Area minimizing vector fields on round 2-spheres

Vincent Borrelli and Olga Gil-Medrano

Abstract. – A vector field V on a n-dimensional round sphere $S^n(r)$ defines a submanifold $V(S^n)$ of the tangent bundle TS^n. The Gluck and Ziller question is to find the infimum of the n-dimensional volume of $V(S^n)$ among unit vector fields. This volume is computed with respect to the natural metric on the tangent bundle as defined by Sasaki. Surprisingly, the problem is only solved for dimension three [10]. In this article we tackle the question for the 2-sphere. Since there is no globally defined vector field on S^2, the infimum is taken on singular unit vector fields without boundary. These are vector fields defined on a dense open set and such that the closure of their image is a surface without boundary. In particular if the vector field is area minimizing it defines a minimal surface of $T^1S^2(r)$. We prove that if this minimal surface is homeomorphic to $\mathbb{R}P^2$ then it must be the *Pontryagin cycle*. It is the closure of unit vector fields with one singularity obtained by parallel translating a given vector along any great circle passing through a given point. We show that Pontryagin fields of the unit 2-sphere are area minimizing.

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1 Introduction and main results

The volume $Vol(V)$ of a smooth unit vector field $V : M^n \to T^1M^n$ is the n-dimensional volume of its image $V(M)$ as a submanifold of the unit tangent bundle T^1M. A natural question, that goes back to the pioneering work of H. Gluck and W. Ziller [10], is to ask for the (absolute) minimizers of the volume. More precisely, given an oriented compact Riemannian manifold (M, g) the question is to determine

$$\inf_{V \in \Gamma^\infty(T^1M)} Vol(V)$$

and to find the minimizers, if this infimum is a minimum. This volume $Vol(V)$ is computed with respect to the Sasaki metric (T^1M, g^{Sas}) which is defined by declaring the orthogonal complement of the vertical distribution to be the horizontal distribution given by the Levi-Civita connection ∇. It is then easy to see that $Vol(V) \geq \mathrm{vol}(M, g)$, with equality if and only if the vector field is parallel $\nabla V = 0$. Therefore the problem is nontrivial only if the unit tangent bundle does not admit any smooth parallel section. Natural examples of such a situation are standard spheres. It turns out that the only dimension for which the problem is solved is three [10]. In that case, the infimum is reached by unit Hopf vector fields *i. e.* those tangent to the fibers of a Hopf fibration. Of course, for two dimensional spheres (and more generally for even dimensional spheres) the space of smooth sections of the unit tangent bundle is empty, so this problem has no interest unless we consider vector fields with singularities.

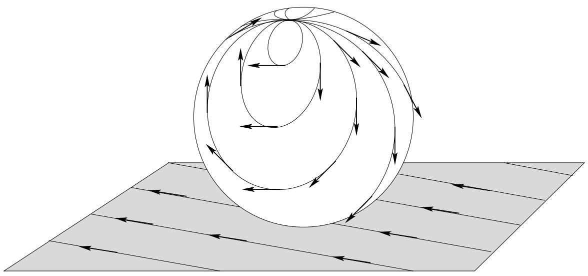
In a manifold M without smooth unit vector fields, the first natural space of sections to consider is the space $\Gamma^{sing}(T^1M)$ of singular unit vector fields, that is, the space of unit smooth vector fields which are defined on a dense open set of M. The new problem is then to determine

$$\inf_{V \in \Gamma^{sing}(T^1M)} Vol(V).$$

Even if M is a standard sphere this last problem is unsolved, partial results have been recently obtained in the case of isolated singularities in [3]. The aim of this paper is to study the Gluck and Ziller problem on an intermediate space between smooth sections and singular ones, namely on the space $\Gamma^{wb}(T^1M)$ of unit vector fields without boundary (see definition below). We solve it when M is the standard 2-sphere.

Theorem 1. – *Among unit vector fields without boundary of* $\mathbb{S}^2(1)$ *those of least area are Pontryagin fields and no others.*

We call a *Pontryagin field* of $\mathbb{S}^n$ any unit vector field P defined in a dense open subset U such that the closure of $P(U)$ is the n-dimensional generalized Pontryagin cycle of $T^1\mathbb{S}^n$. This cycle is the set of all unit vectors obtained by parallel translating a given vector v tangent at x along any great circle passing through x. The resulting field has a single singularity at $-x$ of index 0 or 2 depending on the dimension n of the sphere (see figure below). Pedersen showed that $P(U)$ is a minimal submanifold of $T^1\mathbb{S}^n(1)$ and conjectured that for odd dimensional $S^n(1)$ with $n \geq 5$ the infimum is not reached by any globally defined unit vector field but is the volume of the Pontryagin fields [17].



One of the intriguing points of the Gluck and Ziller problem is the dependence on dilatations of the metric [2]. The reason is that a dilatation on the base manifold produces dilatations with different rates on horizontal or vertical distributions of the tangent bundle. In particular, $T^1\mathbb{S}^2(1)$ is isometric to $\mathbb{R}P^3$ with a round metric while for $r \neq 1$ we will see that $T^1\mathbb{S}^2(r)$ is isometric to a projective space obtained as a quotient of a Berger 3-sphere. In general it is not known if a solution of this problem for $\mathbb{S}^n(1)$ will give a solution for $\mathbb{S}^n(r)$.

If a vector field minimizes the volume among unit vector fields its image must be a minimal submanifold of the tangent bundle [8]. We will show that any great 2-sphere is a minimal surface of the Berger 3-sphere and we obtain as a consequence that the image of a Pontryagin field is a minimal surface of $T^1\mathbb{S}^2(r)$. Moreover, great 2-spheres define on the Berger 3-sphere an open book structure with binding a fiber of the Hopf fibration and with minimal leaves, as defined by Hardt and Rosenberg [12]. Following similar arguments as those used in their work we obtain a unicity result for Pontryagin cycles.

Theorem 2. – *The only minimal surfaces of $T^1\mathbb{S}^2(r)$ homeomorphic to the projective plane arising from vector fields without boundary are Pontryagin cycles.*

Since for $r = 1$ the area minimizing surface is precisely a great $\mathbb{R}P^2$, one can think that minimal surfaces homeomorphic to a projective space are good candidates to be area minimizing, at least for r near 1. The existence of an area minimizing vector field with the topology of the projective plane would imply that this vector field should be the Pontryagin one. Although classical results (see [5], [4]) assure the existence of a smooth area minimizing surface of $T^1\mathbb{S}^2(r)$, there is no reason why this surface should arise from a vector field. The corresponding existence result for vector fields due to Johnson and Smith [15] does not apply since it applies to the class of non singular vector fields. In the proof of Theorem 1 we use the fact that the totally geodesic projective planes in the standard projective 3-space are precisely the surfaces of least area in their homology class but, as far as we know, it is an open question to know if this result is also true for the projective space with the Berger metric. Anyhow the next theorem gives another evidence in favour of the Pontryagin field.

Theorem 3. – *Among unit vector fields of $\mathbb{S}^2(r)$ with one singularity those of least area are Pontryagin ones and no others.*

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2 Geometry of the unit tangent bundle of a sphere

The unit tangent bundle $T^1\mathbb{S}^n(1)$ can be seen as the Stiefel manifold $V_{2,n+1}$ of orthonormal 2-frames of $\mathbb{R}^{n+1}$ and therefore it is diffeomorphic to the homogeneous space $SO(n+1)/SO(n-1)$. If g is the usual metric on $\mathbb{S}^n(r)$, the Sasaki metric g^{Sas} on $T^1\mathbb{S}^n(r)$ can be geometrically described as follows: If $V(t)$ is a curve in $T^1\mathbb{S}^n(r)$ with projection $x(t) = \pi(V(t))$, then

$$g^{Sas}(V'(0), V'(0)) = g(x'(0), x'(0)) + g(\frac{\nabla V}{dt}(0), \frac{\nabla V}{dt}(0)).$$

With respect to this metric, the projection π is a Riemannian submersion and to vary the radius of the sphere is essentially equivalent to perform the canonical variation of this submersion. The Sasaki metric on $T^1\mathbb{S}^n(1)$ is homothetic to the standard homogeneous metric on $SO(n+1)/SO(n-1)$ (see [9]). For $r \neq 1$, although $T^1\mathbb{S}^n(r)$ is the same homogeneous manifold, the Sasaki metric is no longer the standard one.
In the 2-dimensional case, the unit tangent bundle $T^1\mathbb{S}^2(r)$ is the total space of a Riemannian S^1-fibration over a 2-sphere. It is natural to compare it with

the family of Riemannian manifolds obtained as the canonical variation of the Hopf fibration. The manifolds so obtained are known as Berger spheres $(\mathbb{S}^3, g_\mu)$. The metric g_μ for $\mu > 0$ is defined as follows: We denote by J the complex structure of $\mathbb{R}^4 \simeq \mathbb{C}^2$ and by H the Hopf vector field JN, where $N(p) = p$ is the outwards unit normal of $\mathbb{S}^3$, then

$$g_\mu(H, H) = \mu g(H, H) = \mu, \quad (g_\mu)|_{H^\perp} = g|_{H^\perp}, \quad \text{and} \quad g_\mu(H, H^\perp) = 0,$$

where $\perp$ means orthogonal with respect to the metric g.

Proposition 1 *The unit tangent bundle of $\mathbb{S}^2(r)$ is isometric to the projective space $\mathbb{R}P^3$ with the metric obtained by the quotient of a Berger sphere $(\mathbb{S}^3, g^{Ber})$. In particular, the unit tangent bundle of the unit sphere $T^1\mathbb{S}^2(1)$ is isometric to the round $\mathbb{R}P^3$ obtained as the quotient of $\mathbb{S}^3(2)$.*

Proof.
We show first that the map $\psi(x, v) = (x, v, x \wedge v)$ is an isometry from $(T^1\mathbb{S}^2(1), g^{Sas})$ onto $(SO(3), \frac{1}{2} < , >)$ where $< , >$ is the bi-invariant metric on $SO(3)$ defined as

$$\forall A, B \in \mathfrak{so}(3), \qquad < A, B >= tr(A^t B).$$

Let $V(t) = (x(t), v(t))$ be a curve in $T^1\mathbb{S}^2(1)$ with $x(0) = e_1$ and $v(0) = e_2$, we have

$$d\psi_{(e_1,e_2)}(V'(0)) = (x'(0), v'(0), x'(0) \wedge e_2 + e_1 \wedge v'(0))$$

where (e_1, e_2, e_3) is the usual orthonormal basis of $\mathbb{R}^3$. Thus, since $x_1'(0) = 0$, $v_2'(0) = 0$ and $x_2'(0) + v_1'(0) = 0$, we obtain

$$< d\psi_{(e_1,e_2)}(V'(0)), d\psi_{(e_1,e_2)}(V'(0)) >= 2(|x_2'(0)|^2 + |x_3'(0)|^2 + |v_3'(0)|^2).$$

On the other hand

$$\frac{\nabla V}{dt}(0) = V'(0) - g(V'(0), x(0))x(0),$$

therefore

$$g^{Sas}(V'(0), V'(0)) = (|x_2'(0)|^2 + |x_3'(0)|^2) + (|v_1'(0)|^2 + |v_3'(0)|^2 - |v_1'(0)|^2)$$

and $\psi^* < , >= 2g^{Sas}$.

The next step is to show that $T^1\mathbb{S}^2(1)$ is isometric to $\mathbb{R}P^3(2)$. The unit tangent bundle $T^1\mathbb{S}^2(r)$ is diffeomorphic to projective space $\mathbb{R}P^3$ *via* the Euler parametric representation of $SO(3)$

$$\Phi : \mathbb{S}^3 \subset \mathbb{R}^4 \longrightarrow SO(3)$$

with

$$\Phi(\kappa, \lambda, \mu, \nu) = \begin{pmatrix} \kappa^2 + \lambda^2 - \mu^2 - \nu^2 & 2\kappa\nu + 2\lambda\mu & -2\kappa\mu + 2\lambda\nu \\ -2\kappa\nu + 2\lambda\mu & \kappa^2 - \lambda^2 + \mu^2 - \nu^2 & 2\kappa\lambda + 2\mu\nu \\ 2\kappa\mu + 2\lambda\nu & -2\kappa\lambda + 2\mu\nu & \kappa^2 - \lambda^2 - \mu^2 + \nu^2 \end{pmatrix}.$$

The map Φ is a group epimorphism from the subgroup $\mathbb{S}^3$ of the quaternionic field $\mathbb{H}$ onto $SO(Im\ \mathbb{H})$. If we write $q = \kappa + i\lambda + j\mu + k\nu$, then $\Phi(q)$ is the orthogonal transformation of $Im\ \mathbb{H}$ given by

$$u \longmapsto q^{-1}uq.$$

The differential of Φ at $e = 1 \in \mathbb{S}^3$ is a linear map from $T_e\mathbb{S}^3 = Im\ \mathbb{H}$ to the vector space of skewsymmetric endomorphisms of $Im\ \mathbb{H}$. In particular, if $X = X^1 i + X^2 j + X^3 k$, then

$$\begin{array}{rlcl} d\Phi_e(X): & Im\ \mathbb{H} & \longrightarrow & Im\ \mathbb{H} \\ & u & \longmapsto & Xu - uX. \end{array}$$

Since X and u are purely imaginary, the product Xu is again imaginary and can be identified with the usual wedge product in $\mathbb{R}^3 \simeq Im\ \mathbb{H}$. Thus

$$d\Phi_e(X) = 2 \begin{pmatrix} 0 & -X^3 & X^2 \\ X^3 & 0 & -X^1 \\ -X^2 & X^1 & 0 \end{pmatrix}$$

from where $< d\Phi_e(X), d\Phi_e(X) >= 8\|X\|^2$ and $\Phi^* < \, , \, >= 8g$.

Therefore the map Φ induces an isometry between $\mathbb{R}P^3(2) = \mathbb{S}^3(2)/\mathbb{Z}_2$ and $(SO(3), \frac{1}{2} < \, , \, >)$ which composite with ψ^{-1} gives an isometry between $\mathbb{R}P^3(2)$ and $T^1\mathbb{S}^2(1)$.

To finish the proof we only need to see how $T^1\mathbb{S}^2(r)$ is related with $T^1\mathbb{S}^2(1)$. Let h be the map from $T^1\mathbb{S}^2(1)$ to $T^1\mathbb{S}^2(r)$ given by $h(x, v) = (rx, v)$ then $dh_{(x,v)}(V'(0)) = (rx'(0), v'(0))$ and we have

$$g_r^{Sas}(dh_{(x,v)}(V'(0)), dh_{(x,v)}(V'(0))) = r^2 g(x'(0), x'(0)) + g(\frac{\nabla V}{dt}(0), \frac{\nabla V}{dt}(0))$$

where to avoid confusion we have denoted g_r^{Sas} the Sasaki metric on $T^1\mathbb{S}^2(r)$. So the Sasaki metric g_1^{Sas} only differs from the pull-back metric $h^*g_r^{Sas}$ by a factor r^2 on the horizontal distribution. On the other hand, the Berger metric g_μ^{Ber} on $\mathbb{S}^3(2)$ differs from the round metric by a factor μ in the direction of the Hopf fibration. Since the natural structures of $\mathbb{S}^1$-bundle over $\mathbb{S}^2$ of both spaces $\mathbb{S}^3$ and $T^1\mathbb{S}^2(r)$ are preserved by the map Φ, the pull-back metric $h^*g_r^{Sas}$ is equal to $r^2 g_\mu^{Ber}$ with $\mu = \frac{1}{r^2}$. □

Remark.– The last assertion of Proposition 1 was shown by Klingenberg and Sasaki [16].

Proposition 2 *Great* 2*-spheres are minimal surfaces of the* 3*-dimensional Berger sphere.*

Proof
Let $\mathbb{S}^2$ be the great sphere $\mathbb{S}^3 \cap v^\perp$ with $v \in \mathbb{S}^3 \subset \mathbb{R}^4$ and let $\{E_1, E_2\}$ be a local g-orthonormal frame of $\mathbb{S}^2$. We put $E_3(p) = p$ and $E_4(p) = v$ so that $\{E_1, E_2, E_3\}$ is a basis of $v^\perp$ and $\{E_i\}_{i=1}^4$ is a basis of $\mathbb{R}^4$, at each point of the definition domain. The unit normal η of the surface $\mathbb{S}^2 \subset (\mathbb{S}^3, g_\mu)$ must satisfy

$$g(\eta, E_3) = 0, \quad g_\mu(\eta, \eta) = 1 \quad \text{and} \quad g_\mu(\eta, E_i) = 0, i = 1, 2.$$

Since the Hopf vector field is $H = JE_3$, it is not difficult to see that we can choose

$$\eta = \frac{a_\mu}{\mu} E_4 + \frac{(1-\mu)g(H, E_4)}{a_\mu} \sum_{i=1}^{2} g(H, E_i) E_i$$

where $a_\mu = \sqrt{\mu + \mu(\mu - 1)(1 - g(H, E_4)^2)}$; it is easily checked that this function is well defined even if $0 < \mu < 1$. Let us remember that $\nabla_H H = 0$ and $\nabla_X H = JX$ for all $X \in H^\perp$. The Levi-Civita connection ∇^μ of the Berger metric g_μ is related with the connection ∇ of the round sphere by

$$\nabla^\mu_H X = \nabla_H X + (\mu - 1)\nabla_X H, \quad \nabla^\mu_X H = \mu \nabla_X H, \quad \nabla^\mu_X Y = \nabla_X Y,$$

for $X, Y \in H^\perp$. Using these expressions for any Z tangent to $\mathbb{S}^3$ we obtain

$$\nabla^\mu_Z \eta = \nabla_Z \eta + (\mu - 1)\left(g(Z, H)J(\eta - g(\eta, H)H) + g(\eta, H)J(Z - g(Z, H)H\right)$$

and then

$$g(\nabla^\mu_Z \eta, Z) = g(\nabla_Z \eta, Z) + (\mu - 1) g(Z, H) g(J\eta, Z).$$

The mean curvature k_{mean} is given by

$$2k_{mean} = \sum_{i=1}^{2} g(\nabla^{\mu}_{E_i}\eta, E_i) = (\mu - 1)g(H, E_4)g(J\eta, E_4) + \sum_{i=1}^{2} g(\nabla_{E_i}\eta, E_i)$$

where we have used the fact that $g(J\eta, H) = 0$ and $g(J\eta, E_4) = 0$. Moreover, it is easy to see that $g(J\eta, E_4)$ and that $g(\eta, H) = \dfrac{g(H, E_4)}{a_\mu}$ and then the diagonal elements of the covariant derivative of η are

$$\begin{aligned} g(\nabla_{E_i}\eta, E_i) &= (1-\mu)\{E_i(g(\eta, H)g(H, E_i)) - g(\eta, H)g(H, E_j)g(\nabla_{E_i}E_i, E_j)\} \\ &= (1-\mu)\{E_i(g(\eta, H))g(H, E_i) \\ &\qquad\qquad\qquad + g(\eta, H)g(\nabla_{E_i}E_i, H - g(H, E_j)E_j)\} \\ &= (1-\mu)g(H, E_i)E_i(g(\eta, H)), \end{aligned}$$

where either $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$. For the last equality we have used that E_4 is a constant field. We thus have

$$2k_{mean} = (1-\mu)\sum_{i=1}^{2} g(H, E_i)E_i(g(\eta, H)).$$

Since $E_4(g(H, E_4)) = 0$ and $H(g(H, E_4)) = 0$ we have $E_4(g(\eta, H)) = 0$ and $H(g(\eta, H)) = 0$ and finally

$$2k_{mean} = (1-\mu)H(g(\eta, H)) = 0.$$

$\square$

Definition. – For each vector $v \neq 0$ of $\mathbb{R}^3$, let ρ_v be the reflection across the hyperplane $v^\perp$. A *two dimensional Pontryagin cycle* of $SO(3)$ is any subset of the form $\overline{P} = \{\rho_v\rho_{v_0} \ ; \ v \in \mathbb{S}^2\}$ where $v_0 \neq 0$ is any vector of $\mathbb{R}^3$.

It was observed by Pedersen that there is a smooth unit vector field P defined on the whole sphere except one point such that the closure of its image on $T^1\mathbb{S}^2$ is precisely $\overline{P}$. She also defined this vector field analogously on any sphere and gave an alternative description in terms of parallel transport [17].

Definition. A *Pontryagin vector field* of $\mathbb{S}^n$ is any field P defined by parallel translating a given vector $v_0 \in T^1_{p_0}\mathbb{S}^n$ along great circles of $\mathbb{S}^n$ passing through p_0.

Proposition 3 *The Pontryagin fields of $\mathbb{S}^2(r)$ are minimal submanifolds of $T^1\mathbb{S}^2(r)$. Furthermore for $r = 1$ this submanifold is totally geodesic.*

Proof.
Let $(\epsilon_1, \epsilon_2, \epsilon_3)$ be an orthonormal basis of $\mathbb{R}^3$ and let P be the Pontryagin field obtained from $v_0 = -\epsilon_2 \in T^1_{\epsilon_1}\mathbb{S}^2(r)$. At a point $(rx, ry, rz) \in \mathbb{S}^2(r)$, P has the following expression (see [8])

$$P(rx, ry, rz) = -\frac{xy}{1+z}\epsilon_1 + (1 - \frac{y^2}{1+z})\epsilon_2 - y\epsilon_3.$$

It is then easily checked that $\overline{P} = (h \circ \Phi)(\mathbb{S}^2)$ where $\mathbb{S}^2 = \{(\kappa, \lambda, \mu, -\lambda) \in \mathbb{S}^3\} = \mathbb{S}^3 \cap v^\perp$ and $v = (0, 1, 0, 1)$. Since $h \circ \Phi$ is a Riemannian covering it yields from the above proposition that $\overline{P}$ is a minimal submanifold of $(T^1\mathbb{S}^2(r), g^{Sas})$. If $r = 1$ the Berger metric on $\mathbb{S}^3$ is the usual one and great spheres are totally geodesic surfaces, therefore $\overline{P}$ is a totally geodesic $\mathbb{R}P^2$ inside a round $\mathbb{R}P^3(2)$. $\square$

3 Vector fields without boundary

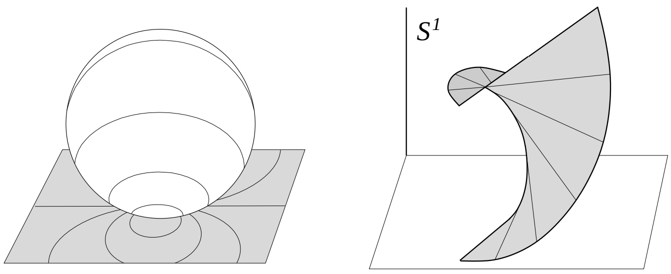
Definition. – Let U be a dense open subset of M and let $V : U \to T^1M$ be a smooth vector field. The field V is said to be *without boundary* if the closure $\overline{V}$ of its image $V(U)$ in T^1M is a smooth submanifold without boundary.

3.1 Pontryagin fields have no boundary

From Proposition 3, a Pontryagin field is a unit vector field without boundary. Here is another way to see that $\overline{P} \simeq \mathbb{R}P^2$. Let D_ρ be the disk of $\mathbb{S}^2$ centered at ϵ_1 and of radius ρ. Since the index of P at the singularity is 2, the image $V(\partial D_\rho)$ is a closed curve that surrounds twice the fiber sitting above the singularity. If $\pi : T^1\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is the projection, we have

$$\overline{P} \cap \pi^{-1}(D_\rho) \simeq \mathbb{M} \text{ (Moebius band).}$$

Since $P(\mathbb{S}^2 \setminus D_\rho)$ is a disk, we obtain $\overline{P} = Disk \cup \mathbb{M} \simeq \mathbb{R}P^2$. $\overline{P}$ is a $\mathbb{R}P^2$ inside $\mathbb{R}P^3$. The figure below shows both the Pontryagin field and the submanifold $\overline{P}$. The picture on the left represents the flow and its stereographic projection from the point antipodal to the sigularity, the one on the right visualizes the image of the vector field in $T^1\mathbb{S}^2$ in the trivialization given by the stereographic projection from the point antipodal to the singularity. The fibers are represented in the vertical axis as intervals, as usual the ends of each interval must be identified.



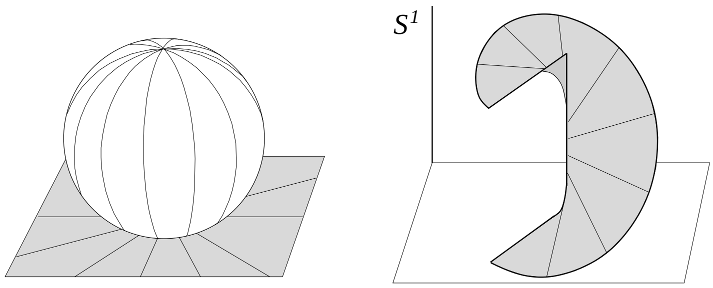
Remark.– For dimension $n > 2$ the Pontryagin field is no longer a vector field without boundary since $\overline{P}$ has an isolated conical point, see [17]. Nevertheless the vector field P has an interesting property: its closure $\overline{P}$ is a $\mathbb{Z}_2$-cycle.

3.2 The radial field has boundary

A radial field R is a unit vector field defined on the complementary of two antipodal points $\{-p, p\}$ of $\mathbb{S}^n$ and which is tangent to great circles passing through p. This field seems to play an important role in the Gluck and Ziller problem. In fact Brito, Chacón and Naveira have shown that the volume of the radial field provides a lower bound for the volume of smooth everywhere defined unit vector field on odd-dimensional spheres [3]. In dimension two, the area of the radial field is lower than the one of the Pontryagin field. Nevertheless the radial field has boundary. Indeed,

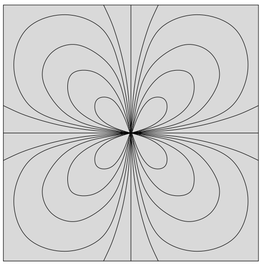
$$\partial\overline{R} = \pi^{-1}(p) \cup \pi^{-1}(-p)$$

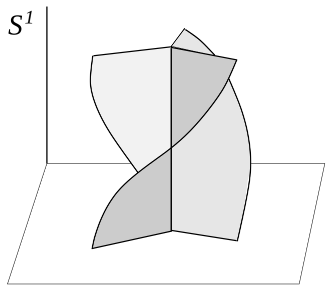
and thus $\overline{R}$ is a cylinder with boundary two fibers. The figure below shows the flow of the radial vector field and a portion of $\overline{R}$ with one boundary component.



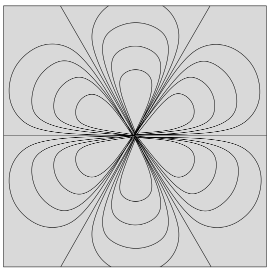
3.3 Examples of singular vector fields with boundary

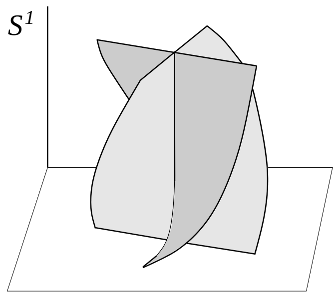
We have seen that the singularities of index one of the radial field give boundary components. Above a singularity of index two, the fiber is covered (at least) twice. So, generically the local situation is that of two (or more) half planes that fit together along the fiber. It can thus happen that the boundary component disappears as it is the case for Pontryagin fields. The following two pictures illustrate some situations where the index is greater than two.





The image of the left is the stereographic projection of the flow from the antipodal point of the singularity. The resulting space in the right has the structure of a CW-complex. In each loop the tangent vector turns on a $\frac{2\pi}{3}$ angle, so the index of the singularity is 3. The figure below shows an example with a singularity of index 4.





3.4 Topology of unit vector fields without boundary

Lemma. – *Let V be a unit vector field of $\mathbb{S}^2$ without boundary and let $[\overline{V}] \in H_2(T^1\mathbb{S}^2, \mathbb{Z}_2)$ be the $\mathbb{Z}_2$-fundamental class of $\overline{V}$. Then $[\overline{V}]$ is the non-trivial class of $H_2(T^1\mathbb{S}^2, \mathbb{Z}_2) = \mathbb{Z}_2$.*

Proof
First observe that the integral 2-homology group of $T^1\mathbb{S}^2$ is trivial while it is not the case with $\mathbb{Z}_2$-coefficient since $H_2(T^1\mathbb{S}^2, \mathbb{Z}_2) = H_2(\mathbb{R}P^3, \mathbb{Z}_2) = \mathbb{Z}_2$ (the generator is the class of $\mathbb{R}P^2 \subset \mathbb{R}P^3$). Let $\pi_* : H_2(T^1\mathbb{S}^2, \mathbb{Z}_2) \longrightarrow H_2(\mathbb{S}^2, \mathbb{Z}_2)$ be the map induced by the projection $\pi : T^1\mathbb{S}^2 \longrightarrow \mathbb{S}^2$. Using the theory of local degree (with $\mathbb{Z}_2$ coefficient) and since V is a vector field it is readily checked that $\pi_*[\overline{V}] = [\mathbb{S}^2]$ where $[\mathbb{S}^2]$ denotes the $\mathbb{Z}_2$-fundamental class of $\mathbb{S}^2$. In particular $[\overline{V}]$ cannot be the trivial class. $\square$

Let V be a unit vector field of $\mathbb{S}^2$ without boundary. We say that V is in general position if V has only a finite number of singular points $p_1, ..., p_N$ and the fibers $\pi^{-1}(p_1)$,...,$\pi^{-1}(p_N)$ are all included in $\overline{V}$.

Lemma. – *Let V be a unit vector field of $\mathbb{S}^2$ without boundary in general position. Then $\overline{V}$ is homeomorphic to $\mathbb{R}P^2 \sharp \mathbb{T}_g$ for some $g \in \mathbb{N}$.*

Proof
As $\overline{V}$ is a submanifold, the indexes of $p_1, ..., p_N$ must be ± 2. Let n_2 (resp. n_{-2}) be the number of singularities with index 2 (resp. with index -2). Since the Euler number of $\mathbb{S}^2$ is 2, we have $n_2 = n_{-2} + 1$. Let D^i_ρ be a disk of $\mathbb{S}^2$ centered at p_i and of small radius ρ, we have

$$\overline{V} \cap \pi^{-1}(D^i_\rho) \simeq \mathbb{M}$$

and thus

$$\overline{V} \simeq \left(\mathbb{S}^2 \setminus \coprod_{i=1}^{N} D^i_\rho \right) \bigcup_{i=1}^{N} \mathbb{M} \simeq \mathbb{R}P^2 \sharp \mathbb{T}_g$$

where $\mathbb{T}_g$ denote the torus with g holes ($g = n_{-2}$) and $N = 2n_{-2} + 1$. $\square$

3.5 Proof of Theorem 1

By the previous result, the closure of every unit vector field without boundary of $\mathbb{S}^2(1)$ belongs to the $\mathbb{Z}_2$-homology class represented by the totally geodesic $\mathbb{R}P^2$. Berger and Fomenko have shown that totally geodesic projective planes are the only area minimizers in their homology class [1], [6]. And every totally geodesic $\mathbb{R}P^2$ is the closure of a Pontryagin field. $\square$

Remark. In [14] Johnson studies the Gluck and Ziller problem in any Riemannian 2-torus. He shows that, for any homotopy class of unit vector fields, there is a smooth field of smallest area. The problem has also been solved in Berger 3-spheres $(\mathbb{S}^3, g_\mu^{Ber})$ with $\mu < 1$, [11]. For $\mu > 1$ some results can be found in [7].

4 Proof of Theorem 2

To prove this theorem we follow the ideas of Hardt and Rosenberg in Theorem 1.2 of [12]. Nevertheless, since our hypothesis is weaker than their, we need to adapt the proof. In particular we need the following lemma.

Lemma *Let V be a vector field without boundary and let F be a fiber of $\pi : T^1\mathbb{S}^2 \to \mathbb{S}^2$ such that $\#(F \cap \overline{V}) \geq 2$. Then at every point $p \in F \cap \overline{V}$ the intersection is not transverse, i. e. $T_pF \subset T_p\overline{V}$.*

Proof.
Assume that there exist $p \in F \cap \overline{V}$ such that $T_p\overline{V}$ is transverse to T_pF. For any other point $q \in F \cap \overline{V}$ the space $T_q\overline{V}$ cannot be transverse to T_qF since this will immediatly lead to a contradiction with the fact that V is a section of $T^1\mathbb{S}^2$ over a dense open set. This point is isolated from $A = (F \cap \overline{V}) \setminus \{p\}$ in F. Indeed, if $(q_n)_{n\in\mathbb{N}}$ is a sequence of points in A converging towards p then the inclusions $T_{q_n}F \subset T_{q_n}\overline{V}$ will imply $T_pF \subset T_p\overline{V}$. Therefore the subset A is closed in F. Moreover it has non empty boundary on F since otherwise the connectedness of the fiber will imply that $A = F$ which is impossible. Let q be an element of this boundary. Since $\overline{V}$ is a submanifold, locally it is a graph over an open ball B of $T_q\overline{V}$, more precisely in a bundle chart of $T^1\mathbb{S}^2$ in which the projection π reads $(x, y, z) \mapsto (x, y)$ there is a neighborhood $U \subset \overline{V}$ of q, a smooth function $f : B \longrightarrow \mathbb{R}$ such that we have

$$U = \{(f(y, z), y, z) \ ; \ (y, z) \in B\}.$$

We also assume that in this chart the coordinates of q are $(0, 0, 0)$. Since $q \in Fr\ A$ there exists a sequence $(0, z_n)_{n\in\mathbb{N}}$ of points of T_qF converging towards q and such that $f(0, z_n) \neq 0$. By the Mean-value theorem, there is a sequence $(w_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, \quad 0 < |w_n| < |z_n| \quad \text{and} \quad \frac{\partial f}{\partial z}(0, w_n) \neq 0.$$

Let $\tilde{q}_n = (f(0, w_n), 0, w_n)$. We thus have $rank\ d\pi_{\tilde{q}_n} = 2$ and π is a diffeomorphism from an open disk D_n centered at $\tilde{q}_n$ onto its image. Since $T_p\overline{V}$ is transverse to T_pF the projection, π is a diffeomorphism from an open disk B centered at p onto its image. Note that we can always assume that

$D_n \cap B = \emptyset$. Since $\tilde{q}_n \to q$ we have $\pi(\tilde{q}_n) \to \pi(q) = \pi(p)$ thus $\pi(B) \cap \pi(D_n)$ is non empty for sufficiently large n. Finally $W = \pi(B) \cap \pi(D_n)$ is an open set for which

$$\forall w \in W, \quad \#(\pi^{-1}(w) \cap \overline{V}) \geq 2.$$

This is a contradiction because V is a vector field on a dense set. $\square$

Theorem 2 *The only minimal surfaces of $T^1\mathbb{S}^2(r)$ homeomorphic to the projective plane arising from vector fields without boundary are Pontryagin cycles.*

Proof.
Let $\overline{V}$ be a minimal smooth surface homeomorphic to $\mathbb{R}P^2$ arising from $V : U \subset \mathbb{S}^2(r) \longrightarrow T^1\mathbb{S}^2(r)$. Since the Euler characteristic of the 2-sphere is 2, the set of singular points

$$Sing(\overline{V}) = \{x \in \mathbb{S}^2 \quad ; \quad \#(\pi^{-1}(x) \cap \overline{V}\ ) \geq 2\}$$

is not empty. Take $x \in Sing(\overline{V})$ and let $\{\overline{P}_\theta \ ; \ \theta \in \mathbb{S}^1\}$ be the family of Pontryagin surfaces whose singular fibre is precisely $\pi^{-1}(x)$. This fibration of $T^1\mathbb{S}^2(r) \setminus \pi^{-1}(x)$ into smooth minimal surfaces is an open book structure for $N = T^1\mathbb{S}^2(r)$ with binding $\Gamma = \pi^{-1}(x)$ and leaves $L_\theta = \overline{P}_\theta \setminus \Gamma$, as defined in [12]. Every Pontryagin cycle $\overline{P}_\theta$ can be seen as the image of a map ψ_θ from a closed 2-disk D^2 to $T^1\mathbb{S}^2(r)$ such that

1. The restriction of ψ_θ to the interior of D^2 is a diffeomorphism onto the leaf L_θ.
2. $\psi_\theta^{-1}(\Gamma) = \partial D^2$.
3. The restriction of ψ_θ to ∂D^2 is $2:1$.

The map $F : N \setminus \Gamma \longrightarrow \mathbb{S}^1$, given by $F(z) = \theta$ if $z \in L_\theta$, is a smooth submersion and then the pull-back under F of an orienting 1-form α of the circle is an smooth closed 1-form $\omega = F^*\alpha$ in $N \setminus \Gamma$. If we denote by $\varphi : \overline{V} \longrightarrow N$ the inclusion map, the pull-back $\varphi^*\omega$ is a closed 1-form defined on $\overline{V} \setminus \overline{V}_\Gamma$, where $\overline{V}_\Gamma = \overline{V} \cap \Gamma$. By construction, $\overline{V}_\Gamma$ contains at least two points and by the previous lemma, $\overline{V}$ and Γ cannot intersect transversely. More precisely, this intersection is of maximal rank at every point, i. e. for all $p \in \overline{V}_\Gamma$ we have $T_p\Gamma \subset T_p\overline{V}$. Therefore, by the maximality of the rank, $\varphi^*\omega$ extends to a smooth 1-form on all of $\overline{V}$ (see [13] p. 478), which is exact since $H_1(\overline{V}, \mathbb{R}) = 0$.

Now we can use the same arguments as in the proof of Theorem 1.2 of [12] to obtain that $\overline{V}$ should be one of the P_θ.

$\square$

5 Proof of Theorem 3

Theorem 3 *Among unit vector fields of $\mathbb{S}^2(r)$ with one singularity those of least area are Pontryagin fields and no others.*

Proof.
For each unit vector field V^r of $\mathbb{S}^2(r)$, let us define a unit vector field of $\mathbb{S}^2(1)$ by $V(p) = V^r(rp)$. The volume of V^r is given by

$$Vol(V^r) \quad = \quad \int_{\mathbb{S}^2(1)} \sqrt{det(r^2 Id +^T \nabla V \circ \nabla V)}\, dvol$$

Let us assume that the single singular point of V is the south pole $s = (0,0,-1)$ and let us choose an orthonormal frame $\{P, P^\perp\}$ on $\mathbb{S}^2(1) \setminus \{s\}$ consisting of two Pontryagin fields

$$P(x,y,z) = \Big(-\frac{xy}{1+z}, 1-\frac{y^2}{1+z}, -y\Big),\ P^\perp(x,y,z) = \Big(1-\frac{x^2}{1+z}, -\frac{xy}{1+z}, -x\Big).$$

Thus there is a function $\theta : \mathbb{S}^2(1) \setminus \{s\} \to \mathbb{R}$ such that on $\mathbb{S}^2(1) \setminus \{s\}$ the vector field is of the form

$$V = \cos\theta P + \sin\theta P^\perp.$$

Lemma. – *If $\{E_1, E_2\}$ is an orthonormal frame on an open set U of a 2-dimensional Riemannian manifold (M^2, g) and $V = \cos\theta E_1 + \sin\theta E_2$ is a unit vector field on U then*

$$^T\nabla V \circ \nabla V = \begin{pmatrix} \beta_1^2 & \beta_1\beta_2 \\ \beta_1\beta_2 & \beta_2^2 \end{pmatrix}$$

where $\beta = d\theta + \alpha$, $\alpha(X) = g(\nabla_X E_1, E_2)$ and $\beta_i = \beta(E_i)$, $i = 1$ or 2.

We omit the proof of this lemma that reduces to an elementary computation. For the local frame $\{E_1, E_2\} = \{P, P^\perp\}$, it is easy to show that

$$\alpha(E_1) = \alpha(P) = \frac{x}{1+z}, \qquad \alpha(E_2) = \alpha(P^\perp) = -\frac{y}{1+z}$$

and thus

$$\begin{aligned} det(r^2 Id +^T \nabla V \circ \nabla V) &= r^2\Big(r^2 + (\theta_1 + \tfrac{x}{1+z})^2 + (\theta_2 - \tfrac{y}{1+z})^2\Big) \\ &= r^2\Big(r^2 + \tfrac{1-z}{1+z} + \tfrac{2}{1+z}(x\theta_1 - y\theta_2) + \theta_1^2 + \theta_2^2\Big) \end{aligned}$$

where $\theta_i = \theta(E_i)$, $i = 1$ or 2. Moreover

$$x\theta_1 - y\theta_2 = xE_1(\theta) - yE_2(\theta) = (xP - yP^\perp)(\theta) = T(\theta)$$

with

$$T(x, y, z) = (-y, x, 0).$$

Finally

$$\begin{aligned} Vol(V^r) &= \int_{\mathbb{S}^2(1)} r\Big(r^2 + \frac{1-z}{1+z} + \frac{2}{1+z}T(\theta) + \theta_1^2 + \theta_2^2\Big)^{\frac{1}{2}}\, dvol \\ &= \int_{\mathbb{S}^2(1)} r\Big(r^2 + \frac{1-z}{1+z}\Big)^{\frac{1}{2}}\Big(1 + h_1(z)T(\theta) + h_2(z)(\theta_1^2 + \theta_2^2)\Big)^{\frac{1}{2}} dvol \end{aligned}$$

where

$$h_1(z) = \frac{2}{(r^2+1) + (r^2-1)z} \quad \text{and} \quad h_2(z) = \frac{1+z}{(r^2+1) + (r^2-1)z}.$$

In particular, if $\theta \equiv 0$, then $V^r = P^r$ and we obtain

$$Vol(P^r) = \int_{\mathbb{S}^2(1)} r\sqrt{r^2 + \frac{1-z}{1+z}}\, dvol.$$

Let $p \mapsto R(p) = p \wedge \dfrac{T(p)}{\sqrt{x^2+y^2}}$ be the unit radial field on $\mathbb{S}^2(1) \subset \mathbb{R}^3$, we have

$$\theta_1^2 + \theta_2^2 = R(\theta)^2 + \frac{T(\theta)^2}{1-z^2}$$

and thus

$$\begin{aligned} Vol(V^r) &= \int_{\mathbb{S}^2(1)} \lambda\Big(1 + h_1(z)T(\theta) + \frac{h_2(z)}{1-z^2}T(\theta)^2 + h_2(z)R(\theta)^2\Big)^{\frac{1}{2}}\, dvol \\ &= \int_{\mathbb{S}^2(1)} \lambda\Big(\Big(1 + \frac{h_1(z)}{2}T(\theta)\Big)^2 + \tilde{h}_3(z)T(\theta)^2 + h_2(z)R(\theta)^2\Big)^{\frac{1}{2}} dvol \end{aligned}$$

where

$$\lambda = r\sqrt{r^2 + \frac{1-z}{1+z}} \text{ and } \tilde{h}_3(z) = \frac{h_2(z)}{1-z^2} - \frac{h_1(z)^2}{4}.$$

Since $\tilde{h}_3(z) > 0$ for $-1 \leq z < 1$ we set

$$h_3(z) := \sqrt{\frac{h_2(z)}{1-z^2} - \frac{h_1(z)^2}{4}} = r\sqrt{\frac{1+z}{1-z}}\frac{h_1(z)}{2}$$

and finally we have

$$Vol(V^r) = \int_{\mathbb{S}^2(1)} \lambda\sqrt{\left(1 + \frac{1}{2}h_1(z)T(\theta)\right)^2 + h_3(z)^2T(\theta)^2 + h_2(z)R(\theta)^2}\, dvol.$$

For $z \geq -1$, $h_2(z) \geq 0$ and then

$$\begin{aligned} Vol(V^r) &\geq \int_{\mathbb{S}^2(1)} \lambda \left|1 + \frac{h_1(z)}{2}T(\theta)\right| dvol \\ &\geq \left|\int_{\mathbb{S}^2(1)} \lambda \left(1 + \frac{h_1(z)}{2}T(\theta)\right) dvol\right| \end{aligned}$$

The integral curves of T are the parallel circles $C_a = \mathbb{S}^2(1) \cap \{z = a\}$ of $\mathbb{S}^2(1)$ and :

$$\int_{C_a} \lambda h_1(z)T(\theta)\, dvol_{C_a} = function(a) \times 2k\pi$$

with $k \in \mathbb{Z}$. The relative integer k is the algebraic number of turns of V in the frame $\{P, P^{\perp}\}$. Since s is the only singular point of V and the index of V at s is the same as the one of P at s, the number k is zero and the above integral vanishes for every $a \in [-1, 1]$. Therefore

$$Vol(V^r) \geq \int_{\mathbb{S}^2(1)} \lambda\, dvol = Vol(P^r).$$

Equality holds if and only if $\theta \equiv cte$ i.e if V^r is a Pontryagin field. □

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V. Borrelli
Institut Camille Jordan, Université Claude Bernard, Lyon 1
43, boulevard du 11 Novembre 1918
69622 Villeurbanne Cedex, France
email : borrelli@math.univ-lyon1.fr

O. Gil-Medrano
Departamento de Geometría y Topología, Facultad de Matemáticas
Universidad de Valencia
46100 Burjassot, Valencia, España
email : Olga.Gil@uv.es