ERROR TERM IN POINTWISE APPROXIMATION OF THE CURVATURE OF A CURVE

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Let $\mathcal{P}$ be a polygonal line approximating a planar curve curve Γ, the discrete curvature $k_d(P)$ at a vertex $P \in \mathcal{P}$ is (usually) defined to be the quotient of the angle between the normals of the two segments with vertex P by the average length of these segments. In this article we give an explicit upper bound of the difference $|k(P) - k_d(P)|$ between the curvature $k(P)$ at P of the curve and the discrete curvature in terms of the polygonal line's data, the supremums over Γ of the curvature function k and its derivative k', and a new geometrical invariant, the *return factor* Ω_Γ. One consequence of this upper bound is that it is not needed to know precisely which curve is passing through the vertices of the polygonal line $\mathcal{P}$ to have a pointwise information on its curvature.

Keywords: Smooth curves; Discrete curvatures, Approximation, Differential Geometry.

1. INTRODUCTION

1.1. **Generalities.** Curvature estimates are often needed in applications from computer vision, computer graphics, geometric modeling and computer aided design. Indeed, they are of a fundamental importance for the intelligence of the geometry of meshes and there is, in applied geometry, a vast literature devoted to their study ([2],[3],[5],[9],[10], for instance). Nevertheless and as far as we know, this literature only focusses on the *convergence issue:* given a sequence of meshes approximating a surface, do discrete curvatures converge towards curvatures of the surface? Yet there is another question, at least as important, the *approximation issue:* given one mesh approximating a surface, can we evaluate the error between discrete curvatures and curvatures of the surface? The goal of this article is to begin the study of this approximation issue in a one-dimensional setting, for curves.

1.2. **Approximation Problem for Curves.** There are many ways to tackle with the problem of pointwise approximation of curvature. The most natural one is probably to start with a curve Γ and a sequence of polygonal lines $(\mathcal{P}_n)_{n\in\mathbb{N}}$ which interpolate the curve more and more closely, then to define a discrete curvature $k_d(P_i^n)$ on every vertex $P_i^n \in \mathcal{P}_n$ such that the numbers $\{k_d(P_i^n)\}_{n\in\mathbb{N}}$ tend toward the curvature $k(P_i)$ of Γ at $P_i = \lim_{n\to\infty} P_i^n$. This approach turns out to be both, relatively easy and efficient: if P_{i-1}, P_i, P_{i+1} are three consecutive vertices of a polygonal line $\mathcal{P}$ then the following number

$$k_d(P_i) = \frac{\pi - \gamma_i}{\overline{\eta_i}},$$

$$\text{with } \gamma_i = \angle(P_iP_{i-1}, P_iP_{i+1}) \in [0, \pi[$$

$$\text{and } \overline{\eta_i} = \frac{1}{2}(|P_iP_{i-1}| + |P_iP_{i+1}|)$$

does the work under very weak hypothesis on the polygonal sequence (see [2] for instance or the proof of [8]). Nevertheless, although natural, this process is inadequate for concrete applications. Indeed, any experimental measurements produce only a finite number of data which give one polygonal line rather than an infinite sequence. Moreover there is also another problem: the curve Γ is unknown. In the ideal setting –when the data are supposed to be infinitely precise– the curve is certainly going through every vertex but this is far from being sufficient to say anything on the geometry of the curve.

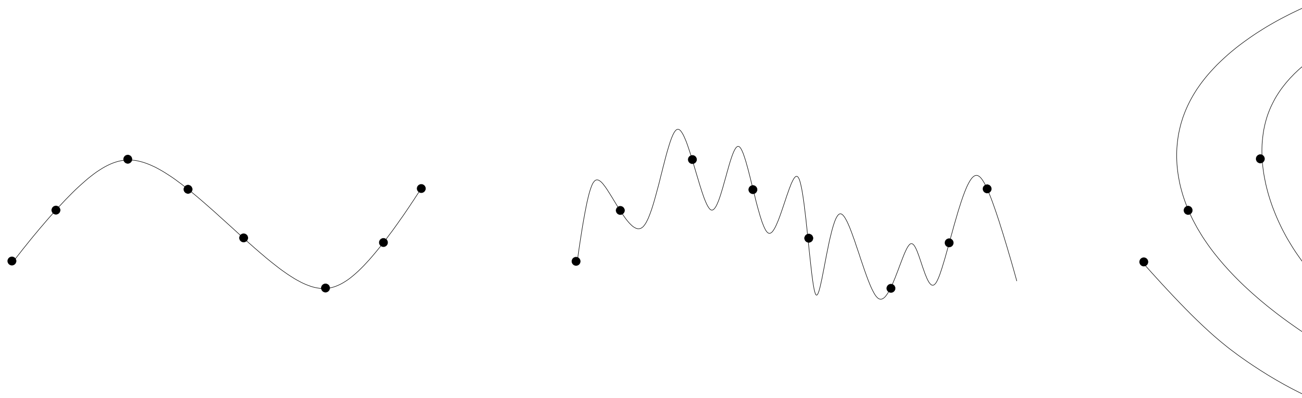
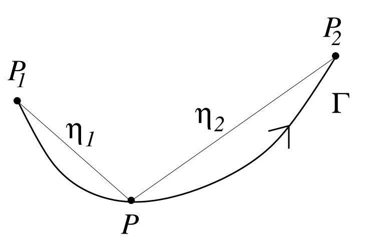


Fig. 1. Curves passing through a given set of points.

In the figures above, the curve drawn on the left seems more "reasonable" than the two others. It simply reflects that our mind is making extra implicit hypothesis on our curve. In other words, we naturally restrict the problem on a smaller subspace of curves, a space of "reasonable" curves. Suppose that we are able to give a precise definition of what is a reasonable curve, the question will then turn to be: is the curvature of any reasonable curve at a point P of a polygonal line has something to do with the discrete curvature $k_d(P)$ defined above? In this article we shall investigate this question in the case of planar curves passing through three given vertices of a polygonal line.

1.3. **Notations and Definitions.** We summarize the situation with the following definition.

Definition. – We call *V-line* a triple $V = (P, P_1, P_2)$ of $(\mathbb{R}^2)^3$. The *geometric realization* $\mathcal{V}$ of V is the polygonal line formed by the two segments P_1P and PP_2. If $\Gamma \subset \mathbb{R}^2$ is any compact connected oriented curve[1] beginning at P_1, passing through P and ending at P_2, we say that Γ is *supported* by V.



[1]We call *curve* a smooth 1-dimensional submanifold of $\mathbb{R}^2$ with or without boundary. In particular, in the definition, Γ has oriented boundary $\partial\Gamma = -\{P_1\} \cup +\{P_2\}$.

Fig. 2. A curve Γ supported by a V-line, the arrow denotes the orientation of Γ.

Let $V = (P, P_1, P_2)$ be a V-line, we denote by η_1 (resp. η_2) the Euclidean distance between P and P_1 (resp. between P and P_2) and we set

$$\bar{\eta} = \frac{\eta_1 + \eta_2}{2}, \ \ \eta_{max} = \max(\eta_1, \eta_2).$$

If Γ is supported by V, we denote by $\gamma : [-L_1, L_2] \to \Gamma$ its arc-length parametrization such that $\gamma(-L_1) = P_1$, $\gamma(0) = P$ and $\gamma(L_2) = P_2$ (note that γ has positive tangent vector with respect to the orientation of Γ). We also set

$$k_{max} = \max_{s \in [-L_1, L_2]} |k(s)|, \quad k'_{\max} = \max_{s \in [-L_1, L_2]} |k'(s)|,$$

where s denotes, as usual, the arc-length, k the curvature[2], k' its derivative with respect to s.

Given two closed curves which can be smooth or piecewise linear, recall that the Length Theorem of [6] produces an upper bound for the discrepancy $|L_2 - L_1|$ of their respective length. This upper bound only depends on the total curvature of the curves and their Fréchet distance. In the spirit of the Length Theorem we are going to obtain an upper bound of $|k(P) - k_d(P)|$ in terms of k_{max}, k'_{max} and an other geometric invariant that will be introduced soon, the return factor Ω_Γ. In particular this last number will provide a control of the Hausdorff distance between the V-line and the curve Γ, therefore it could be thought as playing the role of the Fréchet distance. In fact, only the presence of k'_{max} may sound strange at a first glance, nevertheless it is essential to face up to the following phenomenon: whatever the curve Γ_1 passing through P_1, P, P_2 it is always possible to perform locally a C^0-small perturbation around P so that the resulting curve Γ_2 is still supported by V but has a vanishing curvature at P.

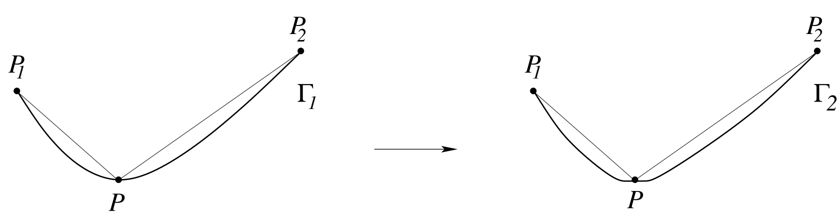


Fig. 3. A flat point phenomenon.

Note that k_{max} remains almost the same, on the other hand the derivative of the curvature function k' dramatically changes, in general:

$$k'_{\max}(\Gamma_2) = \sup_{\Gamma_2} |k'(s_2)| \gg k'_{\max}(\Gamma_1) = \sup_{\Gamma_1} |k'(s_1)|.$$

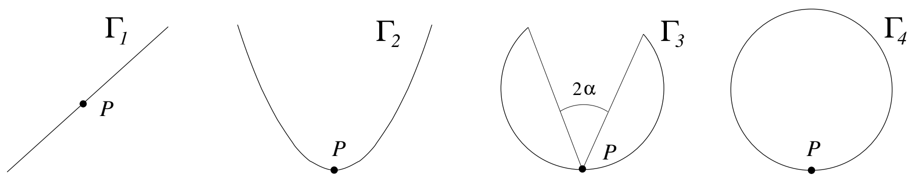
This shows that if we want to state a non-trivial comparison result between $k_d(P)$ and $k(P)$, we need to take into account not only the curvature function but also its 1-jet.

[2] The word *curvature* is a somehow ambiguous since, for plane curves, it is possible to consider a signed version of the curvature. Here, by *curvature* we mean the non-negative number which gives the proportion between the derivative of the unit tangent vector and the principal normal, or equivalently, the norm of the derivative of the unit tangent vector.

Definition. – Let (P,Q) be a pair of points of Γ, we denote by $\alpha_P(Q) \in [0, \frac{\pi}{2}]$ the angle between the line (PQ) and the tangent line $T_Q\Gamma$. The *return factor* of Γ at P is the number

$$\Omega_\Gamma(P) = \sup_{Q\in\Gamma} \frac{1}{\cos\alpha_P(Q)} \in [1,\infty].$$

Here are some examples of curves with their return factor.



Line	*Parabola*	*Arc of circle*	*Circle*
$\Omega_{\Gamma_1}(P) = 1$	$\Omega_{\Gamma_2}(P) = \frac{3}{\sqrt{8}}$	$\Omega_{\Gamma_3}(P) = \frac{1}{\sin\alpha}$	$\Omega_{\Gamma_4}(P) = \infty.$

Fig. 4. Curves and their return factors.

If Γ is supported by V and γ is an arc-length parametrization as above, we put:

$$\begin{array}{cccc} \widetilde{\eta}: & [-L_1,L_2] & \longrightarrow & \mathbb{R}_+ \\ & s & \longrightarrow & sign(s).\mathrm{dist}(P,\gamma(s)) \end{array}$$

where dist(.,.) is the Euclidean distance of $\mathbb{R}^2$. For $s \neq 0$ this function is smooth and a standard computation shows that

$$\left|\frac{\partial\widetilde{\eta}}{\partial s}(s)\right| = \cos\alpha_P(\gamma(s)).$$

Thus, away from P, critical points of $\widetilde{\eta}$ correspond to points such that $\cos\alpha_P(\gamma(s)) = 0$. Such points exist if and only if $\Omega_\Gamma(P) = \infty$. In a neighborhood of $s = 0$, $\widetilde{\eta}$ is increasing and the condition $\Omega_\Gamma(P) = \infty$ is a necessary one to change the monotonicity of $\widetilde{\eta}$. In an interval where $\widetilde{\eta}$ is decreasing, points $\gamma(s)$ are getting closer to P, hence the name *return factor* for $\Omega_\Gamma(P)$.

Suppose that $\Omega_\Gamma(P) < \infty$, then $s \to \widetilde{\eta}(s)$ is an increasing function on the whole interval $[-L_1, L_2]$ and thus

$$\max_{s\in[-L_1,L_2]} |\widetilde{\eta}(s)| = \max\{\eta_1,\eta_2\} = \eta_{max}.$$

In particular, Γ is contained in a disk of radius η_{max} and centered at P.

1.4. **Main Results.** We can now state our main result which gives an upper bound of the difference $|k(P) - k_d(P)|$ in terms of $\Omega_\Gamma(P)$, η_{max}, k_{max} and k'_{max}.

Approximation Theorem. – *Let $V = (P,P_1,P_2)$ be a V-line, $k_d(P) = \dfrac{(\pi-\gamma)}{\bar{\eta}}$ be its discrete curvature at P and Γ be a smooth curve of $\mathbb{R}^2$ supported by V. We have:*

$$|k(P) - k_d(P)| \leq 2\Omega_\Gamma(P)^3 \left(\frac{k^3_{max}}{8}\eta_{max} + \frac{k'_{max}}{3}\Omega_\Gamma(P)\right)\eta_{max}.$$

Remark 1. – The presence of η_{max} as a factor implies that the above theorem is also a convergence theorem. Indeed, if $\Omega_\Gamma(P) < \infty$ the upper bound

$$e(\Gamma, V) = 2\Omega_\Gamma(P)^3 \left(\frac{k_{max}^3}{8}\eta_{max} + \frac{k'_{max}}{3}\Omega_\Gamma(P) \right) \eta_{max}$$

tends toward zero when η_{max} tends toward zero. Note that if $\Omega_\Gamma(P) = \infty$ then in a sufficiently small neighborhood $\mathcal{U}$ of P, the curve $\Gamma \cap \mathcal{U}$ has a finite return factor $\Omega_{\Gamma\cap\mathcal{U}}(P)$.

Remark 2. – The error term $e(\Gamma, V)$ involves three numbers which depend on the curve: k_{max}, k'_{max} and $\Omega_\Gamma(P)$. It is tempting to try to eliminate one of them. In fact, it is easy to find examples showing that a non-trivial upper bound of $|k(P) - k_d(P)|$ using only two of these three numbers can not exist. The figure below illustrates one of these examples which shows that the data of k_{max} and k'_{max} is insufficient to produce a non-trivial bound.

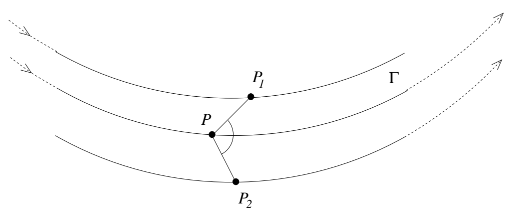


Fig. 5. A large spiral.

The curve Γ is a large spiral so k_{max} and k'_{max} are very small compared with $1/\eta_{max}$, yet the discrepancy $|k(P) - k_d(P)|$ is large since it is approximately equal to $|k_d(P)|$ which is of order $1/\eta_{max}$. Note that in this case $\Omega_\Gamma(P)$ is infinite.

Remark 3. – The error term does not involve any distance between the curve and the geometric realisation $\mathcal{V}$ of the V-line. In fact the Hausdorff distance $\delta(\Gamma, \mathcal{V})$ between Γ and the geometric realization $\mathcal{V}$ is under control.

Proposition. – *We have:* $\delta(\Gamma, \mathcal{V}) \leq \frac{1}{8}\Omega_\Gamma(P) k_{max} \eta_{max}^2$.

The proof of this proposition is given in §3. Coming back to our initial motivation, the experimenter has to decide first which of the spaces

$$\mathcal{G}_{K,K',\Omega}(V) = \{\Gamma \subset \mathbb{R}^2 \mid \Gamma \text{ is supported by } V,\ k_{max}(\Gamma) \leq K,$$
$$k'_{max}(\Gamma) \leq K',\ \Omega_\Gamma(P) \leq \Omega\}$$

is his space of “reasonable curves” to apply the theorem and obtain an upper bound of the error. In particular, he does not need to know precisely which is the “real” curve passing through P_1, P, P_2 to have a pointwise information on its curvature.

1.5. **General Comments on the Approximation Theorem and its Proof.** A way to prove convergence results for a specific discrete curvature k_d is simply to write the Taylor expansion of the function that defines k_d and to make apparent that the “constant“ term is precisely the curvature $k(P)$ of the curve. Here, once the curve Γ and a point P on it are given, the discrete curvature k_d reduces to a function of the two points P_1, P_2 that form the V-line $V = (P, P_1, P_2)$.

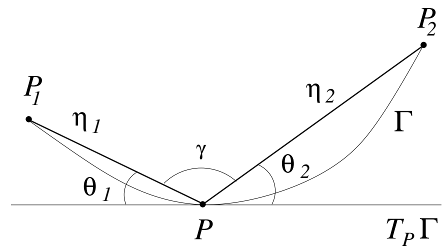


Fig. 6. Smooth curve and inscribed V-line.

Denoting by $(\eta_1, \pi - \theta_1)$ and (η_2, θ_2) the polar coordinates of P_1 and P_2 (*cf.* Figure 6), we then see that k_d appears as a function of η_1, η_2, θ_1 and θ_2:

$$k_d = \Psi(\eta_1, \eta_2, \theta_1, \theta_2).$$

Of course, at least locally, θ_1 and θ_2 are function of η_1 and η_2 respectively, hence k_d is a function of η_1 and η_2 only:

$$k_d = \Psi(\eta_1, \eta_2, \varphi(\eta_1), \varphi(\eta_2)).$$

It then remains to show that $k_d \to k(P)$ when $(\eta_1, \eta_2) \to (0,0)$. This is straightforward since Ψ is explicit and the behaviour of φ around $\eta = 0$ can be deduced by a direct application of the Implicit Function Theorem. In particular, it is easy to see that $\varphi(\eta) \sim \dfrac{k(P)}{2}\eta$ which eventually leads to the following formula, near $(0,0)$:

$$\Psi(\eta_1, \eta_2, \varphi(\eta_1), \varphi(\eta_2)) \sim k(P).$$

The same procedure can be used to obtain an approximation result but then, it will be the first order term of the Taylor expansion which will play the central role rather than the constant term. In fact, if we set

$$\widetilde{\Psi}(\eta_1, \eta_2) := \Psi(\eta_1, \eta_2, \varphi(\eta_1), \varphi(\eta_2))$$

then the difference $k_d - k(P)$ is simply $\widetilde{\Psi}(\eta_1, \eta_2) - \widetilde{\Psi}(0,0)$ and so, if $\widetilde{\Psi}$ is sufficiently regular:

$$|k_d - k(P)| \leq \|d\widetilde{\Psi}\|_\infty \|(\eta_1, \eta_2)\|.$$

Here appears a significant difference with what we have just done for the convergence result: the infinity norm is a supremum and consequently we have to control the *global* behaviour of $d\widetilde{\Psi}$. Technically this will force us to apply a Global Implicit Function Theorem. More, since we want an upper bound in terms of geometrical data, we also have to relate $d\widetilde{\Psi}$ with one geometric invariant of the curve, the curvature for instance. This leads to a non-linear differential equation on the function φ.

It is clear that this procedure will remain valid for every discrete curvature definition of the form $k_d := \Psi(\eta_1, \eta_2, \theta_1, \theta_2)$, but for the discrete curvature:

$$k_d := 2\frac{\pi - \gamma}{\eta_1 + \eta_2} = 2\frac{\theta_1 + \theta_2}{\eta_1 + \eta_2}$$

it turns out that two small "miracles" occur. The first one is that the function $\widetilde{\Psi}$ "nearly" splits in two functions of *one* variable, precisely:

$$\frac{1}{2}(\eta_1 + \eta_2)\widetilde{\Psi}(\eta_1, \eta_2) = \varphi(\eta_1) + \varphi(\eta_2).$$

The second is that the non-linear EDO linking φ and the curvature of the curve can be *explicitly solved* (Lemma 1). This has two interesting consequences. First, since we have an explicit expression for φ, we can easily derive an upper bound for $\|\varphi''\|_\infty$ (just as we needed to know φ at the first order to show that $\widetilde{\Psi}(0,0) = k(P)$, we need to derive φ twice to control the norm of the differential $\|d\widetilde{\Psi}\|_\infty$) and therefore we can assert that the procedure described above for any discrete curvature of the form $k_d := \Psi(\eta_1, \eta_2, \theta_1, \theta_2)$ will succeed in giving an upper bound (although it could be neither the best way to obtain it nor the best conceivable upper bound). We give in §5 more details on how the procedure works for another popular discrete curvature: the circle approximation.

The second consequence is that the result stated in the Approximation Theorem is, in some sense, "flexible". Indeed, any upper bound of $\|\varphi''\|_\infty$ will give, up to a factor $2\eta_{max}$, an upper bound of $|k_d - k(P)|$. Since φ'' is explicit it is possible to produce different upper bounds according to the geometric data that we want to underline. At that step, we have chosen to introduce the return factor Ω_Γ, which is a geometrical invariant that also have a significance in the analysis part of the problem: its finiteness is the necessary and sufficient condition to the existence of a globally defined function $\theta = \varphi(\eta)$ (see Step 1 of §2).

2. Proof of the Theorem

STEP 0: General strategy of the proof. – We assume $\Omega_\Gamma(P) < \infty$ otherwise the theorem is trivial. The proof runs as follow. Let us parametrize Γ in "polar" coordinates

$$\widetilde{\gamma} : \eta \mapsto (\eta \cos\varphi(\eta), \eta \sin\varphi(\eta))$$

such that $0 \mapsto P$. We can always assume that $\varphi(0) = 0$, moreover the first derivative of φ at $\eta = 0$ is half the curvature of Γ at P thus the Taylor-Lagrange expansion of φ leads to

$$\left|\varphi(\eta) - \frac{k(P)}{2}\eta\right| \leq \frac{1}{2}\eta^2 \sup_{[0,\eta]} |\varphi''(t)|.$$

Since $\varphi(-\eta_1) = \theta_1$, $\varphi(\eta_2) = \theta_2$ (see the figure above) and $\pi - \gamma = \theta_1 + \theta_2$, applying twice the above inequality we obtain

$$\begin{aligned} |\pi - \gamma - k(P)\bar{\eta}| &\leq \frac{1}{2}\eta_1^2 \sup_{[-\eta_1,0]} |\varphi''(\eta)| + \frac{1}{2}\eta_2^2 \sup_{[0,\eta_2]} |\varphi''(\eta)| \\ &\leq \eta_{max}^2 \sup_{[-\eta_1,\eta_2]} |\varphi''(\eta)| \end{aligned}$$

thus

$$|k_d - k(P)| \leq 2\eta_{max} \sup_{[-\eta_1,\eta_2]} |\varphi''(\eta)|$$

since $\bar{\eta} \geq \frac{\eta_{max}}{2}$. It remains to obtain an upper bound of the supremum of $|\varphi''(\eta)|$ involving k_{max}, k'_{max} and $\Omega_\Gamma(P)$. This is done by solving the ODE that gives the signed curvature[3] function $\eta \mapsto \tilde{k}_{sgn}(\eta)$ in term of φ.

STEP 1: Existence of the angular function $\eta \mapsto \varphi(\eta)$. – We take an orthonormal frame with P as origin and the tangent at P of Γ as horizontal line. Since Γ is a smooth

[3]Recall that the *signed curvature* k_{sgn} of an oriented curve is the number which gives the proportion between the derivative of the unit tangent vector and its algebraic normal, this last normal being the one obtained by a quarter counterclockwise turn of the unit tangent vector.

curve, there exists a smooth function $F:\mathbb{R}^2\to\mathbb{R}$ such that $\Gamma\subset F^{-1}(0)$ and with a non-vanishing gradient (F_x,F_y) on Γ.

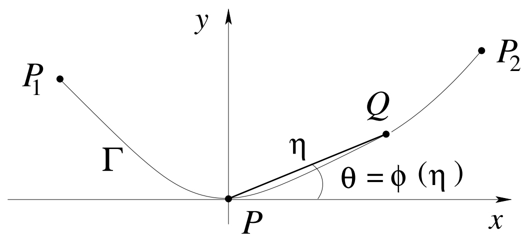


Fig. 7. Polar coordinates.

Our goal in this step is to use the Implicit Function Theorem as in [2] so that we can locally parametrize the curve Γ in a polar form

$$\eta\mapsto(\eta\cos\varphi(\eta),\eta\sin\varphi(\eta))$$

with an angular function $\eta\mapsto\varphi(\eta)$. Note that, since the tangent at P of Γ is the horizontal line this function must satisfy $\varphi(0)=0$ modulo 2π. If such a function exists we thus can assume that $\varphi(0)=0$.

Unfortunately a direct application of the Implicit Function Theorem using function F fails, indeed if

$$\Phi(\eta,\theta)=F(\eta\cos\theta,\eta\sin\theta)=0$$

then

$$\frac{\partial\Phi}{\partial\theta}(\eta,\theta)=\eta\cos\theta F_y-\eta\sin\theta F_x=<\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}F_y\\ -F_x\end{pmatrix}>$$

and $\dfrac{\partial\Phi}{\partial\theta}(\eta=0,\theta=0)=0$. It is possible to overcome this problem by considering the new implicit relation

$$\widetilde{\Phi}(\eta,\theta)=0$$

where $\widetilde{\Phi}:[-\eta_1,\eta_2]\times\mathbb{R}\to\mathbb{R}$ is defined by

$$\widetilde{\Phi}(\eta,\theta)=\begin{cases}\dfrac{F(\eta\cos\theta,\eta\sin\theta)}{\eta} & \text{if } \eta\neq0\\ \cos\theta F_x(0,0)+\sin\theta F_y(0,0) & \text{if } \eta=0.\end{cases}$$

Note that $\widetilde{\Phi}^{-1}(0)$ contains an infinite number of components since

$$\widetilde{\Phi}(\eta,\theta)=\widetilde{\Phi}(-\eta,\theta+(2k+1)\pi)=\widetilde{\Phi}(-\eta,\theta+2k\pi)$$

for $k\in\mathbb{Z}$. Let $\psi:(\eta,\theta)\mapsto(\eta\cos\theta,\eta\sin\theta)$. From the discussion at the end of §1.3, the component G containing $(0,0)$ is such that $\psi(G)=\Gamma$. Using the same arguments than in [2] Appendix A, it is easy to check that $\widetilde{\Phi}$ is C^2. Moreover $\dfrac{\partial\widetilde{\Phi}}{\partial\theta}(0,0)=F_y(P)=|grad_P F|\neq0$ and we can apply the Implicit Function Theorem to obtain an angular function φ which turns out to be defined in the connected component I of

$$\{0\}\cup\{\eta\in\mathbb{R}\mid\exists(x,y)\in\Gamma,\ \eta=sign(x)\sqrt{x^2+y^2}\ \text{ and }\ <\begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}F_y\\ -F_x\end{pmatrix}>\neq0\}$$

containing 0. Indeed $\dfrac{\partial\widetilde{\Phi}}{\partial\theta}(\eta,\theta)=\cos\theta F_y-\sin\theta F_x$ and if $\eta>0$ the vanishing of $\dfrac{\partial\widetilde{\Phi}}{\partial\theta}(\eta,\theta)$ is equivalent to $xF_y-yF_x=0$. Thus the Implicit Function Theorem precisely fails at points

Q where the tangent $T_Q\Gamma$ is orthogonal to the line (PQ) i. e. when $\alpha_P(Q)=\frac{\pi}{2}$, but this is equivalent to the fact that $\Omega_\Gamma(P)=\infty$ and we have assume $\Omega_\Gamma(P)<\infty$. A refined version of the Implicit Function Theorem asserts that φ is defined on the whole interval $I=[-\eta_1,\eta_2]$ (see the exercizes of [7], pp. 275-280 or [1], appendix D).

We denote by $\widetilde{\gamma}\colon[-\eta_1,\eta_2]\to\Gamma\subset\mathbb{R}^2$ the parametrization

$$\widetilde{\gamma}(\eta)=(\eta\cos\varphi(\eta),\eta\sin\varphi(\eta)).$$

We have

$$\left|\frac{\partial\widetilde{\gamma}}{\partial\eta}(\eta)\right|^2=1+\eta^2\varphi'^2(\eta)$$

and so the arc-length function is given by

$$\widetilde{s}(\eta)=-L_1+\int_{-\eta_1}^{\eta}(1+t^2\varphi'^2(t))^{\frac{1}{2}}dt$$

and the arc-length parametrization γ is related to $\widetilde{\gamma}$ *via* $\widetilde{s}$ by $\widetilde{\gamma}=\gamma\circ\widetilde{s}$.

STEP 2: EXPLICIT EXPRESSION FOR φ.

Lemma 1. – *For every* $\eta\in I$, *we have*

$$\varphi'(\eta)=\frac{\displaystyle\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt}{\sqrt{1-\eta^2\left(\displaystyle\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt\right)^2}}$$

where $\widetilde{k}_{sgn}(\eta)=k_{sgn}(\widetilde{s}(\eta))$ *is the signed curvature of* Γ *at the point* $\widetilde{\gamma}(\eta)=\gamma\circ\widetilde{s}(\eta)$.

Proof of Lemma 1. – The computation of the signed curvature leads to the following ODE:

$$\widetilde{k}_{sgn}(\eta)(1+\eta^2\varphi'^2(\eta))^{\frac{3}{2}}=2\varphi'(\eta)+\eta^2\varphi'^3(\eta)+\eta\varphi''(\eta).$$

Initial conditions are $\varphi(0)=0$ and $\varphi'(0)=\dfrac{k_{sgn}(P)}{2}$ (this last condition derives from the Implicit Function Theorem). Due to the term $\eta\varphi''(\eta)$, Cauchy-Lipschitz Theorem only gives uniqueness of solutions over $]0,\eta_2]$ and $[-\eta_1,0[$. Nevertheless it is readily seen using the formal change of variables

$$u=\frac{\eta\varphi'}{(1+\eta^2\varphi'^2)^{\frac{1}{2}}}$$

that this ODE admits a unique C^2-solution (satisfying $\varphi(0)=0$ and $\varphi'(0)=\dfrac{k_{sgn}(P)}{2}$) given by

$$\varphi(\eta)=\int_0^{\eta}\varphi'(t)dt$$

and

$$\varphi'(\eta)=\frac{\displaystyle\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt}{\sqrt{1-\eta^2\left(\displaystyle\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt\right)^2}}.$$

This solution is well-defined on the subinterval $(-\varepsilon_1, \varepsilon_2) \subset [-\eta_1, \eta_2]$ where the denominator of $\varphi'(\eta)$ is non vanishing. It turns out that this subinterval is the whole interval $[-\eta_1, \eta_2]$. Indeed

$$\cos\alpha_P(\widetilde{\gamma}(\eta)) = \langle \frac{\widetilde{\gamma}(\eta)}{|\widetilde{\gamma}(\eta)|}, \frac{\widetilde{\gamma}'(\eta)}{|\widetilde{\gamma}'(\eta)|} \rangle = \frac{1}{\sqrt{1+\eta^2\varphi'^2(\eta)}}$$

and thus

$$\lim_{\eta \overset{>}{\to} -\varepsilon_1} |\varphi'(\eta)| = \infty \quad \text{or} \quad \lim_{\eta \overset{<}{\to} \varepsilon_2} |\varphi'(\eta)| = \infty$$

implies

$$\Omega_\Gamma(P) = \infty.$$

$\square$

STEP 3: UPPER BOUND FOR THE SUPREMUM OF $|\varphi''|$. –

Lemma 2. – *We have:* $|\varphi'(\eta)| \leq \frac{1}{2}\Omega_\Gamma(P)k_{max}$ *and:*

$$|\varphi''(\eta)| \leq \Omega_\Gamma(P)^3\left(\frac{k_{max}^3}{8}\eta_{max} + \frac{k'_{max}}{3}\Omega_\Gamma(P)\right).$$

Proof of Lemma 2. – Note that

$$\cos\alpha_P(\widetilde{\gamma}(\eta)) = \frac{1}{\sqrt{1+\eta^2\varphi'^2(\eta)}} = \sqrt{1-\eta^2\left(\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt\right)^2}$$

thus

$$\varphi'(\eta) = \frac{1}{\cos\alpha_P(\widetilde{\gamma}(\eta))}\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt.$$

It ensues that

$$|\varphi'(\eta)| \leq \Omega_\Gamma(P)k_{max}\int_0^1 t\,dt$$

and finally

$$|\varphi'(\eta)| \leq \frac{1}{2}\Omega_\Gamma(P)k_{max}.$$

It remains to deal with the second derivative of φ.

$$\varphi''(\eta) = \frac{1}{\cos^3\alpha_P(\widetilde{\gamma}(\eta))}\left[\int_0^1 t^2\frac{\partial\widetilde{k}_{sgn}}{\partial\eta}(\eta t)dt + \eta\left(\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt\right)^3\right].$$

Since

$$\frac{\partial\widetilde{k}_{sgn}}{\partial\eta}(\eta) = \frac{\partial\widetilde{s}}{\partial\eta}(\eta)\cdot\frac{\partial k_{sgn}}{\partial s}(\widetilde{s}(\eta)) = \frac{1}{\cos\alpha_P(\widetilde{\gamma}(\eta))}\cdot\frac{\partial k_{sgn}}{\partial s}(\widetilde{s}(\eta))$$

we obtain

$$\varphi''(\eta) = \frac{1}{\cos^3\alpha_P(\widetilde{\gamma}(\eta))}\eta\left[\int_0^1 t\widetilde{k}_{sgn}(\eta t)dt\right]^3 + \frac{1}{\cos^4\alpha_P(\widetilde{\gamma}(\eta))}\left[\int_0^1 t^2\frac{\partial k_{sgn}}{\partial s}(\widetilde{s}(\eta t))dt\right].$$

Thus

$$|\varphi''(\eta)| \leq \Omega_\Gamma^3(P)\eta_{max}\frac{1}{8}k_{max}^3 + \Omega_\Gamma^4(P)\frac{1}{3}k'_{max}.$$

□

3. RELATIONSHIP BETWEEN $\Omega_\Gamma(P)$ AND THE HAUSDORFF DISTANCE

The goal of this section is to prove the proposition stated in §1.4, namely that:

$$\delta(\Gamma,\mathcal{V}) \leq \frac{1}{8}\Omega_\Gamma(P)k_{max}\eta_{max}^2$$

where $\delta(\Gamma,\mathcal{V})$ is the Hausdorff distance between the curve Γ and the V-line $\mathcal{V}$. Here again we assume $\Omega_\Gamma(P) < \infty$ since otherwise the result is trivial.

Let $Q = \gamma(\widetilde{\eta}(s))$ be a point of Γ where we assume $s > 0$. Obviously

$$dist(Q,[PP_2]) \leq dist(Q,Q')$$

where Q' is the intersection point of the segment $[PP_2]$ with the circle of radius $\eta = \widetilde{\eta}(s)$ centered at P. (This intersection point exists since $s \mapsto \widetilde{\eta}(s)$ is an increasing function with

$\widetilde{\eta}(0) = 0$ and $\widetilde{\eta}(L_2) = \eta_2$).

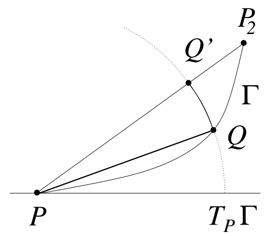


Fig. 8. The arc QQ'.

We have

$$dist(Q, Q') \leq (\varphi(\eta_2) - \varphi(\eta))\eta$$

and since

$$0 \leq \varphi(\eta_2) - \varphi(\eta) \leq (\eta_2 - \eta) \sup_{t \in [\eta, \eta_2]} \varphi'(t)$$

we obtain by Lemma 1

$$dist(Q, [PP_2]) \leq \frac{1}{2}\Omega_\Gamma(P) k_{max} (\eta_2 - \eta)\eta.$$

But $(\eta_2 - \eta)\eta \leq \frac{1}{4}\eta_{max}^2$, therefore

$$dist(Q, [PP_2]) \leq \frac{1}{8}\Omega_\Gamma(P) k_{max} \eta_{max}^2$$

and the directed Hausdorff distance from $\mathcal{V}$ to Γ satisfies

$$\delta_{dir}(\mathcal{V}, \Gamma) = \sup_{Q \in \Gamma} dist(Q, \mathcal{V}) \leq \frac{1}{8}\Omega_\Gamma(P) k_{max} \eta_{max}^2.$$

We now give an upper bound of $\delta_{dir}(\Gamma, \mathcal{V}) = \sup_{Q' \in \mathcal{V}} dist(Q', \Gamma)$. Let Q' be any point of $[PP_2]$, $\mathcal{C}$ be the circle centered at P and of radius PQ' and Γ_+ be the sub-arc $\{\gamma(s) \mid s > 0\}$. Since $s \mapsto \widetilde{\eta}(s)$ is a strictly increasing function with $\widetilde{\eta}(0) = 0$, $\widetilde{\eta}(L_2) = \eta_2$, the intersection $\mathcal{C} \cap \Gamma_+$ is a single point denoted by Q. We have

$$dist(Q', \Gamma) \leq dist(Q, Q').$$

A similar argument than above then shows:

$$dist(Q', \Gamma) \leq \frac{1}{8}\Omega_\Gamma(P) k_{max} \eta_{max}^2$$

and thus

$$\delta_{dir}(\Gamma, \mathcal{V}) \leq \frac{1}{8}\Omega_\Gamma(P) k_{max} \eta_{max}^2.$$

It is well known that the Hausdorff distance is $\delta(\mathcal{V}, \Gamma)$ the maximum between the two directed Hausdorff distances (see [4], p. 252 for instance) hence the proposition.

4. ILLUSTRATIONS

This section discusses two families of examples illustrating the theorem: circles and parabolas supported by same symmetric V-lines (a V-line (P, P_1, P_2) is symmetric if $\eta_1 = \eta_2 = \eta_{max}$, up to rigid motions such a V-line is completely determined by its length η_{max} and its angle at P). It turns out that the majoration of the discrepancy $|k(P) - k_d(P)|$ by

$$e(\Gamma, V) = 2\Omega_\Gamma(P)^3 \left(\frac{k_{max}^3}{8}\eta_{max} + \frac{k'_{max}}{3}\Omega_\Gamma(P) \right) \eta_{max}$$

is in the right order of magnitude if the angle at P is large, approximately greater than $\pi/2$. The study of the various relative errors

$$\varepsilon_r = \frac{|k(P) - k_d(P)|}{k_d(P)}, \quad e_r = \frac{e(\Gamma, V)}{k_d(P)} \quad \text{and} \quad \delta = \frac{e(\Gamma)}{|k(P) - k_d(P)|}$$

shows that they do not depend on η_{max}, hence only depend on the angle at P and are not affected by homotheties of figures. The comparison between circles and parabolas also reveals that for large angles the number δ is better for circles than for parabolas while it is the converse for medium angles.

4.1. **Circles.** We denote by 2α the angle of $V = (P, P_1, P_2)$ at P and by Γ the arc of circle joining P_1, P and P_2 (see the figure below, on the left). It is immediate that

$$\Omega_\Gamma(P) = \frac{1}{\sin\alpha}, \quad k_{max} = \frac{2\cos\alpha}{\eta_{max}} = k(P) \quad \text{and} \quad k_d(P) = \frac{\pi - 2\alpha}{\eta_{max}}.$$

This yields to the following expressions for the relative errors

$$\varepsilon_r(\alpha) = \frac{|\pi - 2\alpha - 2\cos\alpha|}{\pi - 2\alpha} \quad \text{and} \quad e_r(\alpha) = \frac{\cot^3\alpha}{\frac{\pi}{2} - \alpha}.$$

Of course the relative error $\varepsilon_r(\alpha)$ is bounded from above by $e_r(\alpha)$, the figure below (on the right) shows the graphs of these two functions.

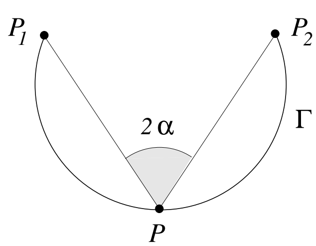


Fig. 9. An arc of circle supported by a symmetric V-line.

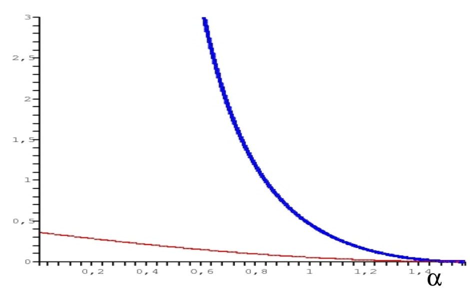


Fig. 10. The relative errors $\varepsilon_r(\alpha)$ and $e_r(\alpha)$ (thin curve and thick curve respectively).

If α is near $\pi/2$ the behaviour is the expected one but for small α, $e_r(\alpha)$ goes to infinity while the relative error $\varepsilon_r(\alpha)$ remains bounded. Note however that the discrete curvature itself k_d also has a “wrong” behaviour for small α. Indeed k_d tends toward the finite value π/η_{max} when α goes to zero...

The discrete curvature gives the correct value of the curvature up to a maximal error of $\varepsilon_r(\alpha) \times 100$ percent. We say that the majoration of the discrepancy by $e(\Gamma, V)$ is in the *right order of magnitude* if $e_r(\alpha) \leq 1$, in other words, when the upper bound allows to state that the error is no more than 100%. For the circle it happens when $\alpha \geq 0.836863...$

corresponds an angle at P greater than 1.67372... Of course the condition $e_r(\alpha) \leq 1$ is arbitrary, the table below gives for more restrictive conditions on $e_r(\alpha)$ the corresponding angles α.

$e_r(\alpha)$	1	0.1	0.01	0.001	0.0001	0.00001	0.000001
α	0.836...	1.268...	1.471...	1.539...	1.560...	1.567...	1.569...

The flatter the angle at P is, the sharper is upper bound.

4.2. **Parabolas.** Let $V = (P, P_1, P_2)$ be a symmetric V-line and Γ be an arc of parabola supported by V and having P as vertex (see the figure below). Straightforward computations lead to

$$\Omega_\Gamma(P) = \begin{cases} 3/\sqrt{8} & \text{if } \alpha \leq \alpha_0 \\ \dfrac{1}{\sin\alpha}\dfrac{\sqrt{1+4\cot^2\alpha}}{1+2\cot^2\alpha} & \text{if } \alpha \geq \alpha_0 \end{cases}$$

where $\alpha_0 = \arctan(\sqrt{2}) = 0.95531...$ We also have $k_{max} = k(P) = 2\dfrac{\cos\alpha}{\sin^2\alpha}\dfrac{1}{\eta_{max}}$ and

$$k'_{max} = \begin{cases} \dfrac{125\cos^2\alpha}{18\sqrt{5}\sin^4\alpha}\dfrac{1}{\eta_{max}^2} & \text{if } \alpha \leq \alpha_1 \\ \dfrac{24\sin\alpha\cos^3\alpha}{(1+3\cos^2\alpha)^3}\dfrac{1}{\eta_{max}^2} & \text{if } \alpha \geq \alpha_1 \end{cases}$$

where $\alpha_1 = \arctan(2\sqrt{5}) = 1.350808...$ It is easy from these expressions to derive the expressions of $\varepsilon_r(\alpha)$, $e_r(\alpha)$ and $\delta(\alpha)$. The figure below (on the right) shows the graphs of ε_r and e_r. The behaviour is the expected one, the difference between $\varepsilon_r(\alpha)$ and $e_r(\alpha)$ is small when the angle at the vertex P is near from being flat.

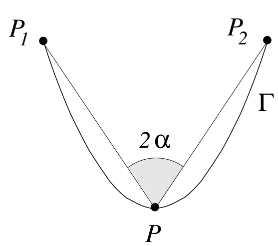


Fig. 11. A portion of parabola supported by a symmetric V-line.

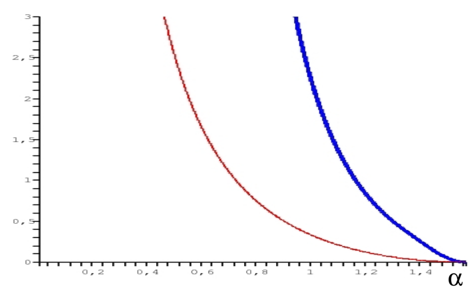


Fig. 12. The relative errors $\varepsilon_r(\alpha)$ and $e_r(\alpha)$ (thin curve and thick curve respectively).

The upper bound $e(\Gamma, V)$ is in the right order of magnitude for $\alpha \geq 1.15789...$, the table below gives the values of α for other conditions on $e_r(\alpha)$.

$e_r(\alpha)$	1	0.1	0.01	0.001	0.0001	0.00001	0.000001
α	1.157...	1.461...	1.537...	1.560...	1.567...	1.569...	1.5704...

Here again precision increases as soon as the angle at P becomes flat.

4.3. **Circles versus parabolas.** Up to now we do not study the number $\delta = \frac{e(\Gamma,V)}{|k(P)-k_d(P)|}$ which precisely measures the sharpness of the upperbound. The figure below shows the graphs of $\delta(\alpha)$ for both the circle and the parabola.

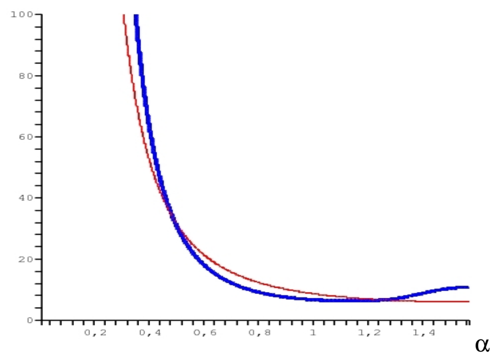


Fig. 13. Functions δ for the circle (thin curve) and the parabola (thick curve).

In both cases, circle and parabola, the number δ is small (≤ 20) for angles α aproximatively greater than 0.6 while it is huge for small angles; this is not surprising according to our previous results. The function $\delta(\alpha)$ converges when α tends to $\pi/2$, the limit value is 6 for the circle and 54/5 for the parabola. For $0.48909... \leq \alpha \leq 1.25732...$ the upper bound $e(\Gamma, V)$ of the discrepancy is better for the parabola than for the circle. Indeed in this range of α, the return factor of the parabola is small enough to compensate the presence of the term involving k'_{max} in $e(\Gamma, V)$.

5. Circle approximation

We give here some details on how to apply the procedure described in §1.5 to obtain an Approximation Theorem for another discrete curvature: the inverse value of the radius of the circle passing through the three vertices of a given V-line. To avoid confusion with the discrete curvature we have considered up to now, we will denote this new one by k_d^*.

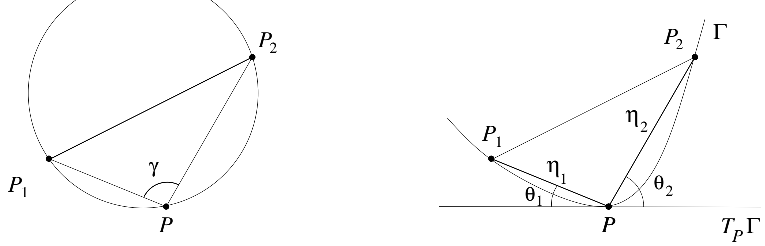


Fig. 14. The circle passing through a V-line

A direct application of the Sine Rule yields to the following formula for k_d^* (see Figure 14 for the notations):

$$k_d^* = 2\frac{\sin\gamma}{\|P_1P_2\|}$$

and a mere computation gives an expression of the form $k_d^* = \widetilde{\Psi}(\eta_1, \eta_2)$ with

$$\widetilde{\Psi}(\eta_1, \eta_2) = \frac{2\sin(\varphi(\eta_1) + \varphi(\eta_2))}{\sqrt{\eta_1^2 + \eta_2^2 + 2\eta_1\eta_2\cos(\varphi(\eta_1) + \varphi(\eta_2))}}.$$

It is easy, but tedious, to show that $\widetilde{\Psi}$ can be continuously extended at $(0,0)$ by $\widetilde{\Psi}(0,0) = k(P)$ and that it is C^1 at that point. Thus

$$|k_d^* - k(P)| \leq \|d\widetilde{\Psi}\|_\infty \sqrt{\eta_1^2 + \eta_2^2}.$$

It remains to obtain an upper bound for $\|d\widetilde{\Psi}\|_\infty$ and this is easy since Lemma 2 gives us a control on φ' and φ''. We just set out the outline of the computations. Let δ denotes the denominator of $\widetilde{\Psi}$, we have

$$\begin{aligned}\delta^3\frac{\partial\widetilde{\Psi}}{\partial\eta_1}(\eta_1,\eta_2) &= 2\varphi'(\eta_1)\cos(\varphi(\eta_1)+\varphi(\eta_2))\left(\eta_1^2+\eta_2^2+2\eta_1\eta_2\cos(\varphi(\eta_1)+\varphi(\eta_2))\right)\\ &\quad -\sin(\varphi(\eta_1)+\varphi(\eta_2))\left(\eta_1+\eta_2\cos(\varphi(\eta_1)+\varphi(\eta_2))\right)\\ &\quad +2\varphi'(\eta_1)\eta_1\eta_2\sin^2(\varphi(\eta_1)+\varphi(\eta_2)).\end{aligned}$$

The denominator δ^3 of $\dfrac{\partial\widetilde{\Psi}}{\partial\eta_1}(\eta_1,\eta_2)$ is of order 3, therefore lines 1 and 2 must be combined so that it appears only terms of order ≥ 3. This can be done by replacing, $\varphi'(\eta_1)$ by $\frac{1}{2}k(P)+\varphi''(c_1')\eta_1$, $c_1'\in]0,\eta_1[$, $\varphi(\eta_1)$ by $\frac{1}{2}k(P)\eta_1+\frac{1}{2}\varphi''(c_1)\eta_1^2$, $c_1\in]0,\eta_1[$, *etc.* It is then easy to produce an explicit constant C such that $C\eta_{max}^3$ is an upper bound of the right hand of the equality. For instance

$$C = 12C'' + \left(\frac{5}{2}+\Omega_\Gamma(P)\right)\Omega_\Gamma(P)^2 k_{max}^3\eta_{max}$$

where C'' denotes the upper bound of $|\varphi''|$ given in Lemma 2, will work. It remains to divide by δ^3 but, due to the presence of the term $2\eta_1\eta_2\cos(\varphi(\eta_1)+\varphi(\eta_2))$, this denominator could be arbitrarily small (note that this technical difficulty does not occur for k_d). This forces us to add an *a priori* upper bound on the sum $\theta_1+\theta_2$, or equivalently, on the minimal value[4] allowed for γ. For example, we can ask γ to be greater than $\frac{\pi}{2}$, then $\delta^{-1}\leq(\eta_1^2+\eta_2^2)^{-\frac{1}{2}}$ and we have:

$$|k_d^*-k(P)|\leq\frac{\sqrt{2}C\eta_{max}^3}{(\eta_1^2+\eta_2^2)^{\frac{3}{2}}}\sqrt{\eta_1^2+\eta_2^2}\leq\sqrt{2}C\eta_{max}.$$

6. Conclusion

This paper establishes an upper bound of the discrepancy between the pointwise curvature of a curve Γ and the discrete curvature of a polygonal approximation of Γ. This upper bound involves data of the polygonal line but also data of the curve, precisely it needs the supremum of curvature k_{max}, the supremum of derivative of the curvature k'_{max} and another geometrical number, the return factor Ω_Γ.

In concrete applications only the polygonal line is known, an *a priori* estimate of the three numbers k_{max}, k'_{max} and Ω_Γ is thus needed to obtain an upper bound of the error between the measure (discrete curvature) and the "true" curvature of Γ. Experimental results suggest that this error is even smaller since the angles of the polygonal line are near to be flat. In an other hand –and this is certainly not a coincidence– discrete curvature behaves badly when the angle tends toward zero. Intuitively the expected limit should be infinite, but nevertheless discrete curvature converges to a finite value. As far as we are interested in convergence processes this unpleasant behaviour has no effect, but it becomes a problem regarding the pointwise approximation question.

From a theoretical perspective our result is half the way toward a similar result for meshes and surfaces. Indeed, the definition of the discrete Gaussian curvature at a vertex of a mesh is very analogous to the one for a polygonal line: it is again a quotient of an angular defect

[4] Such a minimal value is, up to a sinus, what is usually called the *fatness* of a polygonal line.

and an area. It is possible, from the Approximation Theorem to derive an upper bound of the error between the discrete curvature and the Gaussian curvature of the surface. One important consequence of such an upper bound will concern the convergence issue which is, in the surfaces setting, far from being solved. In fact, in a subsequent article, we shall enlarge the convergence results obtained in [2] by controlling the behaviour of this upper bound.

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