Talk I: One dimensional Convex Integration

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June 30, 2013

Convex Integration Theory is a powerful tool for solving differential relations. It was introduced by M. Gromov in his thesis dissertation in 1969, then published in an article [2] in 1973 and eventually generalized in a book [3] in 1986. Nevertheless, reading Gromov is often a challenge since important details are not provided explicitely. Fortunately, there is a good reference that leaves no details in the shadow : the Spring's book [5]. My understanding of Convex Integration Theory primarily comes from this book. I owe it much in this presentation.

1 Two introductory examples

1.1 A first example

Let us consider the following elementary problem.

Problem 1.– Let

$$\begin{array}{rccc} f_0 : & [0,1] & \longrightarrow & \mathbb{R}^3 \\ & t & \longmapsto & (0,0,t) \end{array}$$

be the linear application mapping the segment $[0,1]$ vertically in $\mathbb{R}^3$. The problem is to find $f : [0,1] \overset{C^1}{\longrightarrow} \mathbb{R}^3$ such that:

i) $\forall\, t \in [0,1], \quad |\cos(f'(t), e_3)| < \epsilon$
ii) $\|f - f_0\|_{C^0} < \delta$
where $\epsilon > 0$ and $\delta > 0$ are given.

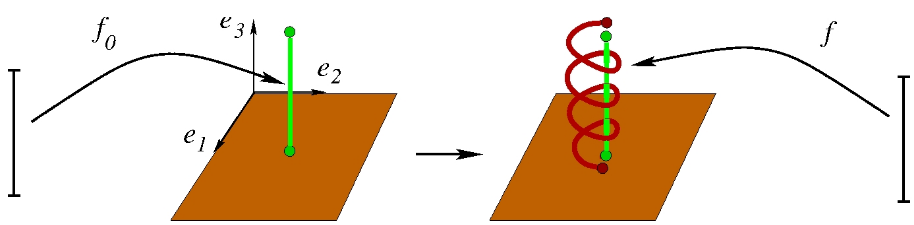
Solution.– At a first glance, the problem seems hopeless since condition i says that the slope is small and then the image has to move far away from the segment before reaching the desired height. After a few seconds of extra thinking, the solution occurs. It is good enough to move along an helix spiralling around the vertical axis:

$$f: \quad [0,1] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \begin{cases} \delta \cos 2\pi N t \\ \delta \sin 2\pi N t \\ t \end{cases}$$

where $N \in \mathbb{N}^*$ is the number of spirals. We have

$$\langle \frac{f'}{\|f'\|}, e_3 \rangle = \frac{1}{\sqrt{1 + 4\pi^2 N^2 \delta^2}}.$$

Therefore, if N is large enough, f fulfills conditions i and ii.

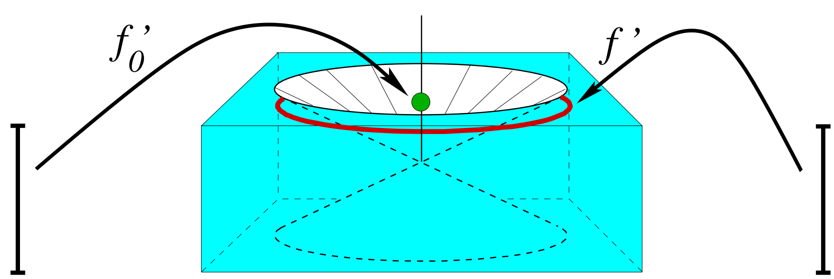


The image of f_0 is the green vertical segment, the solution f is the red helix.

Rephrasing.– The above problem was pretty easy, it will become very informative with a rephrasing of the two conditions. Condition (i) means that the image of f' lies inside the cone:

$$\mathcal{R} = \{v \in \mathbb{R}^3 \setminus \{O\} \mid \left|\langle \frac{v}{|v|}, e_3 \rangle\right| < \epsilon\} \cup \{O\}.$$

By extension, that cone $\mathcal{R}$ is called the *differential relation* of our problem.



The cone $\mathcal{R}$ is pictured in blue, the image of f' is the red circle and the constant image of f_0' the green point outside the cone.

The C^0-closeness required in the second condition, is a consequence of a geometric property of the derivative of f. Indeed, the image of f' in that cone is a circle whose center is the constant image of f_0'. Therefore, the average of f' for each spiral of f is $f_0'(t)$:

$$\frac{1}{Length(I_k)} \int_{I_k} f'(u)du = f_0'(t)$$

where $I_k = [\frac{k}{N}, \frac{k+1}{N}]$ the preimage of one spiral by f. Therefore, when integrating, the two resulting maps are closed together.

1.2 A more general example

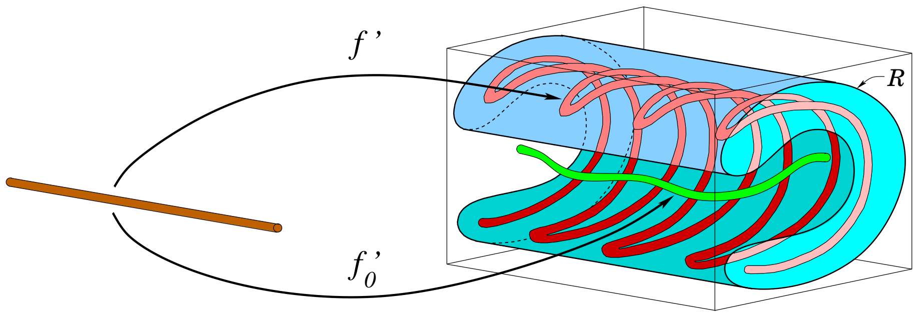
Problem.– Let $\mathcal{R} \subset \mathbb{R}^3$ be a path-connected subset (=our differential relation) and $f_0 : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ be a map such

$$\forall t \in [0,1], \quad f_0'(t) \in \ IntConv(\mathcal{R})$$

where $IntConv(\mathcal{R})$ denotes the interior of the convex hull of $\mathcal{R}$. The problem is to find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ such that :

i) $\forall t \in [0,1], \quad f'(t) \in \mathcal{R}$
ii) $\|f - f_0\|_{C^0} < \delta$
with $\delta > 0$ given.

Solution.– From the hypothesis, the image of f_0' lies in the convex hull of $\mathcal{R}$. The idea is to build f' with an image lying inside $\mathcal{R}$ and such that, on average, it looks like the derivative of f_0. One way to do that is to choose a the f'-image to resemble to a kind of spring. In the spring, each arc as the same effect, on average, as a small piece of the image of the initial map f_0'. So, when integrating, the resulting map will be close to the initial map. As before, we will improve the closeness of f to f_0 by increasing the number of spirals.



The green bended spaghetti[1] *pictures the image of* f_0'*, the half of a spring in rep/pink is the chosen image for* f'*.*

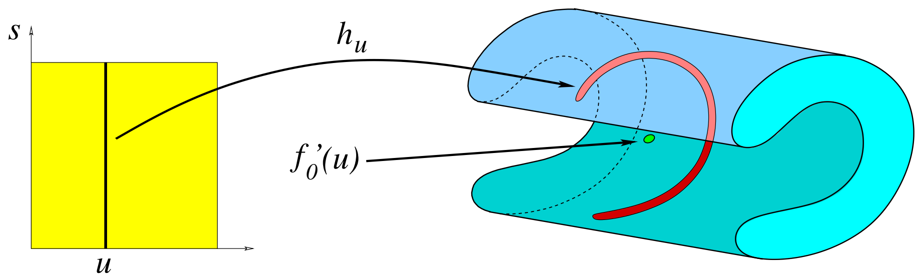
To formally construct a solution f of the problem, it is enough to choose a continuous family of loops of $\mathcal{R}$:

$$
\begin{array}{rccc}
h: & [0,1] & \longrightarrow & C^0(\mathbb{R}/\mathbb{Z}, \mathcal{R}) \\
 & u & \longmapsto & h_u
\end{array}
$$

such that

$$\forall u \in [0,1], \quad \int_{[0,1]} h_u(s)ds = f_0'(u)$$

$i.e$ the average of the loop h_u is $f_0'(u)$.



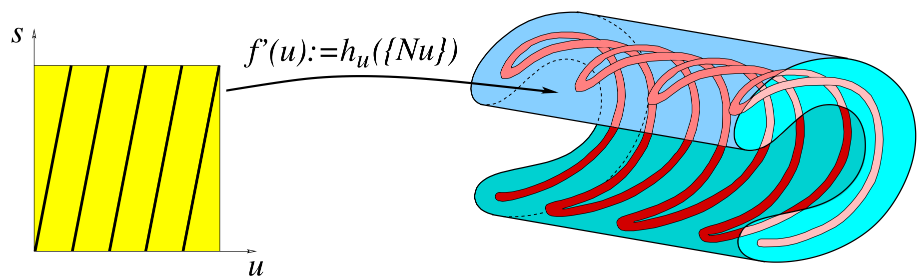
The image of the loop h_u*. In that picture, this image is an arc. This loop is a round-trip starting at one of the endpoint of the arc and arriving at the same endpoint.*

Then, the map f' is extracted from that familly of loops by a simple diagonal process

$$\forall t \in [0,1], \quad f'(t) := h_t(\{Nt\})$$

where $N \in \mathbb{N}^*$ and $\{Nu\}$ is the fractional part of Nt.

[1] Spaghetto ?



The image of f'.

Eventually, it remains to integrate to obtain a solution to our problem:

$$f(t) := f_0(0) + \int_0^t h_u(\{Nu\})du.$$

We say that f is obtained from f_0 by a **convex integration process**. We denote $f := IC(f_0, h, N)$.

2 Finding the loops

In the above problem, we were wilfully blind to the question of the existence of the family of loops $(h_u)_{u\in[0,1]}$ needed to build the solution. We now deal with that issue.

Notation.– Let $A \subset \mathbb{R}^n$ and $a \in A$. We denote by $IntConv(A, a)$ the interior of the convex hull of the connected component of A to which a belongs.

Definition.– A (continuous) loop $g : [0,1] \to \mathbb{R}^n$, $g(0) = g(1)$, *strictly surrounds* $z \in \mathbb{R}^n$ if

$$IntConv(g([0,1])) \supset \{z\}.$$

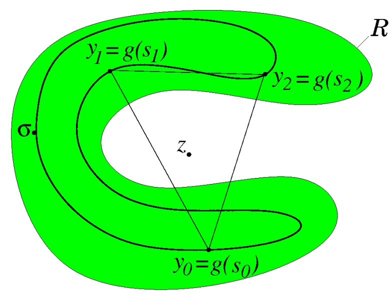
Fundamental Lemma.– *Let $\mathcal{R} \subset \mathbb{R}^n$ be an open set, $\sigma \in \mathcal{R}$ and $z \in IntConv(\mathcal{R}, \sigma)$ There exists a loop $h : [0,1] \xrightarrow{C^0} \mathcal{R}$ with base point σ that strictly surrounds z and such that:*

$$z = \int_0^1 h(s)ds.$$

Proof.– Since $z \in IntConv(\mathcal{R}, \sigma)$, there exists a n-simplex Δ whose vertices $y_0, ..., y_n$ belong to $\mathcal{R}$ and such that z lies in the interior of Δ. Therefore, there also exist

$$(\alpha_0, ..., \alpha_n) \in \,]0,1[\,^{n+1}$$

such that $\sum_{k=0}^{n} \alpha_k = 1$ and $z = \sum_{k=0}^{n} \alpha_k y_k$. Every loop $g : [0,1] \to \mathcal{R}$ with base point σ and passing through $y_0, ..., y_n$ satisfies $IntConv(g([0,1]) \supset \{z\}$ i. e. g surrounds z.



In general

$$z \neq \int_0^1 g(s)ds.$$

Let $s_0, ..., s_n$ be the times for which $g(s_k) = y_k$ and let $f_k : [0,1] \to \mathbb{R}_+^*$ be such that :

i) $f_k < \eta_1$ sur $[0,1] \setminus [s_k - \eta_2, s_k + \eta_2]$,

ii) $\int_0^1 f_k = 1$,

with η_1, η_2 two small positive numbers. We set:

$$z_k := \int_0^1 g(s) f_k(s) ds.$$

The number $\epsilon > 0$ being given, we can choose η_1, η_2 such that:

$$\forall k \in \{0, ..., n\}, \qquad \|z_k - g(s_k)\| \leq \epsilon.$$

Since $\mathcal{R}$ in open and $z \in Int\,\Delta$, for ϵ small enough we have

$$z \in IntConv(z_0, ..., z_n).$$

Therefore, there exist $(p_0, ..., p_n) \in \,]0,1[\,^{n+1}$ such that $\sum_{k=0}^{n} p_k = 1$ and:

$$\begin{aligned} z &= \sum_{k=0}^{n} p_k z_k &&= \sum_{k=0}^{n} p_k \int_0^1 g(s) f_k(s) ds \\ &= \int_0^1 g(s) \sum_{k=0}^{n} p_k f_k(s) ds &&= \int_0^1 g(s) \varphi'(s) ds \end{aligned}$$

where we have set

$$\varphi'(s) := \sum_{k=0}^{n} p_k f_k(s)$$

and

$$\begin{array}{rccc} \varphi: & [0,1] & \longrightarrow & [0,1] \\ & s & \longmapsto & \int_0^s \varphi(u) du. \end{array}$$

We have $\varphi'(s) > 0$, $\varphi(0) = 0$, $\varphi(1) = 1$. Thus φ is a strictly increasing diffeomorphism of $[0,1]$. Let us employ the change of coordinates $s = \varphi^{-1}(t)$, that is $t = \varphi(s)$, we have

$$dt = \varphi'(s) ds$$

therefore:

$$z = \int_0^1 g(s)\varphi'(s) ds = \int_0^1 g \circ \varphi^{-1}(t) dt.$$

Thus $h = g \circ \varphi^{-1}$ is our desired loop. □

Remark.– *A priori* $h \in \Omega_\sigma(\mathcal{R})$, but it is obvious that we can choose h among "round-trips" *i.e* the space:

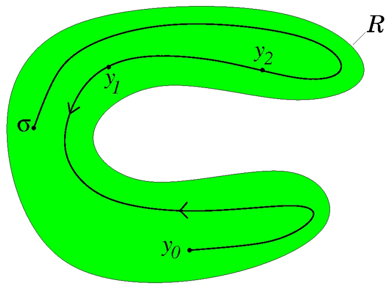
$$\Omega_\sigma^{AR}(\mathcal{R}) = \{h \in \Omega_\sigma(\mathcal{R}) \mid \forall s \in [0,1] \;\; h(s) = h(1-s)\}.$$

The point is that the above space is contractible. For every $u \in [0,1]$ we then denote by $h_u : [0,1] \longrightarrow \mathcal{R}$ the map defined by

$$h_u(s) = \begin{cases} h(s) & \text{if} \quad s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}] \\ h(u) & \text{if} \quad s \in [\frac{u}{2}, 1 - \frac{u}{2}]. \end{cases}$$

This homotopy induces a deformation retract of $\Omega_\sigma^{AR}(\mathcal{R})$ to the constant map

$$\begin{array}{rccc} \widetilde{\sigma}: & [0,1] & \longrightarrow & \mathcal{R} \\ & s & \longmapsto & \sigma. \end{array}$$



3 C^0-density

Let $\mathcal{R} \subset \mathbb{R}^n$ be a arc-connected subset, $f_0 \in C^\infty(I, \mathbb{R}^n)$ be a map such that $f_0'(I) \subset \ IntConv(\mathcal{R})$. From the C^∞ parametric version of the Fundamental Lemma there exists a C^∞-map $h : I \times \mathbb{E}/\mathbb{Z} \longrightarrow \mathcal{R}$ such that

$$\forall t \in I, \ \ f_0'(t) = \int_0^1 h(t, u) du.$$

We set

$$\forall t \in I, \qquad F(t) := f_0(0) + \int_0^t h(s, Ns) ds$$

with $N \in \mathbb{N}^*$.

Definition.– We say that $F \in C^\infty(I, \mathbb{R}^n)$ is obtained from f_0 by an *convex integration process.*

Obviously $F'(t) = h(t, Nt) \in \mathcal{R}$ and thus F is a solution of the differential relation $\mathcal{R}$. One crucial property of the convex integration process is that the solution F can be made arbitrarily close to the initial map f_0..

Proposition (C^0-density).– *We have*

$$\|F - f_0\|_{C^0} \leq \frac{1}{N} \left(2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0} \right)$$

where $\|g\|_{C^0} = \sup_{p \in D} \|g(p)\|_{\mathbb{E}^3}$ *denotes the* C^0 *norm of a function* $g : D \rightarrow \mathbb{E}^3$.

Proof.– Let $t \in [0,1]$. We put $n := [Nt]$ (the integer part of Nt) and set $I_j = [\frac{j}{N}, \frac{j+1}{N}]$ for $0 \leq j \leq n-1$ and $I_n = [\frac{n}{N}, t]$. We write

$$F(t) - f(0) = \sum_{j=0}^{n} S_j \quad \text{and} \quad f_0(t) - f_0(0) = \sum_{j=0}^{n} s_j$$

with $S_j := \int_{I_j} h(v, Nv)\mathrm{d}v$ and $s_j := \int_{I_j} \int_0^1 h(x,u)\mathrm{d}u\mathrm{d}x$. By the change of variables $u = Nv - j$, we get for each $j \in [0, n-1]$

$$S_j = \frac{1}{N} \int_0^1 h(\frac{u+j}{N}, u)\mathrm{d}u = \int_{I_j} \int_0^1 h(\frac{u+j}{N}, u)\mathrm{d}u\mathrm{d}x.$$

It ensues that

$$\|S_j - s_j\|_{\mathbb{E}^3} \leq \frac{1}{N^2} \|\frac{\partial h}{\partial t}\|_{C^0}.$$

The proposition then follows from the obvious inequalities

$$\|S_n - s_n\|_{\mathbb{E}^3} \leq \frac{2}{N} \|h\|_{C^0} \quad \text{and} \quad \|F(t) - f_0(t)\|_{\mathbb{E}^3} \leq \sum_{j=0}^{n} \|S_j - s_j\|_{\mathbb{E}^3}.$$

□

Remark.– Even if $f_0(0) = f_0(1)$, the map F obtained by a convex integration from f_0 does not satisfy $F(0) = F(1)$ in general. This can be easily corrected by defining a new map f with the formula

$$\forall t \in [0,1] \ , \ f(t) = F(t) - t\left(F(1) - F(0)\right).$$

The following proposition shows that the C^0-density property still holds for f and, provided N is large enough, that the map f is still a solution of $\mathcal{R}$.

Proposition.– *We have*

$$\|f - f_0\|_{C^0} \leq \frac{2}{N} \left(2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0} \right)$$

and $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$.

Proof.– The first inequality is obvious. Indeed, from

$$F(1) - F(0) = F(1) - f_0(0) = F(1) - f_0(1)$$

we deduce

$$\|f(t) - f_0(t)\| \leq \|F(t) - f_0(t)\| + \|F(1) - f_0(1)\| \leq 2\|F - f_0\|_{C^0}.$$

Derivating f we have $f'(t) = F'(t) - (F(1) - F(0))$ thus

$$\|f' - F'\|_{C^0} \leq \|F - f_0\|_{C^0} = O\left(\frac{1}{N}\right).$$

Since $\mathcal{R}$ is open, if N is large enough $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$. $\square$

References

[1] M. Ghomi, *h-principles for curves and knots of constant curvature*, Geom. Dedicata 127, 19-35, 2007. http://tinyurl.com/399y78f

[2] M. Gromov, *Convex integration of differential relations I*, Izv. Akad. Nauk SSSR 37 (1973), 329-343.

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[5] D. Spring, *Convex Integration Theory*, Monographs in Mathematics, Vol. 92, Birkhäuser Verlag, 1998.