L1: Nash-Kuiper Theorem

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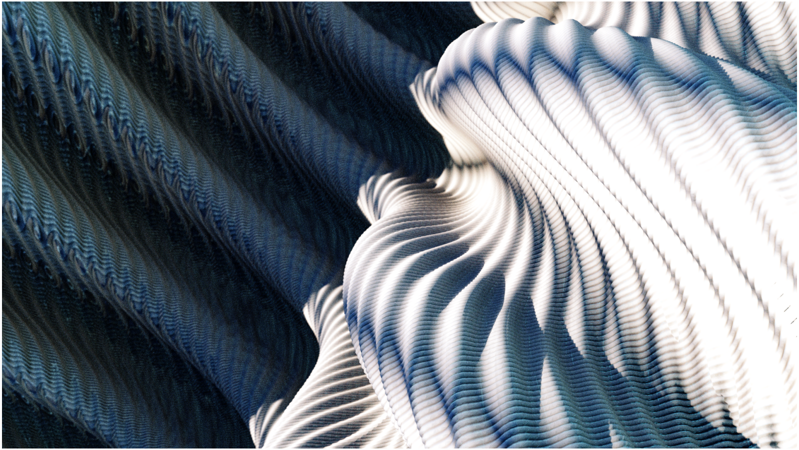

Isometric embeddings

Definition.– A map $f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q = (\mathbb{R}^q, \langle ., . \rangle)$ is *isometric* if $f^* \langle ., . \rangle = g$.

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• In coordinates, the condition $f^*\langle ., . \rangle = g$ reduces to a system of $n(n+1)/2$ equations

$$\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \rangle = g_{ij}$$

of the q unknown functions $f : (x_1, ..., x_n) \mapsto (f_1, ..., f_q)$ with $0 \leq i \leq j \leq n$.

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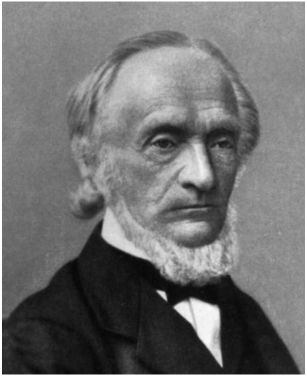
• The number

$$s_n = \frac{n(n+1)}{2}$$

is called the *Janet dimension*.

Schläfli Conjecture



Ludwig Schläfli

Schläfli Conjecture (1873).– *Any n dimensional C^ω Riemannian manifold admits locally an isometric embedding into $\mathbb{E}^{s_n}$.*

Historical perpective

Janet-Cartan Theorem (1926-27).– *Any n dimensional C^ω Riemannian manifold admits locally an isometric embedding into $\mathbb{E}^q$ with $q = s_n$.*

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Nash-Kuiper C^1 Embedding Theorem (1954-1955).– Statement in a couple of minutes...

Historical perpective

Nash C^∞ Embedding Theorem (1956).– *Any C^∞ compact Riemannian manifold admits a C^∞ isometric embedding into $\mathbb{E}^q$ with $q = s_n + 4n$.*

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Theorem (Gromov, Rokhlin, Greene 1970).– *Any C^∞ compact Riemannian manifold admits locally a C^∞ isometric embedding into $\mathbb{E}^q$ with $q = s_n + n$.*

Historical perpective

Theorem (Poznyak, 1973).– *Any* C^∞ *compact Riemannian surface admits locally a* C^∞ *isometric embedding into* $\mathbb{E}^4$.

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$$q = \max\{s_n + 2n, s_n + n + 5\}.$$

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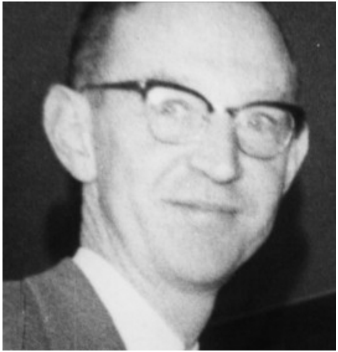
Theorem (Gromov 1989).– *Any C^∞ compact Riemannian surface admits a C^∞ isometric embedding into $\mathbb{E}^5$.*

Nash-Kuiper Theorem

John Nash and Nicolaas Kuiper

Definition.– A map $f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ is said (strictly) *short* if $f^*\langle .,. \rangle \leq Kg$ with $0 < K < 1$.

Nash-Kuiper Theorem

Theorem (1954-55).– *Let* M^n *be a compact manifold and* $f_0 : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$, $q > n$, *be a short embedding. Then, for every* $\epsilon > 0$, *there exists a* C^1*-isometric embedding* $f : (M^n, g) \longrightarrow \mathbb{E}^q$ *such that* $\|f - f_0\|_{C^0} \leq \epsilon$.

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• The C^0-closeness condition appears latter (Kuiper, 1959).

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Corollary (Gromov 1989).– *It is possible to perform an eversion of the 2-sphere through* C^1 *isometric immersions.*

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• Each f_k is built iteratively from f_{k-1}. The parameters of the construction allow to insure that for all k,

$$\|f_{k+1} - f_k\|_{C^0} \leq \frac{1}{2^k} \text{ and } \|df_{k+1} - df_k\|_{C^0} \leq \frac{C}{2^{k/2}}$$

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• Each f_k is an embedding and we show by using the C^0-closeness property that the limit still is an embedding.

Step 1 : Decomposition of Δ

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• The first step is to write the isometric default as a sum of squares of linear forms :

$$\Delta(x) = \sum_{j=1}^{P_0} \rho_j(x) \ell_j \otimes \ell_j$$

with $\rho_j(x) \geq 0$, $j \in \{1, ..., P_0\}$ and $x \in \mathcal{U}_\alpha$.

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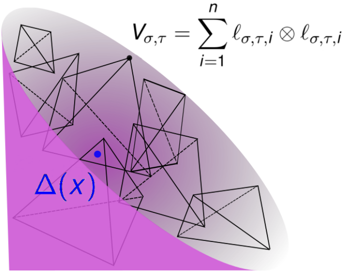

• To do so, we choose a locally finite covering of $\mathcal{S}_2^+(\mathbb{R}^n)$ by open simplices and a partition of unity (φ_σ) subordinated to that covering. Note that, by locally finite, we mean that every point has a neighborhood that intersects a finite number of simplices.

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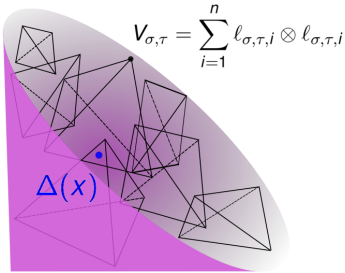

• Furthermore we require this finite number to be uniformly bounded, say by W (we admit the existence of such a covering).

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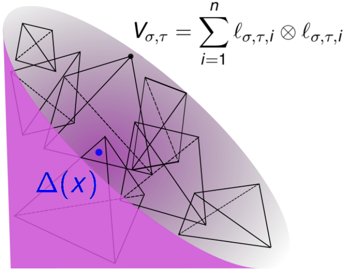

• Each simplex σ has $s_n + 1$ vertices $V_{\sigma,0}, ..., V_{\sigma,s_n}$ and each vertex has a decomposition as a sum of n squares of linear forms

$$V_{\sigma,\tau} = \sum_{i=1}^{n} \ell_{\sigma,\tau,i} \otimes \ell_{\sigma,\tau,i}$$

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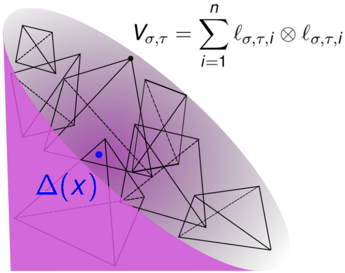

- We then write Δ as a sum of squares of linear forms :

$$\Delta(x) = \sum_{\sigma} \varphi_\sigma(\Delta(x))\Delta(x) = \sum_{\sigma} \varphi_\sigma(\Delta(x)) \sum_{\tau} \alpha_{\sigma,\tau}(x) V_{\sigma,\tau}$$

$$= \sum_{\sigma} \varphi_\sigma(\Delta(x)) \sum_{\tau} \alpha_{\sigma,\tau}(x) \sum_{i=1}^{n} \ell_{\sigma,\tau,i} \otimes \ell_{\sigma,\tau,i}$$

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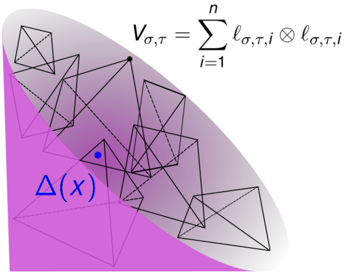

• Since $\Delta(\overline{\mathcal{U}}_\alpha)$ is compact, it intersects a finite number of simplices $Q(\Delta, \overline{\mathcal{U}}_\alpha)$. Reindexing the above sum, we obtain

$$\Delta(x) = \sum_{j=1}^{P_0} \rho_j(x) \ell_j^2$$

with $P_0 = (s_n + 1)nQ(\Delta, \overline{\mathcal{U}}_\alpha)$.

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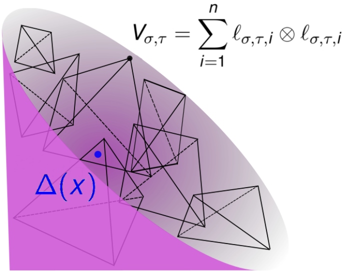


• A crucial observation is that, for each $x \in \overline{\mathcal{U}}_\alpha$ the decomposition

$$\Delta(x) = \sum_{j=1}^{P_0} \rho_j(x)\ell_j^2$$

has at most $(s_n + 1)nW$ non vanishing coefficients $\rho_j(x)$.

Step 2 : Iterations

- We build from f_0 a sequence of maps

$$f_{1,1}, \ldots, f_{1,P_0} = f_1$$

such that

$$g - f_{1,i}^*\langle .,. \rangle \simeq \frac{1}{4}\sum_{j=1}^{i} \rho_j \ell_j^2 + \sum_{j=i+1}^{P_0} \rho_j \ell_j^2$$

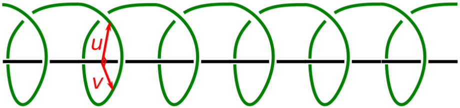
In particular

$$g - f_{1,P_0}^*\langle .,. \rangle \simeq \frac{1}{4}\Delta$$

$\Longrightarrow$

$$\|g - f_1^*\langle .,. \rangle\|_{C^0} \leq \frac{1}{2}\|\Delta\|_{C^0}$$

Step 2 : Iterations



- The maps build by Nash are given iteratively by the formula

$$f_{1,i} = f_{1,i-1} + \frac{\sqrt{3\rho_i}}{2N_{1,i}}\left(\cos(N_{1,i}\,\ell_i)\,\mathbf{u} + \sin(N_{1,i}\,\ell_i)\,\mathbf{v}\right)$$

where $\mathbf{u} = \mathbf{u}_{1,i}$ and $\mathbf{v} = \mathbf{v}_{1,i}$ are two orthogonal unit normal vectors. We have :

$$f_{1,i}^*\langle .,.\rangle - f_{1,i-1}^*\langle .,.\rangle = \frac{3}{4}\rho_i\ell_i^2 + O(1/N_{1,i})$$

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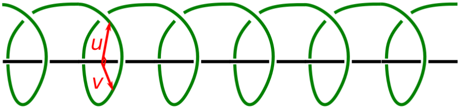


• Therefore

$$g - f_{1,P_0}^*\langle .,.\rangle = \frac{1}{4}\Delta + \sum_{j=1}^{P} O(1/N_{1,j})$$

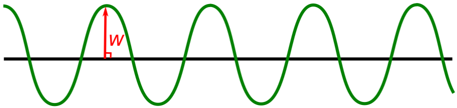
and if the $N_{1,j}$'s are large enough :

$$\|g - f_{1,P_0}^*\langle .,.\rangle\|_{C^0} \leq \frac{\|\Delta\|_{C^0}}{2}$$

Step 2 : Iterations

« Actually the condition $q \geq n + 2$ might be replaced by $q \geq n + 1$. This would come from use of a less easily controlled perturbation process needing only one direction normal to the imbedding. » Nash, 1954

Step 2 : Iterations



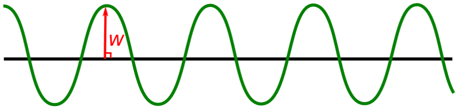
- The maps built by Kuiper are given iteratively by the formula

$$f_{1,i} = f_{1,i-1} - \frac{3\rho_i}{16N_{1,i}}\sin(2N_{1,i}\,\ell_i)\mathbf{t} + \frac{\sqrt{3\rho_i}}{\sqrt{2}N_{1,i}}\sin\left(N_{1,i}\,\ell_i - \frac{3\rho_i}{16}\sin(2N_{1,i}\,\ell_i)\right)\mathbf{w}$$

where $\mathbf{t} = \mathbf{t}_{1,i-1}$ is a (convenient) unit tangent vector and $\mathbf{w} = \mathbf{w}_{1,i-1}$ a unit normal vector. We have

$$f_{1,i}^*\langle .,.\rangle - f_{1,i-1}^*\langle .,.\rangle = \frac{3}{4}\rho_i\ell_i^2 + \text{extra unexpected terms} + O(1/N_{1,i})$$

Step 2 : Iterations



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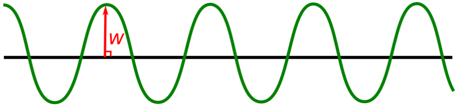
Step 2 : Iterations

« Our proof follows Nash' proof with the exception of a different kind of one step device : a strain. This strain however requires considerations concerning the convergence of the process which are even more delicate then those required with Nash' one step device. We therefore give a complete proof independent of Nash' paper » Kuiper, 1955

Step 2 : Iterations



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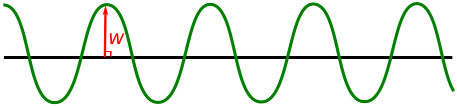

• The passing from the codimension 2 to the codimension 1 shows a real technical problem.

• This problem can be settled by substituting a **Convex Integration** to the Kuiper formula (we shall see how latter).

• The new map $f_{1,i}$ thus defined is such that

$$f_{1,i}^*\langle .,.\rangle - f_{1,i-1}^*\langle .,.\rangle = \frac{3}{4}\rho_i \ell_i^2 + O(1/N_{1,i})$$

Step 3 : Convergence

• We re-do all the process starting with f_1 and decomposing the new isometric default as a sum of P_1 squares of linear forms

$$\Delta_1(x) = g - f_1^*\langle .,. \rangle = \sum_{j=1}^{P_1} \rho_{1,j}(x)\ell_{1,j}^2$$

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• If the $N_{k,i}$'s are large enough, the resulting sequence of maps satisfies :

$$\|g - f_k^*\langle ., . \rangle\|_{C^0} \leq \frac{\|\Delta\|_{C^0}}{2^k}$$

Step 3 : Convergence

• A direct computation shows that the sequence (f_k) is C^1 converging.

Nash :

$$f_{k,i} = f_{k,i-1} + \frac{\sqrt{3\rho_{k,i}}}{2N_{k,i}}\left(\cos(N_{k,i}\,\ell_{k,i})\,\mathbf{u} + \sin(N_{k,i}\,\ell_{k,i})\,\mathbf{v}\right)$$

Kuiper :

$$\begin{aligned} f_{k,i} &= f_{k,i-1} - \frac{3\rho_{k,i}}{16N_{k,i}}\sin(2N_{k,i}\,\ell_{k,i})\mathbf{t} \\ &\quad + \frac{\sqrt{3\rho_{k,i}}}{\sqrt{2}N_{k,i}}\sin\left(N_{k,i}\,\ell_{k,i} - \frac{3\rho_{k,i}}{16}\sin(2N_{k,i}\,\ell_{k,i})\right)\mathbf{w} \end{aligned}$$

(here again $\mathbf{u} = \mathbf{u}_{1,i}$ and $\mathbf{v} = \mathbf{v}_{1,i}$ are two orthogonal unit normal vectors of $f_{k,i-1}$).

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(here again $\mathbf{u} = \mathbf{u}_{1,i}$ and $\mathbf{v} = \mathbf{v}_{1,i}$ are two orthogonal unit normal vectors of $f_{k,i-1}$).

• Thus the limit map f_∞ is C^1 isometric.

Step 4 : The limit map f_∞ is an embedding

• The image of $f_{1,1}$ is a graph above f_0 (lying in a normal neiborhood of f_0), therefore $f_{1,1}$ is an embedding. For the same reason, each f_k is an embedding.

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• Let x_1 and x_2 be two distinct points of M and let $k > 0$, we put

$$d_k(x_1, x_2) := dist(f_k(x_1), f_k(x_2)).$$

Since $\|f_{k+1} - f_k\|_{C_0} \leq \frac{1}{2^k}$ we have

$$dist(f_\infty(x_i) - f_k(x_i)) \leq \frac{1}{2^{k-1}}$$

and thus

$$d_k(x_1, x_2) - \frac{1}{2^{k-2}} \leq dist(f_\infty(x_1), f_\infty(x_2)) = d_\infty(x_1, x_2)$$

Step 4 : The limit map f_∞ is an embedding

• We shall show that, for every couple of distinct points (x_1, x_2), there exists k such that $d_k(x_1, x_2) - \frac{1}{2^{k-2}} > 0$. This will imply that f_∞ is an embedding.

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• Let $(f_{k,i})_{i\in\{1,\dots,P_k\}}$ be the sequence joining f_k to f_{k+1}. We first observe that

$$\lim_{N_{k,i+1}\to+\infty} \|f_{k,i+1} - f_{k,i}\|_{C^0} = 0$$

implies

$$\lim_{N_{k,i+1}\to+\infty} \|d_{k,i+1}(.,.) - d_{k,i}(.,.)\|_{C^0} = 0$$

Thus, for every k and every i, there exists $N_{k,i+1}$ such that

$$d_{k,i+1} \geq \left(\frac{2}{3}\right)^{1/P_k} d_{k,i}$$

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• We have proved the Nash-Kuiper Theorem

The outrageously simple idea

John Nash

The isometric default must be reduced iteratively and not all at once.

The outrageously simple idea

By considering

$$f_{1,i} = f_{1,i-1} + \frac{\sqrt{1\rho_{1,i}}}{N_{1,i}}\left(\cos N_{k,i}\ \ell_{1,i}\right) \mathbf{u} + \sin(N_{1,i}\ \ell_{1,i})\ \mathbf{v})$$

instead of

$$f_{1,i} = f_{1,i-1} + \frac{\sqrt{3\rho_{1,i}}}{2N_{1,i}}\left(\cos N_{1,i}\ \ell_{1,i}\right) \mathbf{u} + \sin(N_{1,i}\ \ell_{1,i})\ \mathbf{v})$$

it is obviously possible to kill the whole isometric default in each direction $\ell_{1,i}$ up to a $O(1/N_{1,i})$:

$$f_{1,i}^*\langle .,. \rangle - f_{1,i-1}^*\langle .,. \rangle = 1 \times \rho_i \ell_i^2 + O(1/N_{1,i})$$

to get a map f_1 approximately isometric

$$g - f_1^*\langle .,. \rangle = \sum_{i=1}^{P_0} O(1/N_{1,i}).$$

The outrageously simple idea

BUT

This leads to a dead end. Indeed there is no control on the sign of $O(1/N_{1,i})$ and consequently f_1 is no longer a short map in general. It lengthens some curves and the helix deformation can not reduce their length.

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NASH

bypasses this difficulty with an iterative approach, dividing the isometric default by 2 at each step rather than trying to reduce it to zero all at once.

That's all folks !

John Nash