H-Principle and Convex Integration Theory

Vincent Borrelli

Université Lyon 1
What is Convex Integration Theory?

A process to solve PDE’s/Differential Relations (mostly arising from Differential Geometry)
What is Convex Integration Theory?

A process to solve PDE’s/Differential Relations (mostly arising from Differential Geometry)

AND
What is Convex Integration Theory?

A process to solve PDE’s/Differential Relations (mostly arising from Differential Geometry)

AND

A tool to detect \( h \)-principles
What is Convex Integration Theory?

A process to solve PDE’s/Differential Relations (mostly arising from Differential Geometry)
What is Convex Integration Theory?

A process to solve PDE’s/Differential Relations (mostly arising from Differential Geometry)

AND
What is Convex Integration Theory?

A process to solve PDE’s/Differential Relations (mostly arising from Differential Geometry)

AND

A tool to detect $h$-principles
What is the \( h \)-Principle?

Problem (\( \sim 1930 \) until 1960).– Given two surfaces of \( \mathbb{R}^3 \), is there exist a regular homotopy joining them? Regular means that at each time the surface remains smooth (no sharp bend, no crease). Self-intersections are allowed.
What is the *h*-Principle?

**Problem (∼ 1930 until 1960).—** Given two surfaces of $\mathbb{R}^3$, is there exist a *regular* homotopy joining them?
What is the $h$-Principle?

**Problem ($\sim$ 1930 until 1960).--** Given two surfaces of $\mathbb{R}^3$, is there exist a *regular* homotopy joining them?

*Regular* means that at each time the surface remains smooth (no sharp bend, no crease).

Self-intersections are allowed.
An example

Obviously impossible. – There is a topological obstruction!
An example

Obviously impossible.— There is a topological obstruction!
Another example
Another example

VIDEO
No obstruction!
What is the *h*-principle?

- Informal definition: We say that the *h*-principle holds on a differential problem if the obstructions to find solutions come from algebraic topology.

*Mikhail Gromov*
What is the $h$-principle?

- Informal definition: We say that the $h$-principle holds on a differential problem if the obstructions to find solutions come from algebraic topology.

- For instance, the $h$-principle holds on the problem of finding regular homotopies.
What is the \( h \)-principle?

- The \( h \)-principle holds for numerous problems: in Symplectic/Contact/Riemannian Geometry, in Foliation Theory...
What is the $h$-principle?

- The $h$-principle holds for numerous problems: in Symplectic/Contact/Riemannian Geometry, in Foliation Theory...

- In these lectures, we shall focus on one of the techniques to detect the presence of the $h$-principle: Convex Integration Theory.
General Philosophy of these lectures

- To present Convex Integration Theory in a historical perspective in order to emphasize the key ideas behind its development.
General Philosophy of these lectures

• To present Convex Integration Theory in a historical perspective in order to emphasize the key ideas behind its development.

• These key ideas were outrageously simple... Sometimes, you do not need tons of abstract and conceptual theories to get a breakthrough.
General Philosophy of these lectures

- To present Convex Integration Theory in a historical perspective in order to emphasize the key ideas behind its development.

- These key ideas were **outrageously simple**... Sometimes, you do not need tons of abstract and conceptual theories to get a breakthrough.

- Lectures on the $h$-principle were already given in 2012 on this MA2 with a classical perspective. It is highly recommended to download the pdf notes on my webpage (section "Enseignement"), to take a look at them and possibly to do some of the exercises.
H-Principle and Convex Integration Theory
V. Borrelli

Content

- L1 : Nash-Kuiper Theorem
- L2 : The Whitney-Graustein Theorem and the $h$-Principle
- L3 : 1D Convex Integration
- L4 : Gromov Theorem on Ample Relations
- L5 : Constructions of $C^1$ isometric maps
- L6 : Theillière’s Formula
Bibliography

Mikhail Gromov

Partial Differential Relations
Bibliography

Hansjörg Geiges et Sinem Onaran
Bibliography

David Spring
Pdf of the lectures "Introduction to the h-principle" on his web page
Patrick Massot: lectures on Differential Topology
Let’s work!

We have some *Wrong Way* signs to explode with $h$-principle dynamites!