} \in A$ such that $\phi(\bar{z}, \bar{a})$ divides $\theta(N)$.

Lemma 5.3. *Let $q \in S(A)$. Then the following hold:*

1. *If q is isolated, then q is locally isolated;*
2. *If q is locally isolated, then q is definable;*
3. *q is locally isolated iff for any ϕ we have $\neg WEC(\phi, q)$.*

Proof. It is obvious. □

Since we work in ordered structures, we can introduce notions of left and right convergences.

Definition 5.4. [B0] We say that the *condition of the left convergence* of a formula $H(y, \bar{u})$ to a type $q \in S_1(A)$ holds and we write $LC(H(y, \bar{u}), q)$ for this, if the left borders of $q(N)$ and $\bigcap\{H(N, \bar{a}) : H(x, \bar{a} \in q(x))\}$ are equal, i.e.

$$q(N)^- = (\bigcap\{H(N, \bar{a}) : H(x, \bar{a}) \in q(x)\})^-$$

We say that the *condition of the right convergence* of a formula $H(y,\bar{u})$ to a type $q \in S_1(A)$ holds and we write $RC(H(y,\bar{u}),q)$ for this, if the right borders of $q(N)$ and $\bigcap\{H(N,\bar{a}) : H(x,\bar{a} \in q(x))\}$ are equal.

It is immediate from the definitions:

Lemma 5.5. *Let $q \in S_1(A)$, and $H(N,\bar{b})$ convex for any $\bar{b} \in N$. Then we have $WEC(H(x,\bar{u}),q)$ iff at least one of the two holds: $LC(H(x,\bar{u}),q)$, $RC(H(x,\bar{u}),q)$.*

Now we can describe F_i if $p_i \in S_1(A)$ is not locally isolated. So, we have $WEC(H(y,\bar{u}),p_i)$ for some formula H. If both $LC(H,p_i)$ and $RC(H,p_i)$ hold, then $F_i(x,\bar{u}) \triangleq H(x,\bar{u})$. Assume that for any H', $RC(H',p_i)$ does not hold. Then by Lemma 5.5, $LC(H,p_i)$ is true. Consider $\phi(x,\bar{z})$. We have that the right border of $\bigcap\{\phi(x,\bar{a}) \in p_i\}$ is greater than $p_i(N)$, then there is a convex A-definable formula $H_1(x)$, whose right border is between the mentioned above right borders. In this case we put $F_i(x,\bar{u}) \triangleq H(x,\bar{u}) \wedge H_1(x)$.

If for any H', $LC(H',p_i)$ does not holds, then we do the same things up to the inverse of the order.

Having found $F_m(x,\bar{u})$ we have to find $\bar{\gamma}_m$ out. Before proving existence of $\bar{\gamma}_m$ we give some auxiliary definition and facts.

Definition 5.6. [B0] Let $\Gamma(\bar{u})$ be a non-isolated consistent set of n-formulae over A. We say that $\Gamma(\bar{u})$ is quasimodel, if for any $\theta(\bar{u}) \in \Gamma(\bar{u})$ there is $\bar{a} \in A$ such that $N \models \theta(\bar{a})$.

When $A = M \cup \bar{\gamma}$, this notion transforms to the standard notion of Shelah: finitely satisfiable in M, and this notion "quasimodel" had been used in this context in the paper of Baizhanov [B0] in the proof of Theorems 56 and 58. For an arbitrary A, Baldwin–Baizhanov in preprint "Notes on cover" used name *TV-type* (Tarskii-Vauhgt-type) instead of the notion of a quasimodel type.

Lemma 5.7. *For any non-isolated quasimodel set Γ of n-formulae over A closed under finite conjunction (more generally, for any finite conjunction there is a formula in Γ, which implies this conjunction), there is a quasimodel type over A extending Γ.*

Proof. Obviously, Γ is locally consistent, and for any A-definable formula $\theta(\bar{u})$ at least one of the following sets $\{\theta\} \cup \Gamma$, $\{\neg\theta\} \cup \Gamma$ is quasimodel. $\square$

Lemma 5.8. *Let $q \in S(A)$. Then the following conditions are equivalent:*

1. *there exists ϕ with $WEC(\phi,q)$;*
2. *there exists $r \in S(A)$, such that r is quasimodel and $r \not\perp^w q$.*

Proof. (1) $\Rightarrow$ (2) Consider the following set of A-definable formulae:

$$\Gamma(\bar{u}) \quad \triangleq \quad \{\Theta_\psi(\bar{u}) : \psi \in q,\ \Theta_\psi(\bar{u}) \triangleq \exists\bar{y}(\psi(\bar{y}) \wedge \phi(\bar{y},\bar{u})) \wedge \exists\bar{y}(\psi(\bar{y}) \wedge \neg\phi(\bar{y},\bar{u}))\}$$

It is clear that Γ is quasimodel and for any finite conjunction of formulae of Γ there exists a formula of Γ, which implies the conjunction. By Lemma 5.7 there is a quasimodel type $r \in S(A)$, which extends Γ. Then for any $\gamma \in r(N) \subseteq \Gamma(N)$, $\gamma \not\perp^w q$, so $r \not\perp^w q$.

(2) $\Rightarrow$ (1) let $\bar{\alpha} \in r(N)$ and $\bar{\alpha} \not\perp^w q$. then for some formula $\phi(\bar{x},\bar{u})$, we have that $\phi(\bar{x},\bar{\alpha})$ divides $q(N)$ and, consequently, it divides any formula ψ of q. Thus,

$N \models \Theta_\psi(\bar{\alpha})$ and $\Theta_\psi \in r$. Since r is quasimodel, there exists $\bar{a} \in A$ such that $N \models \Theta_\psi(\bar{a})$. The last implies $WEC(\phi, q)$. □

Lemma 5.9 (Theorem 56 in [B0]). *Let $q_i \in S_1(A \cup \bar{\alpha}_{i-1})$ for $i \leq m$. Then there exists $\bar{\gamma}_m$ which is suitable for $(q_m, \bar{\alpha})$.*

Proof. We shall proof this lemma by standard compactness arguments.

Since q_m is not locally isolated, in the virtue of Lemma 5.5 there exists an $(A \cup \bar{\alpha}_{i-1})$-definable formula $H(x, \bar{u})$ such that at least one of the two holds: $LC(H(x, \bar{u}), q_m)$, $RC(H(x, \bar{u}), q_m)$. In each of these three possible cases we define an infinite set of formulae as follows, in order to construct $\bar{\gamma}_m$.

First we claim that property of $H(N, \bar{u})$ be big enough in a type $r \in S_1(A)$ for $\bar{\alpha}$ is type-definable over $(A \cup \bar{\alpha})$.

If $LC(H(x, \bar{u}), q_m)$ holds, then

$$\Gamma_{lc(H)}(\bar{u}) \triangleq \{\text{the left border of } H(N, \bar{u}) \text{ is greater than the left border of } \psi(N) : \psi(x) \in q_m(x)\}$$

If $RC(H(x, \bar{u}), q_m)$ holds, then

$$\Gamma_{rc(H)}(\bar{u}) \triangleq \{\text{the right border of } H(N, \bar{u}) \text{ is less than the right border of } \psi(N) : \psi(x) \in q_m(x)\}$$

Let r be an arbitrary type in $S_1(A \cup \bar{\alpha}_i)$, for $i < m$. We define the following kinds of sets of formulae.

If r is isolated by formula $\psi(y)$, then

$$\Gamma_r(\bar{u}) \triangleq \{\exists y(\psi(y) \wedge \theta(y, \bar{u})) \rightarrow \forall y(\psi(y) \rightarrow \theta(y, \bar{u})) : \theta(y, \bar{u}) \text{ is an } (A \cup \bar{\alpha}_i)\text{-definable formula}\}$$

If r is non-isolated, then $\Gamma_r(\bar{u})$ says that $H(N, \bar{u})$ is big enough in r for $\bar{\alpha}^i$.

Claim 5.9.1. (1) *For any $r \in S_1(A \cup \bar{\alpha}_i)$, for any $i \leq m$ and for any $\bar{c} \in A$ we have that $\models \Gamma_r(\bar{c})$.*

(2) *For any quasimodel consistent set $S(\bar{u})$ of $(A \cup \bar{\alpha}_{m-1}, \bar{\alpha}^m)$-definable $l(\bar{u})$-formulae, which is closed under finite conjunctions, for any $r \in S_1(A \cup \bar{\alpha}_i)$, union $\Gamma_r(\bar{u}) \cup S(\bar{u})$ is consistent.*

Proof. (1) is a corollary of the definition of the set of all realizations of a type and of the definition of $QV_r(\bar{\alpha}^{i+1})$, because we cannot distinguish two realizations of a type over $(A \cup \bar{\alpha}_i)$ with an $(A \cup \bar{\alpha}_i)$-definable formula.

(2) is a corollary of (1). □

Claim, for any realization $\bar{\gamma}_m$ of $\bigcup_{i \leq m} \Gamma_{q_i}(\bar{u})$ we have that $tp(\alpha_i / A \cup \bar{\alpha}_{i-1} \cup \bar{\gamma}_m)$ is big enough in q_i for $\bar{\alpha}^i$. Thus we are ready to find $\bar{\gamma}_m$ as a realization of some extension of $\bigcup_{i \leq m} \Gamma_{q_i}(\bar{u})$.

Since type q_m is not locally isolated, it is either irrational (Case 1 below), or quasirational (Case 2 below).

Case 1.1. There are $(A \cup \bar{\alpha}_m)$-definable formulae $H_1(y, \bar{u}_1)$ and $H_2(y, \bar{u}_2)$ such that both $LC(H_1(y, \bar{u}_1), q_m)$ and $RC(H_2(y, \bar{u}_2), q_m)$ hold. Obviously, formula $H(y, \bar{u}) \triangleq H_1(y, \bar{u}_1) \wedge H_2(y, \bar{u}_2)$ has as the left convergence to type q_m, as the right one.

Let $\Gamma(\bar{u}) \triangleq \Gamma_{lc(H)}(\bar{u}) \cup \Gamma_{rc(H)}(\bar{u}) \cup \bigcup_{i \leq m} \Gamma_{q_i}(\bar{u})$. In the virtue of Claim 5.9.1 this set is consistent. Let $\bar{\gamma}_m$ realizes $\Gamma(\bar{u})$. Let $F_m(\bar{\alpha}_{m-1}, z_m, \bar{\gamma}_m) \triangleq H(z_m, \bar{\alpha}_{m-1}, \bar{\gamma}_m)$. It is clear that $\bar{\gamma}_m$ is suitable for $(q_m, \bar{\alpha})$.

Case 1.2. There is an $(A \cup \bar{\alpha}_m)$-definable formula $H(y, \bar{u})$ with $LC(H(y, \bar{u}))$ and for any $(A \cup \bar{\alpha}_m)$-definable formula $H_1(y, \bar{u}_1)$ we have that $RC(H_1(y, \bar{u}_1))$ does not hold.

Let $\Gamma(\bar{u}) \triangleq \Gamma_{lc(H)}(\bar{u}) \cup \bigcup_{i \leq m} \Gamma_{q_i}(\bar{u})$. In the virtue of Claim 5.9.1 this set is consistent. Let $\bar{\gamma}_m$ realizes $\Gamma(\bar{u})$.

Consider an arbitrary formula $\psi \in q_m$. Since $RC(\psi, q_m)$ does not hold, there exists convex formula $\theta_\psi \in q_m$, whose the right border is less than the right border of the convex part of $\psi(N)$, which is consistent with q_m.

Let $F_m(\bar{\alpha}_{m-1}, z_m, \bar{\gamma}_m) \triangleq H(z_m, \bar{\alpha}_{m-1}, \bar{\gamma}_m) \wedge \theta_{\phi^{m+1}}$. It is clear that $\bar{\gamma}_m$ is suitable for $(q_m, \bar{\alpha})$.

Case 1.3. There is an $(A \cup \bar{\alpha}_m)$-definable formula $H(y, \bar{u})$ with $RC(H(y, \bar{u}))$ and for any $(A \cup \bar{\alpha}_m)$-definable formula $H_1(y, \bar{u}_1)$ we have that $LC(H_1(y, \bar{u}_1))$ does not hold.

Consideration is similar to Case 1.2.

Case 2. Assume that q_m is quasirational, say to the right. Since q_m is not locally isolated, there is an $(A \cup \bar{\alpha}_m)$-definable formula $H(y, \bar{u})$ such that $LC(H(y, \bar{u}), q_m)$ holds. Moreover, since the right border of $q_m(N)$ is definable, we may assume that for any $\bar{c} \in A$, the right border of $H(N, \bar{c})$ is equal to the right border of $q_m(N)$.

Let $\Gamma(\bar{u}) \triangleq \Gamma_{lc(H)}(\bar{u}) \cup \bigcup_{i \leq m} \Gamma_{q_i}(\bar{u})$. Then any realization of this set is suitable for $(q_m, \bar{\alpha})$. □

6. General case of the general case

In this section we prove Theorem 1.3. First we describe construction of $\bar{\gamma}_m$ and of formula F_m.

Step n (Basic step). If type $p_n(z_n)$ is locally isolated, then $\bar{\gamma}_n = \bar{\gamma}_{n+1} = \emptyset$ and $F_n(\bar{\alpha}_{n-1}, z_n)$ is any formula in $p_n(z_n)$, which isolates $\phi(x, \bar{y}, \bar{\alpha}_{n-1}, z_n)$-part of type $p_n(z_n)$.

If $p_n(z_n)$ is not locally isolated, then by Lemma 5.9 there exists $\bar{\gamma}'_n$, which is suitable for $(p_n, \bar{\alpha}^n)$. So, we put $\bar{\gamma}_n \triangleq \bar{\gamma}'_n$.

Then there exists $(M \cup \bar{\alpha}_{n-1} \cup \bar{\gamma}_n)$-definable formula $F_n(\bar{\alpha}_{n-1}, z_n, \bar{\gamma}_n)$, which isolates $\phi(x, \bar{y}, \bar{\alpha}_{n-1}, z_n)$-part of type $p_n(z_n)$.

Step m. Denote $q_{m,k}(z_m) = tp(\alpha_m / M \cup \bar{\alpha}_{m-1} \cup \bar{\gamma}_k)$ for $m \leq k < n+1$.

If type $q_{m,m+1}(z_m)$ is locally isolated, then $\bar{\gamma}_m = \bar{\gamma}_{m+1}$ and $F_m(\bar{\alpha}_{n-1}, z_n, \bar{\gamma}_{m+1})$ is any formula in $q_{m,m+1}(z_m)$, which isolates $\phi_{m+1}(x, \bar{y}, \bar{\alpha}_{m-1}, z_m, \bar{\gamma}_{m+1})$-part of type $q_{m,m+1}(z_m)$.

If $q_{m,m+1}(z_m)$ is not locally isolated, then by Lemma 5.9 there exists $\bar{\gamma}'_m$, which is suitable for $(q_{m,m+1}, \bar{\alpha}^m)$. So, we put $\bar{\gamma}_m \triangleq \bar{\gamma}'_m \cup \bar{\gamma}_{m+1}$.

Then there exists $(M \cup \bar{\alpha}_{m-1} \cup \bar{\gamma}_m)$-definable formula $F_m(\bar{\alpha}_{m-1}, z_m, \bar{\gamma}_m)$, which isolates $\phi_{m+1}(x, \bar{y}, \bar{\alpha}_{m-1}, z_n, \bar{\gamma}_{m+1})$-part of type $q_{m,m+1}(z_m)$.

Lemma 6.1. *For any $m \leq n$, formula F_m is big enough in $p_m(N)$.*

Proof. Since formula F_m is big enough in $q_{m,m+1}(N)$ by the construction of $\bar{\gamma}_m$, i.e. $F_m(N) \supset QV_{q_{m,m+1}}(\bar{\alpha}^m)$, it is sufficient to show that

$$QV_{p_m}(\bar{\alpha}^m) = QV_{q_{m,n+1}}(\bar{\alpha}^m) \subseteq QV_{q_{m,n}}(\bar{\alpha}^m) \subseteq \ldots \subseteq QV_{q_{m,m}}(\bar{\alpha}^m)$$

It follows from the fact that for each k with $m \leq k \leq n$, type $q_{m,k}$ is big enough in $q_{m,k+1}$ for $\bar{\alpha}^m$ by the definition of a suitable $\bar{\gamma}_k$. □

The following lemma is a generalization of Lemma 4.2.

Lemma 6.2. *If $\phi(N, \bar{a}, \bar{\alpha}) \cap M = \emptyset$, then $\phi^m(x, \bar{a}, \bar{\alpha}_{m-1}, \bar{\gamma}_m)$ is not consistent with any $r \in \mathbb{O}_m \cup \ldots \mathbb{O}_n$.*

Proof. Note that $F_m(z_i, \bar{\alpha}_{m-1}, \bar{\gamma}_m)$ is big enough in p_m for $\bar{\alpha}^m$ by Lemma 6.1.

Assume the contrary, that $\phi(N)$ has an empty intersection with M and ϕ^m is consistent with some $r \in \mathbb{O}_m$. In the virtue of Lemma 4.1, r has at most two extensions in $S_1(M \cup \bar{\alpha}_{m-1})$, say r_1 and r_2. We have that some convex part of $\phi^m(N)$, and then of $\phi^{m+1}(N)$, is inside $r(N)$. So, some convex part of $\phi^{m+1}(N)$ has non-empty intersection with at least one of $r_i(N)$, say $r_1(N)$. By Lemma 3.7, r_1 has a unique extension to a type $s \in S_1(M \cup \bar{\alpha}_{m-1} \cup \bar{\gamma}_{m+1})$, which is big enough in r_1 for $\bar{\alpha}^m$, and then has non-empty intersection with $\phi^{m+1}(N)$.

Assume that $q_{m,m+1}$ is isolated. Then s is isolated. Since s is big enough in r_1 for $\bar{\alpha}^m$, some convex part of $\phi^{m+1}(x)$ divides $s(N)$. By Corollary 3.4 there is a correspondence τ between the border of the convex part of $\phi^{m+1}(N, \bar{a}, \bar{\alpha}_m, \bar{\gamma}_{m+1})$, belonging to $s(N)$, and some cut in $s(N)$. For simplicity, assume that τ is order preserving and the the right border of the convex part of $\phi^{m+1}(N, \bar{a}, \bar{\alpha}_m, \bar{\gamma}_{m+1})$ belongs to $s(N)$. Then if a cut C is less than $\beta_m \in s(N)$, we have that the right border of $\phi^{m+1}(N, \bar{a}, \bar{\alpha}_{m-1}, \beta_m, \bar{\gamma}_{m+1})$ is less than the right border of $\tau^{-1}(C)$. Since in this case $F_m(N) = q_{m,m+1}(N)$, we obtain that the right border of

$$\bigcap\{\phi^{m+1}(N, \bar{a}, \bar{\alpha}_{m-1}, \beta_m, \bar{\gamma}_{m+1}) : \beta_m \in F_m(N)\}$$

is less than any cut in $s(N)$, thus $\phi^m(N)$ is not consistent with s and then with r_1, because s is big enough in r_1 for $\bar{\alpha}$.

Thus, we can assume that $q_{m,m+1}$ is non-isolated.

In the virtue of Corollary 3.5, if $q_{m,m+1}$ is irrational, then s is irrational. Set $F_m(\bar{\alpha}_{m-1}, N, \bar{\gamma}_m)$ contains two elements β_1 and β_2 such that

$$V_{q_{m,m+1}}(\beta_1) \cap V_{q_{m,m+1}}(\beta_2) = \emptyset$$

In the virtue of Lemma 2.3, $V_s(\beta_1) \cap V_s(\beta_2) = \emptyset$. Note, that $\phi^{m+1}(x, \bar{a}, \bar{\alpha}_m)$ is $(M \cup \bar{\alpha}_m \cup \bar{\gamma}_{m+1})$-definable, so some convex part of $\phi^{m+1}(N, \bar{a}, \bar{\alpha}_m, \bar{\gamma}_{m+1})$ must be in $V_s(\alpha_m)$, then $\phi^{m+1}(N, \bar{a}, \bar{\alpha}_m, \bar{\gamma}_{m+1}) \cap \phi^{m+1}(N, \bar{a}, \bar{\alpha}_{m-1}, \beta_i, \bar{\gamma}_{m+1}) = \emptyset$.

So, let $q_{m,m+1}$ be quasirational, say, to the right. We have that

$$\emptyset = \bigcap\{QV_{q_{m,m+1}}(\beta) : \delta < \beta \text{ and } \beta \in q_{m,m+1}(N)\}$$

By Corollary 3.6, for any type s, which is non weakly orthogonal to $q_{m,m+1}$,

$$\emptyset = \bigcap\{QV_s(\beta) : \delta < \beta \text{ and } \beta \in q_{m,m+1}(N)\}$$

That concludes the proof of the lemma. □

Corollary 6.3. *Theorem 1.3 is proved*

Proof. Assume that $b \in \phi(N, \bar{a}, \bar{\alpha}) \cap M \neq \emptyset$. Since $F(\bar{z})$ isolates $\phi(x, \bar{y}, \bar{z})$-part of type $p(\bar{z})$, we have that $b \in \phi^1(N, \bar{a}, \bar{\gamma}_1)$, so $N \models K_\phi(\bar{a}, \bar{\gamma})$.

Further consider the case $\phi(N, \bar{a}, \bar{\alpha}) \cap M = \emptyset$. By Lemma 6.2, $\phi^1(N, \bar{a}, \bar{\gamma}_1)$ is not consistent with any $r \in \mathbb{O}_1 \cup \ldots \cup \mathbb{O}_n$, then $N \models \neg K_\phi(\bar{a}, \bar{\gamma})$. □

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Author never thought that there is a reason to write such a paper, because he was sure that the original Baizhanov's paper [B0] is readable and correct (up to easily recoverable misprints). However, the author had imprudence to ask the participants of Logic Seminar in Lyon if they wish to get an extended to a complete paper version of the abstract of the author's talks. They did. So, the author would like to thank all of them, because in the opposite case this paper would never be written. It is necessary to say, that the proof presented in this paper is not a really new proof of the Baizhanov's theorem. It is almost the same proof, written just in the different style.

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