Extending partial isomorphisms of finite graphs

In this paper, we call finite graph \((G, R)\) any finite structure \(G\) with one binary symmetric reflexive relation \(R\) (that is \(\forall x \in G \ xRx\) and \(\forall x, y \ xRy \implies yRx\)). We call vertex of such a graph any point of \(G\), and edge, any couple \((x, y)\) such that \(xRy\). Geometrically, a finite graph \((G, R)\) is simply a finite set of points, some of them being linked by edges (see picture 1). A subgraph \((F, R')\) of \((G, R)\) is any subset \(F\) of \(G\) along with the binary relation \(R'\) induced by \(R\) on \(F\).

**Picture 1 — A graph \((G, R)\) and a subgraph \((F, R')\) of \((G, R)\).**

We call isomorphism between two graphs \((G, R)\) and \((G', R')\) any bijection that preserves the binary relations, that is, any bijection \(\sigma\) that sends an edge on an edge along with \(\sigma^{-1}\). If \((G, R) = (G', R')\), then such a \(\sigma\) is called an automorphism of \((G, R)\). A local isomorphism of \((G, R)\) is an isomorphism between two subgraphs of \((G, R)\).

Let us give another example of graph :

**Definition** — Let \(X\) be a finite set and \(n\) a positive integer. We denote by \(G(X, n)\) the graph the vertices of which are the \(n\)-elements subsets of \(X\), with the binary relation \(R\) defined by \(xRy\) if and only if \(x \cap y \neq \emptyset\).

**Picture 2 — The graphs \(G(\{1, 2, 3\}, 2)\) and \(G(\{1, 2, 3, 4\}, 2)\).**

These graphs are quite interesting for two reasons : their vertices are kind of symmetric as each of them has the same number of neighbours. Moreover, they have plenty of automorphisms : any permutation \(\alpha\) of \(X\) induces a natural automorphism on \(G(X, n)\) that we will denote \(\alpha^*\).
1 Extending local isomorphisms to the graph

Let us consider a finite graph \((G, R)\) and a local isomorphism \(\sigma\) of this graph. It is natural to wonder whether there is a way to extend \(\sigma\) to an automorphism of the whole graph \(G\). The answer is clearly negative: just consider the graph \((\{0, 1, 2\}, R = (1, 2))\), and the application \(\sigma\) that sends \(\{2\}\) on \(\{0\}\) (see picture 3). As \(\{0\}\) is part of no edge, there is no way to define \(\sigma(\{1\})\) so as to maintain an edge between the images of \((1, 2)\).

\[\begin{array}{ccc}
0 & \rightarrow & 1 \\
2 & \rightarrow & \sigma
\end{array}\]

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Picture 3 — The graph \((\{0, 1, 2\}, R = (1, 2))\) and a local isomorphism \(\sigma\).

So, generally speaking, there is no way to extend a local isomorphism to an automorphism of the graph. However, are there some special local isomorphisms that could be easily extended to an automorphism? It would be convenient for \(G\) to have a large amount of automorphisms, so as to have better chances to extend \(\sigma\).

We are now going to try to find a family \(\mathcal{F}\) of subgraphs of \(G(X, n)\) where any isomorphism between two elements of \(\mathcal{F}\) could be extended to an automorphism of \(G(X, n)\), the advantage of \(G(X, n)\) being that it has plenty of automorphisms: namely, any permutation \(\alpha\) of \(X\) induces a natural automorphism \(\alpha^*\).

Definition — A subgraph \(G_0\) of \(G(X, n)\) is said to be poor if any couple \(x \neq y\) in \(G_0\) has one element in common in \(X\) at most, and any \(x \in X\) belongs to two different elements of \(G_0\) at most.

This definition has been built to have the following Proposition:

Proposition 1 — Any isomorphism between two poor subgraphs \(F_1\) and \(F_2\) of \(G(X, n)\) extends to an automorphism of \(G(X, n)\).

Proof — Let \(\sigma: F_1 \rightarrow F_2\) be an isomorphism between \(F_1\) and \(F_2\). We build a permutation \(\alpha\) of \(X\) such that \(\alpha^*\) extends \(\sigma\). Let \(x \in X\). There are three cases:

(i) Either \(x\) belongs to two elements \(f\) and \(f'\) in \(F_1\). Then there is nothing but one choice for \(\alpha_1(x)\): it has to be the unique element of \(\sigma(f) \cap \sigma(f')\).

(ii) Or \(x\) belongs to just one element \(f\) in \(F_1\). Then let \(\alpha_f\) be a bijection between those \(x\) in \(f\) and the \(y\) being just in \(\sigma(f)\).

(iii) Or \(x\) is in none of the elements of \(F_1\). Let \(\alpha_2\) be a bijection between those \(x\) and the \(y\) in none of the \(\sigma(f)\).

Then, the union \(\alpha = \bigcup \alpha_i\) defines a permutation of \(X\). \(\square\)
2 Extending the graph to a supergraph

As we generally fail to extend a local isomorphism to an automorphism of the graph, the next natural question is: can we extend the graph $G$ to a larger graph $H$ so that each local isomorphism of $G$ extends to an automorphism of $H$?

Let us have a new look at the graph ($\{0, 1, 2\}, R = (1, 2)$) of picture 3, and the application $\sigma : \{2\} \mapsto \{0\}$. There is no way to extend $\sigma$ because $\{2\}$ and $\{0\}$ do not have the same number of neighbours (points in relation with them). But there is a way to solve this problem, namely by adding a fourth point $\{3\}$ to the graph so that $\{0\}R\{3\}$ (see picture 4). Then $\sigma$ extends in a natural way.

![Picture 4](extending_the_graph.png)

**Picture 4** — *Extending the graph* ($\{0, 1, 2\}, R = (1, 2)$).

We have just seen that a necessary condition for $H$ is that every vertex of $G$ should have the same valency in $H$, that is, the same number of neighbours in $H$:

**Proposition 2** — Any finite graph $(G, R)$ is a subgraph of a graph $(H, R')$ with uniform valency.

**Proof** — Let $n$ be the maximum valency of $G$. One can suppose $n$ odd (or replace $n$ by $n + 1$). Around each vertex $g$ of $G$, let’s add as many new vertices linked to $g$ so that $g$ is surrounded by $n$ neighbours. So, in this new graph, each vertex has valency $n$ or 1. Let’s put these new vertices of valency 1 on a circle, and link each of them with the $(n - 1)/2$ previous and next ones on the circle (on the picture, $n = 5$). Then each new vertex has $1 + 2(n - 1)/2 = n$ neighbours. A problem arises when the $(n - 1)/2$ previous and next ones on the circle are not distinct. This happens when there is less than $n$ points on the circle. But one can always add $n$ new vertices of valency 1 linked to another one. □

But, such a graph $H$ with uniform valency embeds in a $G(X, n)$, which is quite interesting as we saw that those $G(X, n)$ do have plenty of automorphisms. Just take $X$ as the set of all the edges of $H$ and $n$ as the valency of each vertex of $H$ (the application that sends a vertex $x$ of $H$ on the set of $n$ edges adjacent to $x$ is a local isomorphism from $H$ to $G(X, n)$). And the image of $H$ in $G(X, n)$ is poor! Therefore:

**Proposition 3** — Any graph with uniform valency $n \geq 2$ is isomorphic to a poor subgraph of $G(X, n)$.

Then any finite graph $G$ embeds in a poor subgraph $G_0$. Noting that a subgraph of a poor graph is poor, we have answered our second question:
Theorem (Hrushovski) — Any finite graph $G$ embeds in a finite supergraph $H$ so that any local isomorphism of $G$ extends to an automorphism of $H$.

3 Example, and generalization

Let’s give a simple example of the preceding construction. Take the following graph $(G, R)$:

The maximum valency of the vertices is 2, so first build a supergraph $(H, R')$ of $(G, R)$ with uniform valency 2 as shown in picture 5. $(H, R')$ has 5 edges so embeds in $G(\{1, 2, 3, 4, 5\}, 2)$, which is the supergraph we’re looking for. In fact, as this is a fairly simple example, the graph $(H, R')$ would be big enough to extend any local isomorphisms of $(G, R)$.

A graph is nothing but a structure in the language $L = \{R\}$ in the theory of a symmetric, reflexive relation. Whenever the language $L$ should have more than one binary relation, we would speak of multi-colored graphs.

In fact, the theorem we have just given a proof of, not only extends to any multi-colored graph (with a finite number of colors), but also to any structure with a finite relational language:

Theorem (Herwig) — Let $M$ be a finite structure in a finite relational language $L$, and $\sigma$ a local isomorphism of $M$. Then there exists a finite superstructure $N$ and an automorphism $f$ of $N$ extending $\sigma$.

References