

CM 3 : One dimensional Convex Integration

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Convex Integration Theory is a powerful tool for solving differential relations. It was introduced by M. Gromov in his thesis dissertation in 1969, then published in an article [2] in 1973 and eventually generalized in a book [3] in 1986. Nevertheless, reading Gromov is often a challenge since important details are not provided explicitly. Fortunately, there is a good reference that leaves no details in the shadow : the Spring's book [5]. My understanding of Convex Integration Theory primarily comes from this book. I owe it much in this presentation.

1 Two introductory examples

1.1 A first example

Let us consider the following elementary problem.

Problem 1.— Let

$$\begin{aligned} f_0 : [0, 1] &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (0, 0, t) \end{aligned}$$

be the linear application mapping the segment $[0, 1]$ vertically in $\mathbb{R}^3$. The problem is to find $f : [0, 1] \xrightarrow{C^1} \mathbb{R}^3$ such that:

- i) $\forall t \in [0, 1], \quad |\cos(f'(t), e_3)| < \epsilon$
- ii) $\|f - f_0\|_{C^0} < \delta$

where $\epsilon > 0$ and $\delta > 0$ are given.

Solution.— At a first glance, the problem seems hopeless since condition *i* says that the slope is small and then the image has to move far away from the segment before reaching the desired height. After a few seconds of extra thinking, the solution occurs. It is good enough to move along an helix

spiralling around the vertical axis:

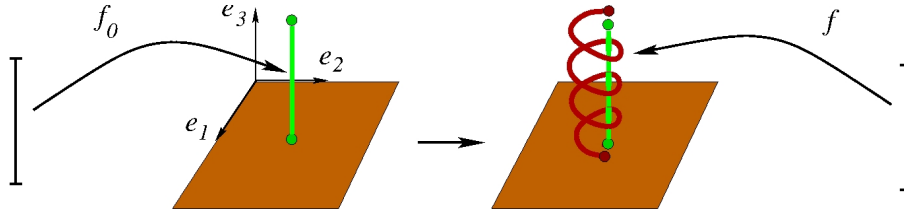
$$f : [0, 1] \longrightarrow \mathbb{R}^3$$

$$t \longmapsto \begin{cases} \delta \cos 2\pi N t \\ \delta \sin 2\pi N t \\ t \end{cases}$$

where $N \in \mathbb{N}^*$ is the number of spirals. We have

$$\left\langle \frac{f'}{\|f'\|}, e_3 \right\rangle = \frac{1}{\sqrt{1 + 4\pi^2 N^2 \delta^2}}.$$

Therefore, if N is large enough, f fulfills conditions *i* and *ii*.

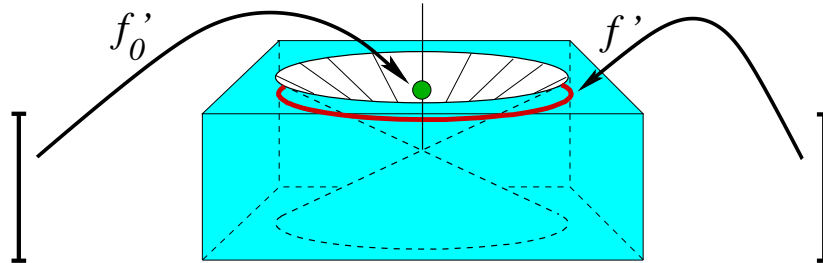


The image of f_0 is the green vertical segment, the solution f is the red helix.

Rephrasing.— The above problem was pretty easy, it will become very informative with a rephrasing of the two conditions. Condition (*i*) means that the image of f' lies inside the cone:

$$\mathcal{R} = \{v \in \mathbb{R}^3 \setminus \{O\} \mid \left| \left\langle \frac{v}{|v|}, e_3 \right\rangle \right| < \epsilon\} \cup \{O\}.$$

By extension, that cone $\mathcal{R}$ is called the *differential relation* of our problem.



The cone $\mathcal{R}$ is pictured in blue, the image of f' is the red circle and the constant image of f_0 the green point outside the cone.

The C^0 -closeness required in the second condition, is a consequence of a geometric property of the derivative of f . Indeed, the image of f' in that cone is a circle whose center is the constant image of f'_0 . Therefore, the average of f' for each spiral of f is $f'_0(t)$:

$$\frac{1}{\text{Long}(I_k)} \int_{I_k} f'(u) du = f'_0(t)$$

where $I_k = [\frac{k}{N}, \frac{k+1}{N}]$ the preimage of one spiral by f . Therefore, when integrating, the two resulting maps are closed together.

1.2 An more general example

Problem.— Let $\mathcal{R} \subset \mathbb{R}^3$ be a path-connected subset (=our differential relation) and $f_0 : [0, 1] \xrightarrow{C^1} \mathbb{R}^3$ be a map such

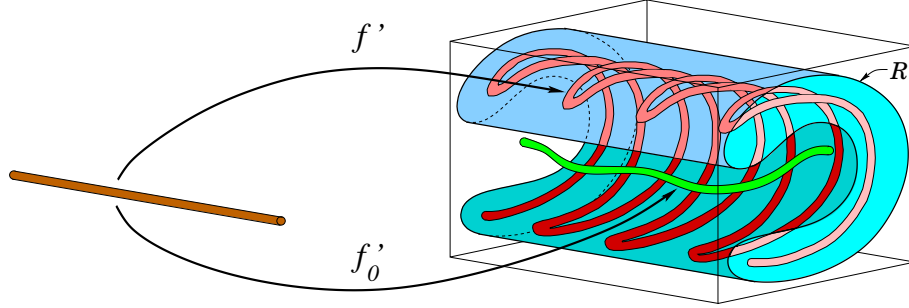
$$\forall t \in [0, 1], \quad f'_0(t) \in \text{IntConv}(\mathcal{R})$$

where $\text{IntConv}(\mathcal{R})$ denotes the interior of the convex hull of $\mathcal{R}$. The problem is to find $f : [0, 1] \xrightarrow{C^1} \mathbb{R}^3$ such that :

- i) $\forall t \in [0, 1], \quad f'(t) \in \mathcal{R}$
- ii) $\|f - f_0\|_{C^0} < \delta$

with $\delta > 0$ given.

Solution.— From the hypothesis, the image of f'_0 lies in the convex hull of $\mathcal{R}$. The idea is to build f' with an image lying inside $\mathcal{R}$ and such that, on average, it looks like the derivative of f_0 . One way to do that is to choose a the f' -image to resemble to a kind of spring. In the spring, each arc as the same effect, on average, as a small piece of the image of the initial map f'_0 . So, when integrating, the resulting map will be close to the initial map. As before, we will improve the closeness of f to f_0 by increasing the number of spirals.



The green bended spaghetti¹ pictures the image of f_0' , the half of a spring in rep/pink is the chosen image for f' .

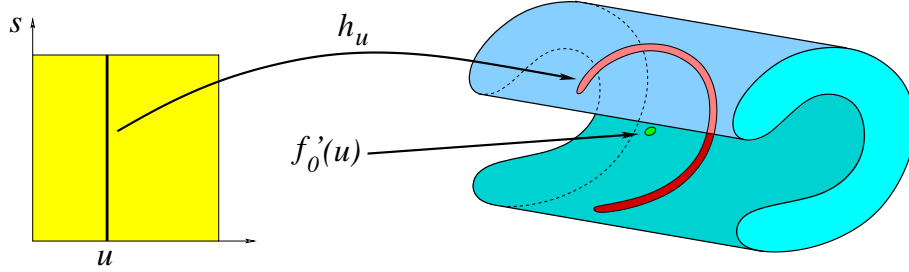
To formally construct a solution f of the problem, it is enough to choose a continuous family of loops of $\mathcal{R}$:

$$\begin{aligned} h : [0, 1] &\longrightarrow C^0(\mathbb{R}/\mathbb{Z}, \mathcal{R}) \\ u &\longmapsto h_u \end{aligned}$$

such that

$$\forall u \in [0, 1], \quad \int_{[0,1]} h_u(s) ds = f_0'(u)$$

i.e the average of the loop h_u is $f_0'(u)$.



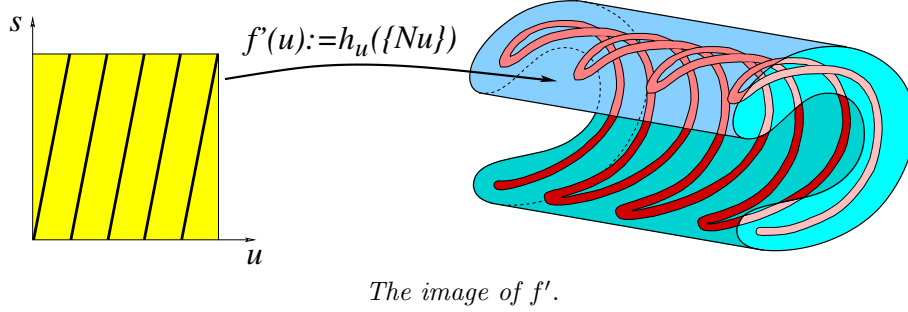
The image of the loop h_u . In that picture, this image is an arc. This loop is a round-trip starting at one of the endpoint of the arc and arriving at the same endpoint.

Then, the map f' is extracted from that family of loops by a simple diagonal process

$$\forall t \in [0, 1], \quad f'(t) := h_t(\{Nt\})$$

where $N \in \mathbb{N}^*$ and $\{Nu\}$ is the fractional part of Nu .

¹Spaghetti ?



Eventually, it remains to integrate to obtain a solution to our problem:

$$f(t) := f_0(0) + \int_0^t h_u(\{Nu\}) du.$$

We say that f is obtained from f_0 by a **convex integration process**.

Example.— Let $f_0 : [0, 1] \rightarrow [0, 1]$, $t \mapsto (0, 0, 1)$, as in problem 1. We define a constant family of loops by putting

$$h : \begin{array}{ccc} [0, 1] \times [0, 1] & \longrightarrow & \mathcal{R} \\ (t, s) & \longmapsto & (-R \sin 2\pi s, R \cos 2\pi s, 1) \end{array}$$

where

$$\mathcal{R} = \{v \in \mathbb{R}^3 \setminus \{O\} \mid \left| \left\langle \frac{v}{|v|}, e_3 \right\rangle \right| < \epsilon\} \cup \{O\}$$

and $R \in \mathbb{R}_+^*$ such that $\frac{1}{\sqrt{R^2+1}} < \epsilon$. We obviously have

$$\int_0^1 h(t, s) ds = f'_0(t) = e_1.$$

Whence, the map f obtained from f_0 by a convex integration convex has the following expression

$$\begin{aligned} f(t) &= f_0(0) + \int_0^t h_u(\{Nu\}) du \\ &= 0 + \int_0^t (-R \sin 2\pi N s, R \cos 2\pi N s, 1) du \\ &= \left(\frac{R}{2\pi N} \cos 2\pi N s, \frac{R}{2\pi N} \sin 2\pi N s, t \right) \end{aligned}$$

This is the solution f to the first introductory problem ($\delta = \frac{R}{2\pi N}$).

2 Finding the loops

In the above problem, we were wilfully blind to the question of the existence of the family of loops $(h_u)_{u \in [0,1]}$ needed to build the solution. We now deal with that issue.

Notation.— Let $A \subset \mathbb{R}^n$ and $a \in A$. We denote by $IntConv(A, a)$ the interior of the convex hull of the connected component of A to which a belongs.

Definition.— A (continuous) loop $g : [0, 1] \rightarrow \mathbb{R}^n$, $g(0) = g(1)$, *strictly surrounds* $z \in \mathbb{R}^n$ if

$$IntConv(g([0, 1])) \supset \{z\}.$$

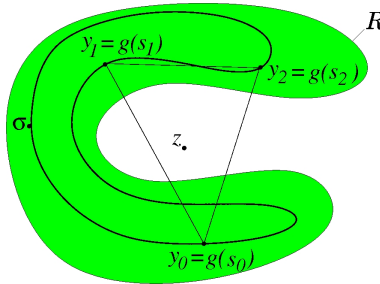
Fundamental Lemma.— Let $\mathcal{R} \subset \mathbb{R}^n$ be an open set, $\sigma \in \mathcal{R}$ and $z \in IntConv(\mathcal{R}, \sigma)$. There exists a loop $h : [0, 1] \xrightarrow{C^0} \mathcal{R}$ with base point σ that strictly surrounds z and such that:

$$z = \int_0^1 h(s) ds.$$

Proof.— Since $z \in IntConv(\mathcal{R}, \sigma)$, there exists a n -simplex Δ whose vertices $y_0, \dots, y_n$ belong to $\mathcal{R}$ and such that z lies in the interior of Δ . Therefore, there also exist

$$(\alpha_0, \dots, \alpha_n) \in]0, 1[^{n+1}$$

such that $\sum_{k=0}^n \alpha_k = 1$ and $z = \sum_{k=0}^n \alpha_k y_k$. Every loop $g : [0, 1] \rightarrow \mathcal{R}$ with base point σ and passing through $y_0, \dots, y_n$ satisfies $IntConv(g([0, 1])) \supset \{z\}$ i. e. g surrounds z .



In general

$$z \neq \int_0^1 g(s) ds.$$

Let $s_0, \dots, s_n$ be the times for which $g(s_k) = y_k$ and let $f_k : [0, 1] \rightarrow \mathbb{R}_+^*$ be such that :

$$\text{i) } f_k < \eta_1 \text{ sur } [0, 1] \setminus [s_k - \eta_2, s_k + \eta_2],$$

$$\text{ii) } \int_0^1 f_k = 1,$$

with η_1, η_2 two small positive numbers. We set:

$$z_k := \int_0^1 g(s) f_k(s) ds.$$

The number $\epsilon > 0$ being given, we can choose η_1, η_2 such that:

$$\forall k \in \{0, \dots, n\}, \quad \|z_k - g(s_k)\| \leq \epsilon.$$

Since $\mathcal{R}$ is open and $z \in \text{Int } \Delta$, for ϵ small enough we have

$$z \in \text{IntConv}(z_0, \dots, z_n).$$

Therefore, there exist $(p_0, \dots, p_n) \in]0, 1[^{n+1}$ such that $\sum_{k=0}^n p_k = 1$ and:

$$\begin{aligned} z &= \sum_{k=0}^n p_k z_k &= \sum_{k=0}^n p_k \int_0^1 g(s) f_k(s) ds \\ &= \int_0^1 g(s) \sum_{k=0}^n p_k f_k(s) ds &= \int_0^1 g(s) \varphi'(s) ds \end{aligned}$$

where we have set

$$\varphi'(s) := \sum_{k=0}^n p_k f_k(s)$$

and

$$\begin{aligned} \varphi : [0, 1] &\longrightarrow [0, 1] \\ s &\longmapsto \int_0^s \varphi(u) du. \end{aligned}$$

We have $\varphi'(s) > 0$, $\varphi(0) = 0$, $\varphi(1) = 1$. Thus φ is a strictly increasing diffeomorphism of $[0, 1]$. Let us employ the change of coordinates $s = \varphi^{-1}(t)$, that is $t = \varphi(s)$, we have

$$dt = \varphi'(s) ds$$

therefore:

$$z = \int_0^1 g(s) \varphi'(s) ds = \int_0^1 g \circ \varphi^{-1}(t) dt.$$

Thus $h = g \circ \varphi^{-1}$ is our desired loop. $\square$

Remark.— *A priori* $h \in \Omega_\sigma(\mathcal{R})$, but it is obvious that we can choose h among "round-trips" i.e the space:

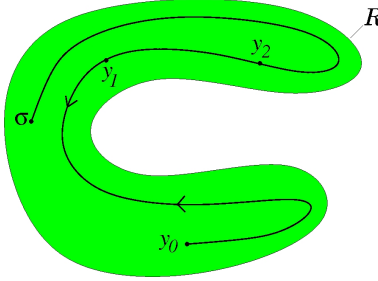
$$\Omega_\sigma^{AR}(\mathcal{R}) = \{h \in \Omega_\sigma(\mathcal{R}) \mid \forall s \in [0, 1] \ h(s) = h(1 - s)\}.$$

The point is that the above space is contractible. For every $u \in [0, 1]$ we then denote by $h_u : [0, 1] \rightarrow \mathcal{R}$ the map defined by

$$h_u(s) = \begin{cases} h(s) & \text{if } s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}, 1] \\ h(u) & \text{if } s \in [\frac{u}{2}, 1 - \frac{u}{2}]. \end{cases}$$

This homotopy induces a deformation retract of $\Omega_\sigma^{AR}(\mathcal{R})$ to the constant map

$$\begin{array}{ccc} \tilde{\sigma} : [0, 1] & \longrightarrow & \mathcal{R} \\ s & \longmapsto & \sigma. \end{array}$$



Parametric version of the Fundamental Lemma. — *Let P be a compact manifold, $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a trivial bundle, and $\mathcal{R} \subset E$ be a set such that*

$$\forall p \in P, \quad \mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R} \text{ is an open set of } \mathbb{R}^n$$

Let $\sigma \in \Gamma(\mathcal{R})$ and $z \in \Gamma(E)$ such that:

$$\forall p \in P, \quad z(p) \in \text{IntConv}(\mathcal{R}_p, \sigma(p)).$$

Then, there exists $h : P \times [0, 1] \xrightarrow{C^\infty} \mathcal{R}$ such that:

$$h(., 0) = h(., 1) = \sigma \in \Gamma^\infty(\mathcal{R}), \quad \forall p \in P, \quad h(p, .) \in \Omega_{\sigma(p)}^{AR}(\mathcal{R}_p)$$

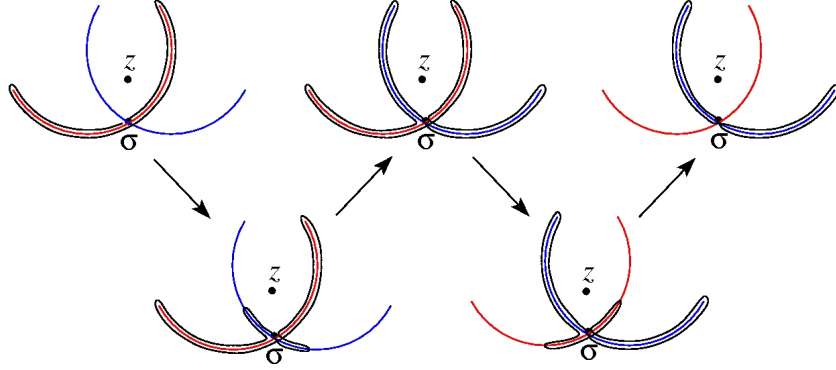
and

$$\forall p \in P, \quad z(p) = \int_0^1 h(p, s) ds.$$

Proof.— The proof is rather long and technical. The main problem is the following: the result of the previous lemma rests on the existence of points $y_0, \dots, y_n$ of $\mathcal{R}$ such that $z \in \text{IntConv}(\{y_0, \dots, y_n\})$. If we want to mimic the previous proof while adding, we need to be able to follow continuously the points over P , that is, we need to show the existence of $(n + 1)$ continuous maps $y_0, \dots, y_n : P \longrightarrow \mathbb{R}^n$ such that

$$\forall p \in P, \quad z(p) \in \text{IntConv}(\{y_0(p), \dots, y_n(p)\}).$$

Locally, it is easy to obtain maps $h_{\mathcal{U}} : \mathcal{U} \times [0, 1] \xrightarrow{C^\infty} \mathcal{R}$ over open sets $\mathcal{U}$, the true problem is to glue them together. In order to do that, we take advantage of the contractibility of the round-trip loops. The following sequence of pictures should be enlightning.



A homotopy among loops surrounding z and joining $h_{\mathcal{U}}$ (red) to $h_{\mathcal{V}}$ (blue).

We then obtain a globally defined continuous map $h : P \times [0, 1] \xrightarrow{C^\infty} \mathcal{R}$ such that

$$\forall p \in P, \quad z(p) \in \text{IntConv}(h(p, [0, 1]))$$

and

$$h(., 0) = h(., 1) = \sigma \in \Gamma^\infty(\mathcal{R}), \quad \forall p \in P, \quad h(p, .) \in \Omega_{\sigma(p)}^{AR}(\mathcal{R}_p).$$

It eventually remains to reparametrize the map h so that

$$\forall p \in P, \quad z(p) = \int_0^1 h(p, s) ds.$$

For more details, see [5] p. 29-31. □

C^∞ parametric version of the Fundamental Lemma. – *Let P be a compact manifold, $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ a trivial bundle and $\mathcal{R} \subset E$ be a set such that*

$$\forall p \in P, \quad \mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R} \text{ is an open set of } \mathbb{R}^n$$

Let $\sigma \in \Gamma^\infty(\mathcal{R})$ and $z \in \Gamma^\infty(E)$ such that

$$\forall p \in P, \quad z(p) \in \text{IntConv}(\mathcal{R}_p, \sigma(p)).$$

Then there exists $h : P \times [0, 1] \xrightarrow{C^\infty} \mathcal{R}$ such that

$$h(., 0) = h(., 1) = \sigma \in \Gamma(\mathcal{R}), \quad \forall p \in P, \quad h(p, .) \in \Omega_{\sigma(p)}^{AR}(\mathcal{R}_p)$$

and

$$\forall p \in P, \quad z(p) = \int_0^1 h(p, s) ds.$$

Proof.— Let $(\rho_\epsilon : [0, 1] \rightarrow \mathbb{R})_{\epsilon > 0}$ be a sequence of mollifiers. For every $p \in P$ we define a C^∞ map by the formula

$$\begin{aligned} h_\epsilon(p, .) : \quad [0, 1] &\longrightarrow \mathbb{R}^n \\ t &\longmapsto (h(p, .) * \rho_\epsilon)(t). \end{aligned}$$

We set

$$z_\epsilon(p) := \int_0^1 h_\epsilon(p, t) dt$$

and we define $H_\epsilon : P \times \mathbb{R} \rightarrow \mathbb{R}^n$ by

$$H_\epsilon(p, t) := h_\epsilon(p, t) + z(p) - z_\epsilon(p).$$

We have

$$\int_0^1 H_\epsilon(p, t) dt = z(p).$$

If ϵ is small enough, the image of the map $t \mapsto H_\epsilon(p, t)$ lies inside $\mathcal{R}_p$. Thanks to the compactness of P the choice of the ϵ can be made independently of $p \in P$. □

3 C^0 -density

Let $\mathcal{R} \subset \mathbb{R}^n$ be a arc-connected subset, $f_0 \in C^\infty(I, \mathbb{R}^n)$ be a map such that $f'_0(I) \subset \text{IntConv}(\mathcal{R})$. From the C^∞ parametric version of the Fundamental Lemma there exists a C^∞ -map $h : I \times \mathbb{E}/\mathbb{Z} \rightarrow \mathcal{R}$ such that

$$\forall t \in I, \quad f'_0(t) = \int_0^1 h(t, u) du.$$

We set

$$\forall t \in I, \quad F(t) := f_0(0) + \int_0^t h(s, Ns) ds$$

with $N \in \mathbb{N}^*$.

Definition.— We say that $F \in C^\infty(I, \mathbb{R}^n)$ is obtained from f_0 by an *convex integration process*.

Obviously $F'(t) = h(t, Nt) \in \mathcal{R}$ and thus F is a solution of the differential relation $\mathcal{R}$. One crucial property of the convex integration process is that the solution F can be made arbitrarily close to the initial map f_0 .

Proposition (C^0 -density).— *We have*

$$\|F - f_0\|_{C^0} \leq \frac{1}{N} \left(2\|h\|_{C^0} + \left\| \frac{\partial h}{\partial t} \right\|_{C^0} \right)$$

where $\|g\|_{C^0} = \sup_{p \in D} \|g(p)\|_{\mathbb{E}^n}$ denotes the C^0 norm of a function $g : D \rightarrow \mathbb{E}^n$.

Proof.— Let $t \in [0, 1]$. We put $n := [Nt]$ (the integer part of Nt) and set $I_j = [\frac{j}{N}, \frac{j+1}{N}]$ for $0 \leq j \leq n-1$ and $I_n = [\frac{n}{N}, t]$. We write

$$F(t) - f(0) = \sum_{j=0}^n S_j \quad \text{and} \quad f_0(t) - f_0(0) = \sum_{j=0}^n s_j$$

with $S_j := \int_{I_j} h(v, Nv) dv$ and $s_j := \int_{I_j} \int_0^1 h(x, u) du dx$. By the change of variables $u = Nv - j$, we get for each $j \in [0, n-1]$

$$S_j = \frac{1}{N} \int_0^1 h\left(\frac{u+j}{N}, u\right) du = \int_{I_j} \int_0^1 h\left(\frac{u+j}{N}, u\right) du dx.$$

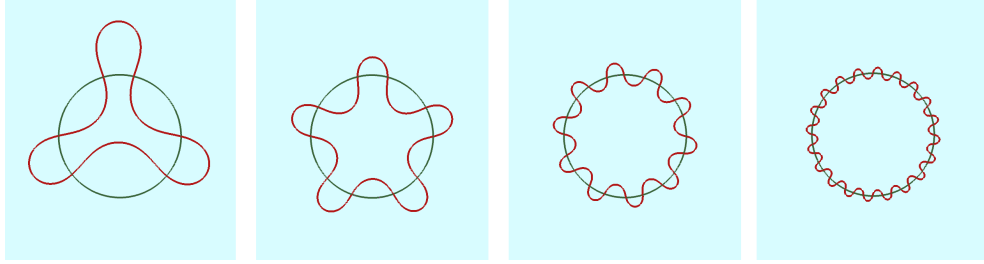
It ensues that

$$\|S_j - s_j\|_{\mathbb{E}^3} \leq \frac{1}{N^2} \left\| \frac{\partial h}{\partial t} \right\|_{C^0}.$$

The proposition then follows from the obvious inequalities

$$\|S_n - s_n\|_{\mathbb{E}^3} \leq \frac{2}{N} \|h\|_{C^0} \quad \text{and} \quad \|F(t) - f_0(t)\|_{\mathbb{E}^3} \leq \sum_{j=0}^n \|S_j - s_j\|_{\mathbb{E}^3}.$$

□



The increase of the C^0 closeness with N .

In a multi-variables setting, the convex integration formula take the following natural form:

$$f(c_1, \dots, c_m) := f_0(c_1, \dots, c_{m-1}, 0) + \int_0^{c_m} h(c_1, \dots, c_{m-1}, s, Ns) ds$$

where $(c_1, \dots, c_m) \in [0, 1]^m$. This expression is nothing else but the parametric formula of a convex integration process with parameter space $P = [0, 1]^{m-1}$. It turns out that the above C^0 -density property can then be enhanced to a $C^{1, \hat{m}}$ -density property where the notation $C^{1, \hat{m}}$ means that the closeness is measured with the following norm

$$\|f\|_{C^{1, \hat{m}}} = \max(\|f\|_{C^0}, \|\frac{\partial f}{\partial c_1}\|_{C^0}, \dots, \|\frac{\partial f}{\partial c_{m-1}}\|_{C^0}),$$

that is the C^1 -norm without the $\|\frac{\partial f}{\partial c_m}\|_{C^0}$ term.

Proposition ($C^{1, \hat{m}}$ -density).— *Let $\mathcal{R} \subset \mathbb{R}^n$ be an open set, $E = C \times \mathbb{R}^n \xrightarrow{\pi} C$ be the trivial bundle over the cube $C = [0, 1]^m$, $\sigma \in \Gamma(\mathcal{R})$ and let $f_0 : C \rightarrow \mathbb{R}^n$ be a map such that:*

$$\forall c = (c_1, \dots, c_m) \in [0, 1]^m, \quad \frac{\partial f_0}{\partial c_m}(c) \in \text{IntConv}(\mathcal{R}_c, \sigma(c))$$

where $\mathcal{R}_c = \pi^{-1}(c) \cap \mathcal{R}$. Then, for every $\epsilon > 0$, there exists $f : C \rightarrow \mathbb{R}^n$ such that:

- i) $\frac{\partial f}{\partial c_m} \in \Gamma(\mathcal{R})$
- ii) $\frac{\partial f}{\partial c_m}$ is homotopic to σ in $\Gamma(\mathcal{R})$
- iii) $\|f - f_0\|_{C^{1,\hat{m}}} = O\left(\frac{1}{N}\right)$.

Proof.— We have

$$\frac{\partial f}{\partial c_m}(c_1, \dots, c_m) = h(c_1, \dots, c_{m-1}, c_m, Nc_m) \in \mathcal{R}_c$$

and $\frac{\partial f}{\partial c_m}(c_1, \dots, c_m)$ is homotopic to $\sigma(c)$ via

$$\sigma_u(c) := h_u(c_1, \dots, c_{m-1}, c_m, Nc_m)$$

where h_u is the contracting map described just below the proof of the Fundamental Lemma. Mimicking the proof of the C^0 -density property, it is easy to show that

$$\left\| \frac{\partial f}{\partial c_j} - \frac{\partial f_0}{\partial c_j} \right\|_{C^0} = O\left(\frac{1}{N}\right)$$

for every $j \in \{1, \dots, m-1\}$. □

Remark.— Even if $f_0(0) = f_0(1)$, the map F obtained by a convex integration from f_0 does not satisfy $F(0) = F(1)$ in general. This can be easily corrected by defining a new map f with the formula

$$\forall t \in [0, 1] \quad , \quad f(t) = F(t) - t(F(1) - F(0)).$$

The following proposition shows that the C^0 -density property still holds for f and, provided N is large enough, that the map f is still a solution of $\mathcal{R}$ (which is assumed to be open).

Proposition.— We have

$$\|f - f_0\|_{C^0} \leq \frac{2}{N} \left(2\|h\|_{C^0} + \left\| \frac{\partial h}{\partial t} \right\|_{C^0} \right)$$

and $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$.

Proof.— The first inequality is obvious. Indeed, from

$$F(1) - F(0) = F(1) - f_0(0) = F(1) - f_0(1)$$

we deduce

$$\|f(t) - f_0(t)\| \leq \|F(t) - f_0(t)\| + \|F(1) - f_0(1)\| \leq 2\|F - f_0\|_{C^0}.$$

Derivating f we have $f'(t) = F'(t) - (F(1) - F(0))$ thus

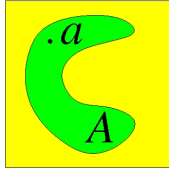
$$\|f' - F'\|_{C^0} \leq \|F - f_0\|_{C^0} = O\left(\frac{1}{N}\right).$$

Since $\mathcal{R}$ is open, if N is large enough $f'(\mathbb{R}/\mathbb{Z}) \subset \mathcal{R}$. □

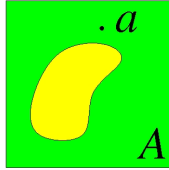
Remark.— It is of course easy to produce a parametric version of that proposition.

4 One dimensional h -principle

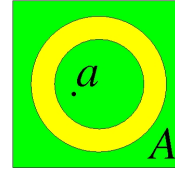
Definition.— A subset $A \subset \mathbb{R}^n$ is *ample* if for every $a \in A$ the interior of the convex hull of the connected component to which a belongs is $\mathbb{R}^n$ i. e. : $IntConv(A, a) = \mathbb{R}^n$ (in particular $A = \emptyset$ is ample).



A is not ample



A is ample



A is not ample.

Example.— The complement of a linear subspace $F \subset \mathbb{R}^n$ is ample if and only if $\text{Codim } F \geq 2$.

Definition.— Let $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a fiber bundle, a subset $\mathcal{R} \subset E$ is said to be *ample* if, for every $p \in P$, $\mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R}$ is ample in $\mathbb{R}^n$.

Remark.— If $\mathcal{R} \subset E$ is ample, then, for every $p \in P$, the condition $z(p) \in Conv(\mathcal{R}_p, \sigma(p))$ necessarily holds.

Proposition.— Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ be a trivial bundle and let $\mathcal{R} \subset E$ be an open and ample differential relation. Then, for every $\sigma \in \Gamma(\mathcal{R})$, there exists $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^n$ such that

- i) $f' \in \Gamma(\mathcal{R})$, i. e. $f \in \text{Sol}(\mathcal{R})$,
- ii) f' is homotopic to σ in $\Gamma(\mathcal{R})$.

Remark.— As a consequence, the natural

$$\pi_0(\text{Sol}(\mathcal{R})) \rightarrow \pi_0(\Gamma(\mathcal{R}))$$

is onto.

Proof.— Let $f_0 : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^n$ be a C^1 map. Since $\mathcal{R}$ is ample, we have

$$\forall t \in \mathbb{R}/\mathbb{Z}, \quad f'_0(t) \in \mathbb{R}^n = \text{IntConv}(\mathcal{R}_t, \sigma(t)).$$

If N is large enough, the map $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^n$ obtained from f_0 by a convex integration (with gluing)

$$\forall t \in [0, 1], \quad f(t) := f_0(0) + \int_0^t h(s, Ns) ds - t \int_0^1 h(s, Ns) ds$$

is a solution of $\mathcal{R}$. Thus, the point i . For all $u \in [0, 1]$, we define $f_u : [0, 1] \rightarrow \mathbb{R}^n$ by

$$\forall t \in [0, 1], \quad f_u(t) := f_0(0) + \int_0^t h_u(s, Ns) ds - u.t \int_0^1 h(s, Ns) ds$$

where $h_u : \mathbb{R}/\mathbb{Z} \times [0, 1] \rightarrow \mathcal{R}$ is the natural deformation retract

$$h_u(t, s) = \begin{cases} h(t, s) & \text{if } s \in [0, \frac{u}{2}] \cup [1 - \frac{u}{2}, 1] \\ h(t, u) & \text{if } s \in [\frac{u}{2}, 1 - \frac{u}{2}]. \end{cases}$$

Of course $h_1(t, s) = h(t, s)$ and $h_0(t, s) = \sigma(t)$. The map f_u does not descend to the quotient $\mathbb{R}/\mathbb{Z} = [0, 1]/\partial[0, 1]$. But its derivative

$$f'_u(t) = h_u(t, Nt) - u \int_0^1 h(s, Ns) ds$$

induces a map from $\mathbb{R}/\mathbb{Z}$ in $\mathbb{R}^n$ since

$$f'_u(0) = h_u(0, 0) - \int_0^1 h_u(s, Ns) ds = \sigma(0) - u \int_0^1 h(s, Ns) ds$$

$$f'_u(1) = h_u(1, N) - \int_0^1 h_u(s, Ns)ds = \sigma(1) - u \int_0^1 h(s, Ns)ds$$

and thus $f'_u(0) = f'_u(1)$ because $\sigma(0) = \sigma(1)$. Hence, $\sigma_u := f'_u$ is a homotopy joining $f' = f'_1$ to σ . Since

$$\left\| \int_0^1 h(s, Ns)ds \right\| = \|F(1) - f_0(1)\| = O\left(\frac{1}{N}\right)$$

for every $u \in [0, 1]$ and $t \in \mathbb{R}/\mathbb{Z}$, the point $\sigma_u(t)$ is as close as desired to $h_u(t, Nt) \in \mathcal{R}$. Since $\mathcal{R}$ is open, it exists N such that, for all $u \in [0, 1]$, we have $\sigma_u \in \Gamma(\mathcal{R})$. This shows the point *ii*. $\square$

A parametric version of that proof allows to obtain the following theorem:

Theorem (One-dimensional h -principle).— *Let $E = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ be a un trivial bundle and let $\mathcal{R} \subset E$ be a open and ample differential relation, then the map*

$$J : \text{Sol}(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$$

is a weak homotopy equivalence.

Observation.— Obviously, in the above theorem, $\mathbb{R}/\mathbb{Z}$ can be replaced by an interval.

5 Two applications of one-dimensional convex integration

5.1 Whitney-Graustein Theorem

Whitney-Graustein Theorem (1937). — *We have : $\pi_0(I(\mathbb{S}^1, \mathbb{R}^2)) \simeq \mathbb{Z}$, with an identification given by the tangential degree.*

Proof.— The theorem is a direct application of the 1-dimensional h -principle with $n = 2$ and $\mathcal{R} = \mathbb{R}/\mathbb{Z} \times (\mathbb{R}^2 \setminus \{(0, 0)\})$ which is open and ample. We then have

$$\text{Sol}(\mathcal{R}) = I(\mathbb{S}^1, \mathbb{R}^2), \quad \Gamma(\mathcal{R}) = C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2 \setminus \{(0, 0)\})$$

and

$$\begin{array}{ccc} J : \text{Sol}(\mathcal{R}) & \longrightarrow & \Gamma(\mathcal{R}) \\ \gamma & \longmapsto & \gamma' \end{array}$$

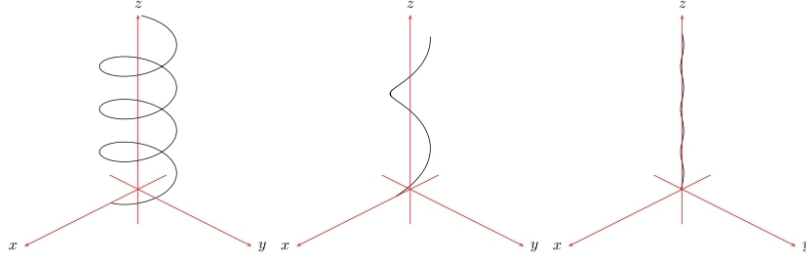
induces a bijection at the π_0 -level. Note that the components of $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{R}^2 \setminus \{(0,0)\})$ are in one to one correspondance with $\mathbb{Z}$, the bijection being given by the turning number. It ensues that $\pi_0(J)$ is the tangential degree. $\square$

5.2 A theorem of Ghomi

Theorem (Ghomi 2007).— *Let $f_0 \in I(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ be a curve with curvature function k_0 and let c be a real number such that $c > \max k_0$. Then, for every $\epsilon > 0$, there exists $f_1 \in I(\mathbb{R}/\mathbb{Z}, \mathbb{R}^3)$ of constant curvature c and such that*

$$\|f_1 - f_0\|_{C^1} = \|f_1 - f_0\|_{C^0} + \|f'_1 - f'_0\|_{C^0} \leq \epsilon.$$

An example.— How to C^1 approximate a line by curve with an arbitrarily large constant curvature ? The answer lies in an picture :



Just a little comment however (from [4]: let us parametrize the line as a vertical segment in the three dimensional Euclidean space

$$f_0(t) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$$

with $t \in [0, 1]$. The theorem asserts that there exists a curve with constant curvature c which is C^1 -close to f_0 . A starting point is to begin by approximating the segment with an helix, for instance:

$$f_1(t) = \begin{pmatrix} \epsilon \cos \alpha t \\ \epsilon \sin \alpha t \\ t \end{pmatrix}$$

where $\alpha > 0$ and $\epsilon > 0$. The C^0 closeness of f_1 to f_0 is ruled by ϵ . Regarding the curvature, it is constant and can be made as large as we want by

decreasing α . However, as the number α is becoming large, the derivative moves far away from the derivative of the initial function. It ensues that the helix is not C^1 close to f_0 . To correct that point, we need to reduce the horizontal variations of the function. Let $k > 0$ and τ be two numbers, we set

$$f_{k,\tau}(t) = \begin{pmatrix} \frac{k}{k^2 + \tau^2} \cos \sqrt{k^2 + \tau^2} t \\ \frac{k}{k^2 + \tau^2} \sin \sqrt{k^2 + \tau^2} t \\ \frac{\tau}{\sqrt{k^2 + \tau^2}} t \end{pmatrix}.$$

This is an helix with constant curvature k and constant torsion τ . It is then visible that we have to choose a torsion notably bigger to the curvature to ensure a quasi-vertical derivative.

Skecth of the proof.— This is a good example of use of the 1-dimensional convex integration even if it is not an direct application of the 1-dimensional h -principle theorem. Here are the main steps:

1) First, reduce the problem to the case where the parametrization of f_0 is given by the arc-length. Then, the curvature is the norm of the second derivative, that is the speed of $T_0 := f'_0 : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{S}^2$.

2) Find $T_1 : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{S}^2$ with constant speed (in order to have a constant curvature) which is C^0 -close to T_0 (to ensure that $\|f'_1 - f'_0\|_{C^0}$ is small) and close in average to T_0 (to get a small norm $\|f_1 - f_0\|_{C^0}$).

3) Technically, T_1 should complete small loops with constant speed in a neighborhood of $T_0(\mathbb{R}/\mathbb{Z})$ in $\mathbb{S}^2$ and such that the average on each loop is close to the one of T_0 in the corresponding interval.

For more details, see [1]. □

References

- [1] M. GHOMI, *h-principles for curves and knots of constant curvature*, Geom. Dedicata 127, 19-35, 2007. <http://tinyurl.com/399y78f>
- [2] M. GROMOV, *Convex integration of differential relations I*, Izv. Akad. Nauk SSSR 37 (1973), 329-343.

- [3] M. GROMOV, *Partial differential relations*, Springer-Verlag, New York, 1986.
- [4] M. KOURGANOV *h-principe pour les courbes à courbure constante*, rapport de stage ENS-Lyon, <http://math.univ-lyon1.fr/~borrelli/Jeunes.html>
- [5] D. SPRING, *Convex Integration Theory*, Monographs in Mathematics, Vol. 92, Birkhäuser Verlag, 1998.