
1 Two introductory examples

1.1 A first example

Let us consider the following elementary problem.

**Problem 1.**– Let

\[
\begin{align*}
    f_0 : [0, 1] &\rightarrow \mathbb{R}^3 \\
    t &\mapsto (0, 0, t)
\end{align*}
\]

be the linear application mapping the segment \([0, 1]\) vertically in \(\mathbb{R}^3\). The problem is to find \(f : [0, 1] \xrightarrow{C^1} \mathbb{R}^3\) such that:

\[
\begin{align*}
    i) & \quad \forall t \in [0, 1], \quad |\cos(f'(t), e_3)| < \epsilon \\
    ii) & \quad \|f - f_0\|_{C^0} < \delta
\end{align*}
\]

where \(\epsilon > 0\) and \(\delta > 0\) are given.

**Solution.**– At a first glance, the problem seems hopeless since condition \(i\) says that the slope is small and then the image has to move far away from the segment before reaching the desired height. After a few seconds of extra thinking, the solution occurs. It is good enough to move along an helix.
spiralling around the vertical axis:

\[ f : \ [0,1] \rightarrow \mathbb{R}^3 \]

\[ t \mapsto \begin{cases} 
\delta \cos 2\pi Nt \\
\delta \sin 2\pi Nt \\
t
\end{cases} \]

where \( N \in \mathbb{N}^* \) is the number of spirals. We have

\[ \left\langle \frac{f'}{\|f'\|}, e_3 \right\rangle = \frac{1}{\sqrt{1 + 4\pi^2 N^2 \delta^2}}. \]

Therefore, if \( N \) is large enough, \( f \) fulfills conditions \( i \) and \( ii \).

Rephrasing.-- The above problem was pretty easy, it will become very informative with a rephrasing of the two conditions. Condition \( (i) \) means that the image of \( f' \) lies inside the cone:

\[ \mathcal{R} = \{ v \in \mathbb{R}^3 \setminus \{ O \} \mid \left\langle \frac{v}{\|v\|}, e_3 \right\rangle < \epsilon \} \cup \{O\}. \]

By extension, that cone \( \mathcal{R} \) is called the differential relation of our problem.
The $C^0$-closeness required in the second condition, is a consequence of a geometric property of the derivative of $f$. Indeed, the image of $f'$ in that cone is a circle whose center is the constant image of $f'_0$. Therefore, the average of $f'$ for each spiral of $f$ is $f'_0(t)$:

$$\frac{1}{\text{Long}(I_k)} \int_{I_k} f'(u) du = f'_0(t)$$

where $I_k = \left[ \frac{k}{N}, \frac{k+1}{N} \right]$ the preimage of one spiral by $f$. Therefore, when integrating, the two resulting maps are closed together.

### 1.2 An more general example

**Problem.**— Let $\mathcal{R} \subset \mathbb{R}^3$ be a path-connected subset (=our differential relation) and $f_0 : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ be a map such

$$\forall t \in [0,1], \quad f'_0(t) \in \text{IntConv}(\mathcal{R})$$

where $\text{IntConv}(\mathcal{R})$ denotes the interior of the convex hull of $\mathcal{R}$. The problem is to find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ such that :

1. $\forall t \in [0,1], \quad f'(t) \in \mathcal{R}$
2. $\|f - f_0\|_{C^0} < \delta$

with $\delta > 0$ given.

**Solution.**— From the hypothesis, the image of $f'_0$ lies in the convex hull of $\mathcal{R}$. The idea is to build $f'$ with an image lying inside $\mathcal{R}$ and such that, on average, it looks like the derivative of $f_0$. One way to do that is to choose a the $f'$-image to resemble to a kind of spring. In the spring, each arc as the same effect, on average, as a small piece of the image of the initial map $f'_0$. So, when integrating, the resulting map will be close to the initial map. As before, we will improve the closeness of $f$ to $f_0$ by increasing the number of spirals.
The green bended spaghetti\textsuperscript{3} pictures the image of $f'_0$, the half of a spring in rep/pink is the chosen image for $f'$.

To formally construct a solution $f$ of the problem, it is enough to choose a continuous family of loops of $\mathcal{R}$:

$$h : [0,1] \rightarrow C^0(\mathbb{R}/\mathbb{Z}, \mathcal{R})$$

such that

$$\forall u \in [0,1], \int_{[0,1]} h_u(s)ds = f'_0(u)$$

i.e the average of the loop $h_u$ is $f'_0(u)$.

Then, the map $f'$ is extracted from that family of loops by a simple diagonal process

$$\forall t \in [0,1], \quad f'(t) := h_t(\{Nt\})$$

where $N \in \mathbb{N}^*$ and $\{Nu\}$ is the fractional part of $Nt$.

\textsuperscript{3}Spaghetti ?
Eventually, it remains to integrate to obtain a solution to our problem:

\[ f(t) := f_0(0) + \int_0^t h_u(\{Nu\}) du. \]

We say that \( f \) is obtained from \( f_0 \) by a convex integration process.

### 2 Finding the loops

In the above problem, we were wilfully blind to the question of the existence of the family of loops \((h_u)_{u \in [0,1]}\) needed to build the solution. We now deal with that issue.

**Notation.** Let \( A \subset \mathbb{R}^n \) and \( a \in A \). We denote by \( IntConv(A,a) \) the interior of the convex hull of the connected component of \( A \) to which \( a \) belongs.

**Definition.** A (continuous) loop \( g : [0,1] \to \mathbb{R}^n \), \( g(0) = g(1) \), strictly surrounds \( z \in \mathbb{R}^n \) if

\[ IntConv(g([0,1])) \supset \{ z \}. \]

**Fundamental Lemma.** Let \( \mathcal{R} \subset \mathbb{R}^n \) be an open set, \( \sigma \in \mathcal{R} \) and \( z \in IntConv(\mathcal{R}, \sigma) \) There exists a loop \( h : [0,1] \xrightarrow{C^0} \mathcal{R} \) with base point \( \sigma \) that strictly surrounds \( z \) and such that:

\[ z = \int_0^1 h(s) ds. \]

**Proof.** Since \( z \in IntConv(\mathcal{R}, \sigma) \), there exists a \( n \)-simplex \( \Delta \) whose vertices \( y_0, ..., y_n \) belong to \( \mathcal{R} \) and such that \( z \) lies in the interior of \( \Delta \). Therefore, there also exist

\[ (\alpha_0, ..., \alpha_n) \in ]0,1[^{n+1} \]
such that $\sum_{k=0}^{n} \alpha_k = 1$ and $z = \sum_{k=0}^{n} \alpha_k y_k$. Every loop $g : [0, 1] \to \mathcal{R}$ with base point $\sigma$ and passing through $y_0, ..., y_n$ satisfies $IntConv(g([0, 1])) \supset \{ z \}$ i. e. $g$ surrounds $z$.

In general

$$z \neq \int_0^1 g(s)ds.$$

Let $s_0, ..., s_n$ be the times for which $g(s_k) = y_k$ and let $f_k : [0, 1] \to \mathbb{R}_+^*$ be such that:

i) $f_k < \eta_1$ sur $[0, 1] \setminus [s_k - \eta_2, s_k + \eta_2]$, 

ii) $\int_0^1 f_k = 1$,

with $\eta_1, \eta_2$ two small positive numbers. We set:

$$z_k := \int_0^1 g(s)f_k(s)ds.$$

The number $\epsilon > 0$ being given, we can choose $\eta_1, \eta_2$ such that:

$$\forall k \in \{0, ..., n\}, \quad \| z_k - g(s_k) \| \leq \epsilon.$$

Since $\mathcal{R}$ in open and $z \in Int \Delta$, for $\epsilon$ small enough we have

$$z \in IntConv(z_0, ..., z_n).$$

Therefore, there exist $(p_0, ..., p_n) \in ]0, 1[^{n+1}$ such that $\sum_{k=0}^{n} p_k = 1$ and:

$$z = \sum_{k=0}^{n} p_k z_k = \sum_{k=0}^{n} p_k \int_0^1 g(s)f_k(s)ds = \int_0^1 g(s)\sum_{k=0}^{n} p_k f_k(s)ds = \int_0^1 g(s)\varphi'(s)ds.$$
where we have set
\[ \varphi'(s) := \sum_{k=0}^{n} p_k f_k(s) \]
and
\[ \varphi : [0, 1] \rightarrow [0, 1] \]
\[ s \mapsto \int_0^s \varphi(u)du. \]
We have \( \varphi'(s) > 0, \varphi(0) = 0, \varphi(1) = 1. \) Thus \( \varphi \) is a strictly increasing diffeomorphism of \([0, 1]\). Let us employ the change of coordinates \( s = \varphi^{-1}(t) \), that is \( t = \varphi(s) \), we have
\[ dt = \varphi'(s)ds \]
therefore:
\[ z = \int_0^1 g(s) \varphi'(s)ds = \int_0^1 g \circ \varphi^{-1}(t)dt. \]
Thus \( h = g \circ \varphi^{-1} \) is our desired loop.

Remark.– A priori \( h \in \Omega_{\sigma}(\mathcal{R}) \), but it is obvious that we can choose \( h \) among "round-trips" i.e the space:
\[ \Omega^{AR}_{\sigma}(\mathcal{R}) = \{ h \in \Omega_{\sigma}(\mathcal{R}) | \forall s \in [0, 1] \ h(s) = h(1 - s) \}. \]
The point is that the above space is contractible. For every \( u \in [0, 1] \) we then denote by \( h_u : [0, 1] \rightarrow \mathcal{R} \) the map defined by
\[ h_u(s) = \begin{cases} 
  h(s) & \text{if } s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}] \\
  h(u) & \text{if } s \in [\frac{u}{2}, 1 - \frac{u}{2}].
\end{cases} \]
This homotopy induces a deformation retract of \( \Omega^{AR}_{\sigma}(\mathcal{R}) \) to the constant map
\[ \overline{\sigma} : [0, 1] \rightarrow \mathcal{R} \]
\[ s \mapsto \sigma. \]
**Parametric version of the Fundamental Lemma.** — Let \( P \) be a compact manifold, \( E = P \times \mathbb{R}^n \stackrel{\pi}{\to} P \) be a trivial bundle, and \( \mathcal{R} \subset E \) be a set such that

\[
\forall p \in P, \quad \mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R} \text{ is an open set of } \mathbb{R}^n
\]

Let \( \sigma \in \Gamma(\mathcal{R}) \) and \( z \in \Gamma(E) \) such that:

\[
\forall p \in P, \quad z(p) \in \text{IntConv}(\mathcal{R}_p, \sigma(p)).
\]

Then, there exists \( h : P \times [0,1] \to \mathcal{R} \) such that:

\[
h(.,0) = h(.,1) = \sigma \in \Gamma(\mathcal{R}), \quad \forall p \in P, \quad h(p,.) \in \Omega_{\sigma(p)}^{\theta}(\mathcal{R}_p)
\]

and

\[
\forall p \in P, \quad z(p) = \int_0^1 h(p,s)ds.
\]

**Proof.**— The proof is rather long and technical. The main problem is the following: the result of the previous lemma rests on the existence of points \( y_0, ..., y_n \) of \( \mathcal{R} \) such that \( z \in \text{IntConv}(\{y_0, ..., y_n\}) \). If we want to mimic the previous proof while adding, we need to be able to follow continuously the points over \( P \), that is, we need to show the existence of \((n+1)\) continuous maps \( y_0, ..., y_n : P \to \mathbb{R}^n \) such that

\[
\forall p \in P, \quad z(p) \in \text{IntConv}(\{y_0(p), ..., y_n(p)\}).
\]

Locally, it is easy to obtain maps \( h_U : U \times [0,1] \to \mathcal{R} \) over open sets \( U \), the true problem is to glue them together. In order to do that, we take advantage of the contractibility of the round-trip loops. The following sequence of pictures should be enlightening.

A homotopy among loops surrounding \( z \) and joining \( h_U \) (red) to \( h_V \) (blue).
We then obtain a globally defined continuous map $h : P \times [0, 1] \xrightarrow{\mathcal{C}^\infty} \mathcal{R}$ such that

$$\forall p \in P, \quad z(p) \in \text{IntConv}(h(p, [0, 1]))$$

and

$$h(., 0) = h(., 1) = \sigma \in \Gamma^\infty(\mathcal{R}), \quad \forall p \in P, \ h(p, .) \in \Omega_{\sigma(p)}^{AR}(\mathcal{R}_p).$$

It eventually remains to reparametrize the map $h$ so that

$$\forall p \in P, \ z(p) = \int_0^1 h(p, s)ds.$$ 


**$\mathcal{C}^\infty$ parametric version of the Fundamental Lemma.** – Let $P$ be a compact manifold, $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ a trivial bundle and $\mathcal{R} \subset E$ be a set such that

$$\forall p \in P, \ \mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R} \text{ is an open set of } \mathbb{R}^n$$

Let $\sigma \in \Gamma^\infty(\mathcal{R})$ and $z \in \Gamma^\infty(E)$ such that

$$\forall p \in P, \ z(p) \in \text{IntConv}(\mathcal{R}_p, \sigma(p)).$$

Then there exists $h : P \times [0, 1] \xrightarrow{\mathcal{C}^\infty} \mathcal{R}$ such that

$$h(., 0) = h(., 1) = \sigma \in \Gamma(\mathcal{R}), \quad \forall p \in P, \ h(p, .) \in \Omega_{\sigma(p)}^{AR}(\mathcal{R}_p)$$

and

$$\forall p \in P, \ z(p) = \int_0^1 h(p, s)ds.$$ 

**Proof.** – Let $(\rho_\epsilon : [0, 1] \longrightarrow \mathbb{R})_{\epsilon > 0}$ be a sequence of mollifiers. For every $p \in P$ we define a $\mathcal{C}^\infty$ map by the formula

$$h_\epsilon(p, .) : \ [0, 1] \longrightarrow \mathbb{R}^n \quad t \longmapsto (h(p, .) * \rho_\epsilon)(t).$$

We set

$$z_\epsilon(p) := \int_0^1 h_\epsilon(p, t)dt$$
and we define \( H_\epsilon : P \times \mathbb{R} \longrightarrow \mathbb{R}^n \) by
\[
H_\epsilon(p, t) := h_\epsilon(p, t) + z(p) - z_\epsilon(p).
\]

We have
\[
\int_0^1 H_\epsilon(p, t) dt = z(p).
\]

If \( \epsilon \) is small enough, the image of the map \( t \mapsto H_\epsilon(p, t) \) lies inside \( \mathcal{R}_p \).

Thanks to the compactness of \( P \) the choice of the \( \epsilon \) can be made independently of \( p \in P \).

\[\square\]

3 \( C^0 \)-density

Let \( \mathcal{R} \subset \mathbb{R}^n \) be a arc-connected subset, \( f_0 \in C^\infty(I, \mathbb{R}^n) \) be a map such that \( f_0'(I) \subset \text{IntConv}(\mathcal{R}) \). From the \( C^\infty \) parametric version of the Fundamental Lemma there exists a \( C^\infty \)-map \( h : I \times \mathbb{E}/\mathbb{Z} \longrightarrow \mathcal{R} \) such that
\[
\forall t \in I, \quad f_0'(t) = \int_0^1 h(t, u) du.
\]

We set
\[
\forall t \in I, \quad F(t) := f_0(0) + \int_0^t h(s, Ns) ds
\]

with \( N \in \mathbb{N}^* \).

**Definition.**– We say that \( F \in C^\infty(I, \mathbb{R}^n) \) is obtained from \( f_0 \) by an convex integration process.

Obviously \( F'(t) = h(t, Nt) \in \mathcal{R} \) and thus \( F \) is a solution of the differential relation \( \mathcal{R} \). One crucial property of the convex integration process is that the solution \( F \) can be made arbitrarily close to the initial map \( f_0 \).

**Proposition (\( C^0 \)-density).**– We have
\[
\|F - f_0\|_{C^0} \leq \frac{1}{N} \left( 2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0} \right)
\]

where \( \|g\|_{C^0} = \sup_{p \in D} \|g(p)\|_{\mathbb{E}^3} \) denotes the \( C^0 \) norm of a function \( g : D \rightarrow \mathbb{E}^3 \).
Proof.– Let \( t \in [0, 1] \). We put \( n := \lfloor Nt \rfloor \) (the integer part of \( Nt \)) and set \( I_j = \left[ \frac{j}{N}, \frac{j+1}{N} \right] \) for \( 0 \leq j \leq n - 1 \) and \( I_n = \left[ \frac{n}{N}, t \right] \). We write
\[
F(t) - f(0) = \sum_{j=0}^{n} S_j \quad \text{and} \quad f_0(t) - f_0(0) = \sum_{j=0}^{n} s_j
\]
with \( S_j := \int_{I_j} h(v, Nv)dv \) and \( s_j := \int_{I_j} \int_{0}^{1} h(x, u)du \ dx \). By the change of variables \( u = Nv - j \), we get for each \( j \in [0, n - 1] \)
\[
S_j = \frac{1}{N} \int_{0}^{1} h\left( \frac{u + j}{N}, u \right)du = \int_{I_j} \int_{0}^{1} h\left( \frac{u + j}{N}, u \right)du \ dx.
\]
It ensues that
\[
\|S_j - s_j\|_{\mathbb{E}^3} \leq \frac{1}{N^2} \left\| \frac{\partial h}{\partial t} \right\|_{C^0}.
\]
The proposition then follows from the obvious inequalities
\[
\|S_n - s_n\|_{\mathbb{E}^3} \leq \frac{2}{N} \left\| h \right\|_{C^0} \quad \text{and} \quad \|F(t) - f_0(t)\|_{\mathbb{E}^3} \leq \sum_{j=0}^{n} \|S_j - s_j\|_{\mathbb{E}^3}.
\]

□

The increase of the \( C^0 \) closeness with \( N \).

In a multi-variables setting, the convex integration formula take the following natural form:
\[
f(c_1, ..., c_m) := f_0(c_1, ..., c_{m-1}, 0) + \int_{0}^{c_m} h(c_1, ..., c_{m-1}, s, Ns)ds
\]
where \((c_1, ..., c_m) \in [0, 1]^m\). This expression is nothing else but the parametric formula of a convex integration process with parameter space \( P = \)
It turns out that the above $C^0$-density property can then be enhanced to a $C^{1,\tilde{m}}$-density property where the notation $C^{1,\tilde{m}}$ means that the closeness is measured with the following norm

$$
\|f\|_{C^{1,\tilde{m}}} = \max(\|f\|_{C^0}, \|\frac{\partial f}{\partial c_1}\|_{C^0}, \ldots, \|\frac{\partial f}{\partial c_{m-1}}\|_{C^0}),
$$

that is the $C^1$-norm without the $\|\frac{\partial f}{\partial c_m}\|_{C^0}$ term.

**Proposition ($C^{1,\tilde{m}}$-density).**— Let $\mathcal{R} \subset \mathbb{R}^n$ be an open set, $E = C \times \mathbb{R}^n \rightarrow C$ be the trivial bundle over the cube $C = [0,1]^m$, $\sigma \in \Gamma(\mathcal{R})$ and let $f_0 : C \rightarrow \mathbb{R}^n$ be a map such that:

- $\forall c = (c_1, \ldots, c_m) \in [0,1]^m$, $\frac{\partial f_0}{\partial c_m}(c) \in \text{IntConv}(\mathcal{R}_c, \sigma(c))$

where $\mathcal{R}_c = \pi^{-1}(c) \cap \mathcal{R}$. Then, for every $\epsilon > 0$, there exists $f : C \rightarrow \mathbb{R}^n$ such that:

1. $\frac{\partial f}{\partial c_m} \in \Gamma(\mathcal{R})$
2. $\frac{\partial f}{\partial c_m}$ is homotopic to $\sigma$ in $\Gamma(\mathcal{R})$
3. $\|f - f_0\|_{C^{1,\tilde{m}}} = O \left(\frac{1}{N}\right)$.

**Proof.**— We have

$$
\frac{\partial f}{\partial c_m}(c_1, \ldots, c_m) = h(c_1, \ldots, c_{m-1}, c_m, Nc_m) \in \mathcal{R}_c
$$

and $\frac{\partial f}{\partial c_m}(c_1, \ldots, c_m)$ is homotopic to $\sigma(c)$ via

$$
\sigma_u(c) := h_u(c_1, \ldots, c_{m-1}, c_m, Nc_m)
$$

where $h_u$ is the contracting map described just below the proof of the Fundamental Lemma. Mimicking the proof of the $C^0$-density property, it is easy to show that

$$
\|\frac{\partial f}{\partial c_j} - \frac{\partial f_0}{\partial c_j}\|_{C^0} = O \left(\frac{1}{N}\right)
$$

for every $j \in \{1, \ldots, m-1\}$. 

\[\square\]
Remark.— Even if $f_0(0) = f_0(1)$, the map $F$ obtained by a convex integration from $f_0$ does not satisfy $F(0) = F(1)$ in general. This can be easily corrected by defining a new map $f$ with the formula

$$\forall t \in [0,1], \ f(t) = F(t) - t(F(1) - F(0)).$$

The following proposition shows that the $C^0$-density property still holds for $f$ and, provided $N$ is large enough, that the map $f$ is still a solution of $\mathcal{R}$.

Proposition.— We have

$$\|f - f_0\|_{C^0} \leq \frac{2}{N} \left(2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0}\right)$$

and $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$.

Proof.— The first inequality is obvious. Indeed, from

$$F(1) - F(0) = F(1) - f_0(0) = F(1) - f_0(1)$$

we deduce

$$\|f(t) - f_0(t)\| \leq \|F(t) - f_0(t)\| + \|F(1) - f_0(1)\| \leq 2\|F - f_0\|_{C^0}.$$ Derivating $f$ we have $f'(t) = F'(t) - (F(1) - F(0))$ thus

$$\|f' - F'\|_{C^0} \leq \|F - f_0\|_{C^0} = O\left(\frac{1}{N}\right).$$

Since $\mathcal{R}$ is open, if $N$ is large enough $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$. \qed

Remark.— It is of course easy to produce a parametric version of that proposition.

4 One dimensional $h$-principle

Definition.— A subset $A \subset \mathbb{R}^n$ is ample if for every $a \in A$ the interior of the convex hull of the connected component to which $a$ belongs is $\mathbb{R}^n$, i.e.: $\text{IntConv}(A, a) = \mathbb{R}^n$ (in particular $A = \emptyset$ is ample).
Example.— The complement of a linear subspace \( F \subset \mathbb{R}^n \) is ample if and only if \( \text{Codim } F \geq 2 \).

Definition.— Let \( E = P \times \mathbb{R}^n \stackrel{\pi}{\longrightarrow} P \) be a fiber bundle, a subset \( R \subset E \) is said to be ample if, for every \( p \in P \), \( R_p := \pi^{-1}(p) \cap R \) is ample in \( \mathbb{R}^n \).

Remark.— If \( R \subset E \) is ample, then, for every \( p \in P \), the condition \( z(p) \in \text{Conv}(R_p, \sigma(p)) \) necessarily holds.

Proposition.— Let \( E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} \) be a trivial bundle and let \( R \subset E \) be an open and ample differential relation. Then, for every \( \sigma \in \Gamma(R) \), there exists \( f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^n \) such that

1. \( f' \in \Gamma(R) \), i.e. \( f \in \text{Sol}(R) \),
2. \( f' \) is homotopic to \( \sigma \) in \( \Gamma(R) \).

Remark.— As a consequence, the natural

\[ \pi_0(\text{Sol}(R)) \longrightarrow \pi_0(\Gamma(R)) \]

is onto.

Proof.— Let \( f_0 : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^n \) be a \( C^1 \) map. Since \( R \) is ample, we have

\[ \forall t \in \mathbb{R}/\mathbb{Z}, \quad f'_0(t) \in \mathbb{R}^n = \text{IntConv}(R_t, \sigma(t)) \).

If \( N \) is large enough, the map \( f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^n \) obtained from \( f_0 \) by a convex integration (with gluing)

\[ \forall t \in [0, 1], \quad f(t) := f_0(0) + \int_0^t h(s, Ns)ds - t \int_0^1 h(s, Ns)ds \]

is a solution of \( R \). Thus, the point \( i \). For all \( u \in [0, 1] \), we define \( f_u : [0, 1] \longrightarrow \mathbb{R}^n \) by

\[ \forall t \in [0, 1], \quad f_u(t) := f_0(0) + \int_0^t h_u(s, Ns)ds - u.t \int_0^1 h(s, Ns)ds \]
where \( h_u : \mathbb{R}/\mathbb{Z} \times [0, 1] \to \mathcal{R} \) is the natural deformation retract

\[
h_u(t, s) = \begin{cases} h(t, s) & \text{if } s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}] \\ h(t, u) & \text{if } s \in [\frac{u}{2}, 1 - \frac{u}{2}]. \end{cases}
\]

Of course \( h_1(t, s) = h(t, s) \) and \( h_0(t, s) = \sigma(t) \). The map \( f_u \) does not descend to the quotient \( \mathbb{R}/\mathbb{Z} = [0, 1]/\partial[0, 1] \). But its derivative

\[
f'_u(t) = h_u(t, Nt) - u \int_0^1 h(s, Ns)ds
\]

induces a map from \( \mathbb{R}/\mathbb{Z} \) in \( \mathbb{R}^n \) since

\[
f'_u(0) = h_u(0, 0) - \int_0^1 h_u(s, Ns)ds = \sigma(0) - u \int_0^1 h(s, Ns)ds
\]

\[
f'_u(1) = h_u(1, N) - \int_0^1 h_u(s, Ns)ds = \sigma(1) - u \int_0^1 h(s, Ns)ds
\]

and thus \( f'_u(0) = f'_u(1) \) because \( \sigma(0) = \sigma(1) \). Hence, \( \sigma_u := f'_u \) is a homotopy joining \( f' = f'_1 \) to \( \sigma \). Since

\[
\left\| \int_0^1 h(s, Ns)ds \right\| = \|F(1) - f_0(1)\| = O\left(\frac{1}{N}\right)
\]

for every \( u \in [0, 1] \) and \( t \in \mathbb{R}/\mathbb{Z} \), the point \( \sigma_u(t) \) is as close as desired to \( h_u(t, Nt) \in \mathcal{R} \). Since \( \mathcal{R} \) is open, it exists \( N \) such that, for all \( u \in [0, 1] \), we have \( \sigma_u \in \Gamma(\mathcal{R}) \). This shows the point ii. \( \square \)

A parametric version of that proof allows to obtain the following theorem:

**Theorem (One-dimensional h-principle).**— Let \( E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}/\mathbb{Z} \) be a un trivial bundle and let \( \mathcal{R} \subset E \) be a open and ample differential relation, then the map

\[
J : \text{Sol}(\mathcal{R}) \to \Gamma(\mathcal{R})
\]

is a weak homotopy equivalence.

**Observation.**— Obviously, in the above theorem, \( \mathbb{R}/\mathbb{Z} \) can be replaced by an interval.

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5 Two applications of one-dimensional convex integration

5.1 Whitney-Graustein Theorem

Whitney-Graustein Theorem (1937). – We have : \( \pi_0(I(S^1, \mathbb{R}^2)) \simeq \mathbb{Z} \), with an identification given by the tangential degree.

Proof.– The theorem is a direct application of the 1-dimensional \( h \)-principle with \( n = 2 \) and \( \mathcal{R} = \mathbb{R}/\mathbb{Z} \times (\mathbb{R}^2 \setminus \{(0,0)\}) \) which is open and ample. We then have

\[
\text{Sol}(\mathcal{R}) = I(S^1, \mathbb{R}^2), \quad \Gamma(\mathcal{R}) = C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2 \setminus \{(0,0)\})
\]

and

\[
J : \text{Sol}(\mathcal{R}) \rightarrow \Gamma(\mathcal{R}) \quad \gamma \mapsto \gamma'
\]

induces a bijection at the \( \pi_0 \)-level. Note that the components of \( C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2 \setminus \{(0,0)\}) \) are in one to one correspondance with \( \mathbb{Z} \), the bijection being given by the turning number. It ensues that \( \pi_0(J) \) is the tangential degree.

5.2 A theorem of Ghomi

Theorem (Ghomi 2007).– Let \( f_0 \in I(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) \) be a curve with curvature function \( k_0 \) and let \( c \) be a real number such that \( c > \max k_0 \). Then, for every \( \epsilon > 0 \), there exists \( f_1 \in I(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3) \) of constant curvature \( c \) and such that

\[
\|f_1 - f_0\|_{C^1} = \|f_1 - f_0\|_{C^0} + \|f'_1 - f'_0\|_{C^0} \leq \epsilon.
\]

An example.– How to \( C^1 \) approximate a line by curve with an arbitrarily large constant curvature ? The answer lies in an picture :
Just a little comment however (from [4]): let us parametrize the line as a vertical segment in the three dimensional Euclidean space

\[ f_0(t) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \]

with \( t \in [0,1] \). The theorem asserts that there exists a curve with constant curvature \( c \) which is \( C^1 \)-close to \( f_0 \). A starting point is to begin by approximating the segment with an helix, for instance:

\[ f_1(t) = \begin{pmatrix} \epsilon \cos \alpha t \\ \epsilon \sin \alpha t \\ t \end{pmatrix} \]

where \( \alpha > 0 \) and \( \epsilon > 0 \). The \( C^0 \) closeness of \( f_1 \) to \( f_0 \) is ruled by \( \epsilon \). Regarding the curvature, it is constant and can be made as large as we want by decreasing \( \alpha \). However, as the number \( \alpha \) is becoming large, the derivative moves far away from the derivative of the initial function. It ensues that the helix is not \( C^1 \) close to \( f_0 \). To correct that point, we need to reduce the horizontal variations of the function. Let \( k > 0 \) and \( \tau \) be two numbers, we set

\[ f_{k,\tau}(t) = \begin{pmatrix} \frac{k}{k^2 + \tau^2} \cos \sqrt{k^2 + \tau^2} t \\ \frac{k}{k^2 + \tau^2} \sin \sqrt{k^2 + \tau^2} t \\ \tau \sqrt{k^2 + \tau^2} t \end{pmatrix} \]

This is an helix with constant curvature \( k \) and constant torsion \( \tau \). It is then visible that we have to choose a torsion notably bigger to the curvature to ensure a quasi-vertical derivative.

**Sketch of the proof.**— This is a good example of use of the 1-dimensional convex integration even if it is not an direct application of the 1-dimensional \( h \)-principle theorem. Here are the main steps:

1) First, reduce the problem to the case where the parametrization of \( f_0 \) is given by the arc-length. Then, the curvature is the norm of the second derivative, that is the speed of \( T_0 := f_0' : \mathbb{R}/\mathbb{Z} \rightarrow S^2 \).

2) Find \( T_1 : \mathbb{R}/\mathbb{Z} \rightarrow S^2 \) with constant speed (in order to have a constant curvature) which is \( C^0 \)-close to \( T_0 \) (to ensure that \( \| f_1' - f_0' \|_{C^0} \) is small)
and close in average to $T_0$ (to get a small norm $\|f_1 - f_0\|_{C^0}$).

3) Technically, $T_1$ should complete small loops with constant speed in a neighborhood of $T_0(\mathbb{R}/\mathbb{Z})$ in $\mathbb{S}^2$ and such that the average on each loop is close to the one of $T_0$ in the corresponding interval.

For more details, see [1]. □

References


