Hyperbolicity of minimisers for the stochastic Burgers equation

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1D Periodic Stochastic Burgers Equation

$$u_t + uu_x = (\nu u_{xx}) + \eta, \ \ t \geq 0, \ \ x \in S^1 = \mathbb{R}/\mathbb{Z}. \qquad \text{(1DB)}$$

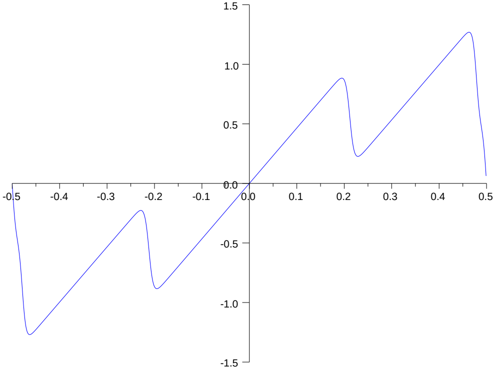
$\eta(t, x) = \eta^\omega(t, x)$: smooth in space random force, white or "kicked" in time.
Initial condition $u_0 = u(0, \cdot) \in L_1(S^1)$.

For simplicity, we assume that $\int_{S^1} \eta(t, \cdot) = 0, \forall t \, ; \ \int_{S^1} u(0, \cdot) = 0$.
Thus $\int_{S^1} u(t, \cdot) = 0, \ \forall t$.

Same type of nonlinearity and dissipation as (NSE) ; no pressure. Therefore, natural model for (NSE). Studied by physicists such as Burgers, Kida, Kraichnan, Zeldovich, Frisch, Parisi, Gotoh, Polyakov... We could consider a more general nonlinearity $f'(u)u_x$ (we need convexity and some growth assumptions for f).

Shocks after a finite time.
Unless otherwise stated, we consider the case $\nu = 0$.

Typical Profile of a Burgers Solution



Amplitude of solution ~ 1. Shocks : number of shocks ~ 1, jump ~ -1.
Burgers turbulence or "Burgulence" : see [Bec-Khanin 2007].
Ramp-cliff structure $\Rightarrow$ intermittency.

Different types of forcing

-Kicked force :

$$\eta^\omega(x) = \sum_{i=1}^{+\infty} \delta_{t=i} \zeta_i^\omega(x),$$

where ζ_i^ω nontrivial i.i.d. smooth r.v.'s in L_2.

-White force : $\eta^\omega(x) = w_t^\omega(x)$, where w^ω is a spatially smooth L_2-valued Wiener process.

Variational description

By taking the primitive in x of the Burgers equation :

$$u_t + uu_x = \eta,$$

we obtain the Hamilton-Jacobi equation :

$$\phi_t + \frac{1}{2}\phi_x^2 = F.$$

The Legendre transform of the Hamiltonian $H(t, x, \phi_x) = \phi_x^2/2 - F$ gives :

$$L(t, x, \dot{x}) = \frac{1}{2}\dot{x}^2 + F$$

For an initial condition $g(x)$ at time t_1, we have the following variational description :

$$\phi(t_2, x) = \min_{\gamma(t_2)=x} \left(g(\gamma(t_1)) + \int_{t_1}^{t_2} L(t, \gamma, \dot{\gamma}) \right).$$

Minimisers

For an initial condition $g(x)$ at time t_1, we have the following variational description :

$$\phi(t_2, x) = \min_{\gamma} \left(g(\gamma(t_1)) + \int_{t_1}^{t_2} L(t, \gamma, \dot{\gamma}) \right).$$

Here, we minimise over the curves γ s.t. $\gamma(t_2) = x$. Such curves are called minimisers.

We can also define one- or two-sided infinite minimisers (without the initial condition and minimising over perturbations with compact support in time).

Random dynamics

Two questions :
-Existence-uniqueness-properties of the stationary measure
-Properties of the minimisers (namely existence, uniqueness and hyperbolicity of a global minimiser).

[Sinai '91] (heavy use of Cole-Hopf) ;
[E-Khanin-Mazel-Sinai '00] (hyperbolicity).
Multi-d case : [Iturriaga-Khanin '03],
[Gomes-Iturriaga-Khanin-Padilla '05].
Some additional assumptions (needed for properties of the stationary measure, but not for its existence/uniqueness).

Minimisers and shocks (I)

For $s < t$, there is a well-defined map $S_s^t : S^1 \to S^1$ ("Generalised Lagrangian flow"). It depends on the initial condition at time $s-1$.
If, at time s, a point y belonging to S^1 is reached by a ψ-minimizer on $[s-1, t]$ starting in x at time t, then $S_s^t(y)$ is equal to the point x.
If a point y is not reached by such a ψ-minimizer, then it belongs to a closed interval corresponding to a shock at time t. In this case $S_s^t(y)$ is the corresponding shock position.
Minimizers on the time interval $[s-1, t]$ cannot intersect except at the endpoints $s-1$ and t. Therefore S_s^t is well-defined and monotone (problems in multi-d setting).

Minimisers and shocks (II)

Definition

For a fixed function $\psi(\cdot, s-1) : S^1 \to \mathbb{R}$, let $\Omega_{s,t}$ be the set of points reached, at the time s, by minimisers on $[s-1, t]$.

We note that $\Omega_{s,t}$ is closed.

For a closed subset Z of S^1, the diameter of Z is defined as :

$$d(Z) = 1 - m(Z),$$

where $m(Z)$ is the maximal length of a connected component of $S^1 - Z$.

Another possible definition : the minimal length of a closed interval on S^1 containing Z.

Conditions on potentials

Idea : we want to have enough freedom so that with positive probability we can create "small potentials" (automatic in the white noise case) and "potentials which are minimised on ANY of three pairwise disjoint intervals". The goal is to prove exponential convergence of the minimisers to the global minimiser.

For instance : a forcing generated by $(\cos\ 4\pi kx,\ \sin\ 4\pi kx)$ does not work (periodicity) ; $(\cos\ 2\pi kx,\ \sin\ 2\pi kx)$ works.

The separation property

Property

There exist $\alpha_0 > 0$, three pairwise disjoint open intervals J_i, $i = 1, 2, 3$, and three potentials $\tilde{F}_i$, $i = 1, 2, 3$ s.t. :
1) The potentials $\tilde{F}_i$ are in the support of forcing.
2) Each of the functions $\tilde{F}_i$ reaches its minimum, denoted by m_i, at a single point x_i.
3) For every α, $0 < \alpha \leq \alpha_0$, there exist three open intervals $I_i(\alpha)$, $I_i \subset J_i$, $i = 1, 2, 3$ s.t.

$$\tilde{F}_i(S^1 - I_i) \subset [m_i + \alpha, +\infty).$$

For every i and α, the point x_i where $\min(\tilde{F}_i)$ is reached belongs to I_i.

Lemma

Assumptions 1 or 2 imply the separation property.

The shrinking lemma and the theorem

Here, we fix $-\infty < s \leq t < +\infty$.

Lemma (EKMS '00, Bor-Khanin '13)

There exist constants $c > 0$, $T > 0$ such that, a.s. :

$$\mathbb{P}(d(\Omega_{s,t+T}) \leq d(\Omega_{s,t})/2 \mid \mathcal{B}_t) \geq c.$$

Theorem

There exist constants $\lambda > 0.$, $\tilde{C} > 0$ such that :

$$\mathbb{E}(d(\Omega_{s,t})) \leq \tilde{C} \exp(-\lambda(t-s)).$$

Corollary

Fix $s \in \mathbb{R}$. Then, for a.e. ω, there exists a random constant $\tilde{C}(s,\omega) > 0$ with all moments finite s.t. :

$$d(\Omega_{s,t}) \leq \tilde{C}(s,\omega) \exp(-\lambda(t-s)/2), \quad t \geq s.$$

Hyperbolicity

By the Pesin theory, Corollary 1 implies that the global minimiser is hyperbolic for the Euler-Lagrange dynamics in the phase space (x, u, t).

What does and does not hold in multi-d

What does hold : Existence of a unique global minimiser ; hyperbolicity [Khanin-Zhang].

What does not hold : "Exponential shrinking" property ; nice well-understood structure of shocks.

Extensions to the noncompact case

[Hoang-Khanin '03] : Forcing with a well-localised minimum.

[Bakhtin-Cator-Khanin '13] : Poisson process in space-time. Exponentially coalescing minimisers.

Stationary measure

The arguments here hold uniformly in the viscosity coefficient $\nu \geq 0$.

The solutions u of the stochastic Burgers equations define a Markov process in $L_1(S^1)$. We have nice upper bounds (taking the limit $\nu \to 0$ in [Bor12, Bor13]) : the Bogolyubov-Krylov argument implies the existence of a stationary measure.

Speed of convergence

The semigroup G_t^ω corresponding to the Markov process is contracting :

$$|G_t^\omega u_0 - G_t^\omega \tilde{u}_0|_1 \leq |u_0 - \tilde{u}_0|_1.$$

Uniqueness of the stationary measure, and the algebraic rate of convergence to it, follow from a coupling argument such as in [Kuksin-Shirikyan '12].
Idea : distance between solutions is made small since the solutions themselves become small, and then this distance is nonincreasing.

No need for any non-degeneracy condition (holds even if no force !)

Perspectives

Prove that under conditions 1 or 2, the convergence is exponential (uniformly in ν). Get rid of these conditions ?

Conditions on mimimisers (I)

Assumptions

"Kicked" case :

(i) The kicked potentials at integer times j are of the form

$$F^{\omega}(j) = \sum_{k=1}^{K} c_k^{\omega}(j) F^k,$$

where F^k are smooth on $S^1 = \mathbb{R}/\mathbb{Z}$. The vectors $(c_k^{\omega}(j))_{1 \leq k \leq K}$ i.i.d. $\mathbb{R}^K$-valued r.v.'s.
Their distribution on $\mathbb{R}^K$, denoted by μ, AC w.r.t. $\mu_{Lebesgue}$.
(ii) $0 \in Supp\ \mu$.
(iii) The mapping

$$x \mapsto (F^1(x), ..., F^K(x))$$

is an embedding (injective and homeomorphism onto its image).

Conditions on mimimisers (II)

Assumptions

White force case :

(i) The forcing potential has the form

$$F^\omega(x,t) = \sum_{k=1}^{K} \dot{W}_k^\omega(t) F^k(x),$$

where F^k are smooth on S^1, and $\dot{W}_k^\omega$ are independent white noises.
(ii) The mapping

$$x \mapsto (F^1(x), ..., F^K(x))$$

is an embedding.

Proof of the shrinking lemma (I)

Property

There exist $\alpha_0 > 0$, three pairwise disjoint open intervals J_i, $i = 1, 2, 3$, and three potentials $\tilde{F}_i$, $i = 1, 2, 3$ s.t. :
1) "Kicked" case : $\tilde{F}_i \in Supp\ \mu$ for every i. White force case : each $\tilde{F}_i$ is a linear combination of the F^k.
2) Each of the functions $\tilde{F}_i$ reaches its minimum, denoted by m_i, at a single point x_i.
3) For every $\alpha,\ 0 < \alpha \leq \alpha_0$, there exist three open intervals $I_i(\alpha)$, $I_i \subset J_i,\ i = 1, 2, 3$ s.t.

$$\tilde{F}_i(S^1 - I_i) \subset [m_i + \alpha, +\infty).$$

Proof of the shrinking lemma (II)

Shrinking lemma :

Lemma

There exist constants $c > 0, \ T > 0$ *such that if* $-\infty < s \leq t < +\infty$, *then a.s. :*

$$\mathbb{P}\left(d(\Omega_{s,t+T}) \leq \frac{d(\Omega_{s,t})}{2} \mid \mathcal{B}_t \right) \geq c.$$

Idea : on $[t, t+T/2]$, very weak noise ; on $[t+T/2, t+T]$, "well-chosen" noise.

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