L2: From Nash-Kuiper to Gromov

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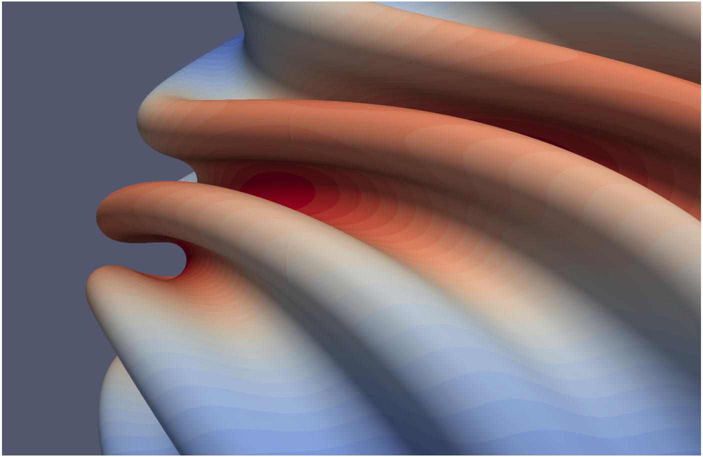
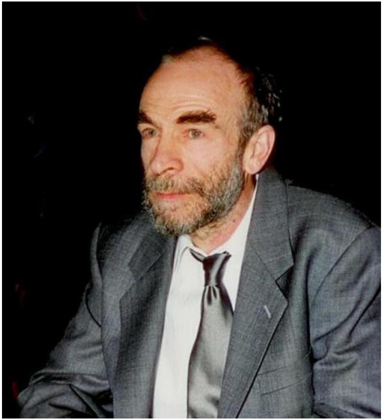
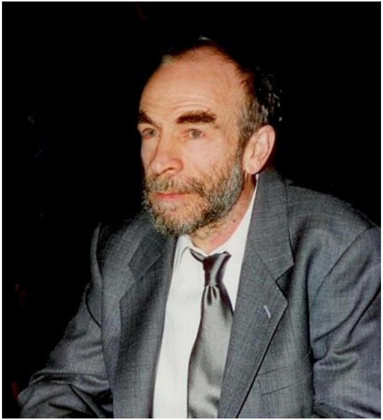


An incredible result



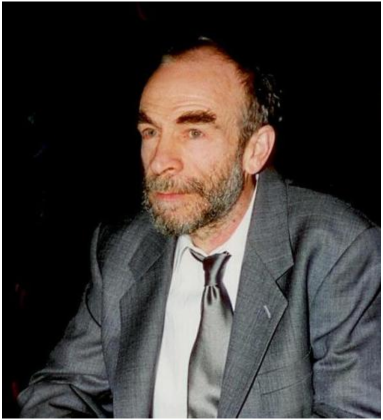
« *At first, I looked at one of Nash's papers and thought it was just nonsense [...] It was incredible. It could not be true but it was true* ».

An incredible



« *I was thinking about this for several years, trying to understand the mechanism behind [the Nash's proof]* »

An inspirational source



The Nash's proof was an inspirational source for the Gromov's Convex Integration Theory

Back to the Nash-Kuiper's proof

The step 2 problem.– Let

- $f_0 : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{E}^q$ be an immersion,
- $\rho : \mathcal{U} \to \mathbb{R}_{\geq 0}$
- $\ell : \mathbb{R}^n \to \mathbb{R}$ be a linear form,
- $\epsilon > 0$

Find $f : \mathcal{U} \to \mathbb{E}^q$ such that :

i) $f^*\langle .,. \rangle = f_0^*\langle .,. \rangle + \rho \ell \otimes \ell$

ii) $\|f - f_0\|_{C^0} < \epsilon$

Back to the Nash-Kuiper's proof

The step 2 problem (rephrasing+codimension at least 2).– Let

- $f_0 : [0,1]^n \to \mathbb{E}^q$ be an immersion with $q \geq n+2$,
- $\rho : [0,1]^n \to \mathbb{R}_{\geq 0}$
- $\ell = dx_1$
- $\epsilon > 0$

Find $f : [0,1]^n \to \mathbb{E}^q$ such that :

i) $f^*\langle .,. \rangle = f_0^*\langle .,. \rangle + \rho dx_1 \otimes dx_1$

ii) $\|f - f_0\|_{C^0} < \epsilon$

Solution

- If we apply condition i) to (∂_1, ∂_1) we obtain

$$f^*\langle \partial_1, \partial_1 \rangle = f_0^*\langle \partial_1, \partial_1 \rangle + \rho dx_1(\partial_1) dx_1(\partial_1)$$

i.e.

$$\|\partial_1 f(x)\|^2 = \|\partial_1 f_0(x)\|^2 + \rho(x)$$

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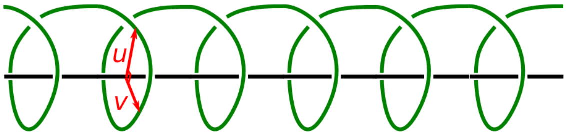
i.e.

$$\|\partial_1 f(x)\|^2 = \|\partial_1 f_0(x)\|^2 + \rho(x)$$

• We first build a map $\partial_1 f(x)$ satisfying the above equation and then we define f to be a primitive of that map :

$$f(x) = f_0(0, x_2, ..., x_n) + \int_0^{x_1} \partial_1 f(u, x_1, ..., x_n)du$$

Solution

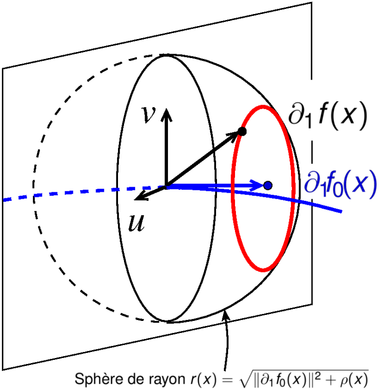


• To define $\partial_1 f$ we follow the Nash's approach. Given two unit normal vectors of f_0 :

$$\mathbf{u}, \mathbf{v} : [0,1]^n \to \mathbb{E}^q$$

such that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, we look for a solution $\partial_1 f$ behaving as a tangent vector to an helix.

Convex Integration

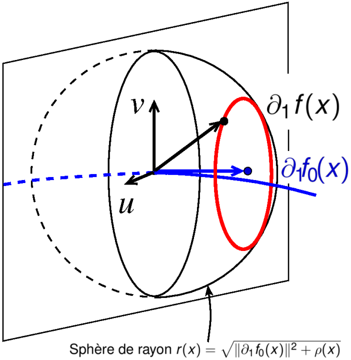


- We put

$$\partial_1 f(x) = \sqrt{\rho(x)}\; e^{i\theta(x)} + \partial_1 f_0(x)$$

where $e^{i\theta} := \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{v}$ and $\theta : [0,1]^n \longrightarrow \mathbb{R}$ will be chosen latter.

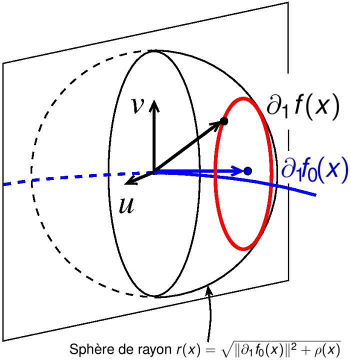
Convex Integration



- Since $(\mathbf{u}, \mathbf{v})$ are unit normal vectors, we have

$$\partial_1 f(x) = \sqrt{\rho(x)}\, e^{i\theta(x)} + \partial_1 f_0(x) \quad \Longrightarrow \quad \|\partial_1 f(x)\|^2 = \rho(x) + \|\partial_1 f_0(x)\|^2$$

Convex Integration



- A possible choice for θ is

$$\theta(x) = 2\pi N x_1$$

where $N \in \mathbb{N}^*$ is a free parameter (= the number of spirals) .

C^0-density

- For every $x \in [0,1]^n$ we set

$$\begin{aligned} f(x) &:= f_0(0, x_2, ..., x_n) + \int_0^{x_1} \sqrt{\rho(u, x_2, ..., x_n)} e^{i2\pi N u} + \partial_1 f_0(u, x_2, ..., x_n)\, \mathrm{d}u \\ &= f_0(x) + \int_0^{x_1} \sqrt{\rho(u, x_2, ..., x_n)} e^{i2\pi N u}\, \mathrm{d}u \end{aligned}$$

C^0-density

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&= \; f_0(x) + \int_0^{x_1} \sqrt{\rho(u, x_2, ..., x_n)} e^{i2\pi N u}\, \mathrm{d}u
\end{aligned}
$$

- Since $\int_0^1 e^{i2\pi u} du = 0$, we have (see the lemma below)

$$
\|f - f_0\|_{C^0} = O(\frac{1}{N}) \quad \text{and} \quad \|\partial_j f - \partial_j f_0\|_{C^0} = O(\frac{1}{N})
$$

for every $j \geq 2$.

C^0-density

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$$f(x) := f_0(0, x_2, ..., x_n) + \int_0^{x_1} \sqrt{\rho(u, x_2, ..., x_n)} e^{i2\pi N u} + \partial_1 f_0(u, x_2, ..., x_n)\, \mathrm{d}u$$
$$= \quad f_0(x) + \int_0^{x_1} \sqrt{\rho(u, x_2, ..., x_n)} e^{i2\pi N u}\, \mathrm{d}u$$

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$$\|f - f_0\|_{C^0} = O(\frac{1}{N}) \quad \text{and} \quad \|\partial_j f - \partial_j f_0\|_{C^0} = O(\frac{1}{N})$$

for every $j \geq 2$.

- Thus, if N is large enough, f fulfills the C^0-closeness condition *ii*).

A useful lemma

Lemma.– *Let* $f:[a,b]\to\mathbb{E}^q$ *be a* C^1 *function and* $h:\mathbb{R}\to\mathbb{R}$ *be a continuous* T*-periodic function then*

$$\int_a^b f(s)h(Ns)ds = \left(\int_a^b f(s)ds\right)\left(\frac{1}{T}\int_0^T h(s)ds\right) + O\left(\frac{1}{N}\right)$$

In particular, if $\overline{h}=0$, *then*

$$\int_a^b f(s)h(Ns)ds = O\left(\frac{1}{N}\right)$$

A useful lemma

Proof.– Let $g = h - \overline{h}$. We have

$$\begin{aligned}\int_a^b f(s)\left(h(Ns) - \overline{h}\right) ds &= \int_a^b f(s) g(Ns) ds \\ &= \left[f(s)\frac{G(Ns)}{N}\right]_a^b - \int_a^b f'(s)\frac{G(Ns)}{N} ds\end{aligned}$$

where G is the primitive of g given by $G(t) = \int_0^t g(s)ds$.

A useful lemma

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Since

$$\int_0^T g(s)ds = \int_0^T (h(s) - \overline{h})ds = 0$$

the primitive G is T-periodic thus bounded.

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Since

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the primitive G is T-periodic thus bounded. We conclude

$$\int_a^b f(s)\left(h(Ns) - \overline{h}\right) ds = O\left(\frac{1}{N}\right)$$

□

The isometric condition

- By construction, for $j = 1$, we have

$$\|\partial_1 f(x)\|^2 = \|\partial_1 f_0\|^2 + \rho(x)$$

for every $x \in [0,1]^n$. Thus, f fulfills Condition i) for the couple (∂_1, ∂_1).

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- However, for the other couples (∂_i, ∂_j), $(i,j) \neq (1,1)$, we have

$$\begin{aligned}\langle \partial_i f(x), \partial_j f(x)\rangle &= \langle \partial_i f_0(x), \partial_j f_0(x)\rangle + O(\frac{1}{N}) \\ &= \langle \partial_i f_0(x), \partial_j f_0(x)\rangle + \rho(x) dx_1(\partial_i) dx_1(\partial_j) + O(\frac{1}{N})\end{aligned}$$

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• Finally, we have solved condition i) approximately

$$f^*\langle .,. \rangle = f_0^*\langle .,. \rangle + \rho dx_1 \otimes dx_1 + O(\frac{1}{N})$$

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- Finally, we have solved condition i) approximately

$$f^*\langle .,. \rangle = f_0^*\langle .,. \rangle + \rho dx_1 \otimes dx_1 + O(\frac{1}{N})$$

- Note that Nash also solved this condition approximately.

Convex Integration Formula

Definition.– Let

- $\gamma : [0,1]^n \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^q$ be a family of loops
- $f_0 : [0,1]^n \to \mathbb{R}^q$ a map
- $N > 0$

We define a new map $F : [0,1]^n \to \mathbb{R}^q$ by setting

$$F(x) := f_0(0, x_2, ..., x_n) + \int_0^{x_1} \gamma(u, x_2, ..., x_n; Nu)du$$

for every $x \in [0,1]^n$. The map F is said to be obtained from f_0 by **Convex Integration.** It is denoted by $F = CI_\gamma(f_0, \partial_1, N)$.

Convex Integration Formula

Example.– The map f previously built is obtained by convex integration from f_0 and with the following choice for γ :

$$\gamma(x,t) = \sqrt{\rho(x)}e^{2i\pi t} + \partial_1 f_0(x)$$

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• Observe that

$$\int_0^1 \gamma(x,t)dt = \partial_1 f_0(x).$$

In average, the effect of γ and $\partial_1 f_0$ are the same. This is the reason why f is C^0-close to f_0.

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Definition.– A family of loops $\gamma : [0,1]^n \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^q$ *satisfies the average condition with respect to* f_0 *and in the direction of* ∂_1 if

$$\forall x \in [0,1]^n, \quad \int_0^1 \gamma(x,t)dt = \partial_1 f_0(x).$$

Convex Integration Formula

Proposition.– *If γ satisfies the average condition with respect to f_0 and in the direction ∂_1 then the following properties hold for $F = CI_\gamma(f_0, \partial_1, N)$:*

$$(P_1)\ \ \|f_0 - F\|_{C^0} = O(1/N),$$

$$(P_2)\ \ \|\partial_i f_0 - \partial_i F\|_{C^0} = O(1/N) \text{ for every } i \neq 1,$$

$$(P_3)\ \ \forall x \in [0,1]^n, \quad \partial_1 F(x) = \gamma(x, N x_1).$$

Convex Integration Formula

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$$(P_3)\ \ \forall x \in [0,1]^n, \quad \partial_1 F(x) = \gamma(x, Nx_1).$$

Proof.– Postponed to the lecture devoted to the 1D Convex Integration.

Improving the Kuiper Formula

The step 2 problem (codimension 1).– We assume $q = n+1$. Given $\epsilon > 0$ we want to construct $f : [0,1]^n \to \mathbb{E}^{n+1}$ such that

$$i) \quad \|f^*\langle .,. \rangle - (f_0^*\langle .,. \rangle + \rho dx_1 \otimes dx_1)\|_{C^0} < \epsilon$$

$$ii) \quad \|f - f_0\|_{C^0} < \epsilon$$

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$$ii) \quad \| f - f_0 \|_{C^0} < \epsilon$$

Solution.– We are going to build f by a convex integration from f_0 in the direction ∂_1. Any such map will satisfy property $(P2)$:

$$\|\partial_i f - \partial_i f_0\|_{C^0} = O(1/N) \text{ pour tout } i \neq 1$$

which implies that

$$\langle \partial_i f, \partial_j f \rangle = \langle \partial_i f_0, \partial_j f_0 \rangle + O(1/N)$$

for every $i \neq 1$, $j \neq 1$.

Improving the Kuiper Formula

- It remains to solve (i) for the couples $(1, i)$, $i \in \{1, ..., n\}$, i. e.

$$\left\{ \begin{array}{l} \langle \partial_1 f, \partial_i f \rangle = \langle \partial_1 f_0, \partial_i f_0 \rangle + O(1/N) \text{ for every } i \neq 1 \\ \|\partial_1 f\|^2 = \|\partial_1 f_0\|^2 + \rho + O(1/N) \end{array} \right.$$

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Or equivalently

$$\begin{cases} \langle \partial_1 f, \partial_i f_0 \rangle = \langle \partial_1 f_0, \partial_i f_0 \rangle + O(1/N) \text{ for every } i \neq 1 \\ \|\partial_1 f\|^2 = \|\partial_1 f_0\|^2 + \rho + O(1/N) \end{cases}$$

since

$$\|\partial_i f - \partial_i f_0\|_{C^0} = O(1/N) \text{ pour tout } i \neq 1$$

- For every $x \in [0,1]^n$, we put

$$\mathcal{R}_x = \left\{ v \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \langle v, \partial_i f_0(x) \rangle = \langle \partial_1 f_0(x), \partial_i f_0(x) \rangle \text{ for every } i \neq 1 \\ \|v\|^2 = \|\partial_1 f_0(x)\|^2 + \rho(x) \end{array} \right\}$$

Improving the Kuiper Formula

$$\mathcal{R}_x = \left\{ v \in \mathbb{R}^{n+1} \mid \begin{array}{l} \langle v, \partial_i f_0(x) \rangle = \langle \partial_1 f_0(x), \partial_i f_0(x) \rangle \text{ for every } i \neq 1 \\ \|v\|^2 = \|\partial_1 f_0(x)\|^2 + \rho(x) \end{array} \right\}$$

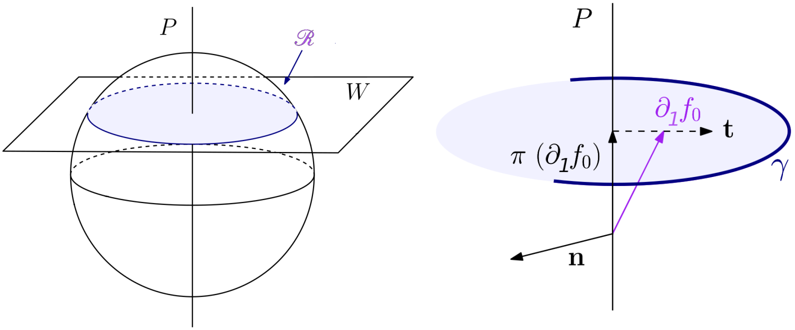
- The set $\mathcal{R}_x$ is the intersection of a hypersphere $\mathbb{S}^n(R)$ of radius

$$R = \sqrt{\|\partial_1 f_0(x)\|^2 + \rho(x)}$$

and of an affine 2-plane

$$W = \{ v \in \mathbb{R}^{n+1} \mid \langle v, \partial_i f_0(x) \rangle = \langle \partial_1 f_0(x), \partial_i f_0(x) \rangle \text{ for every } i \neq 1 \}$$

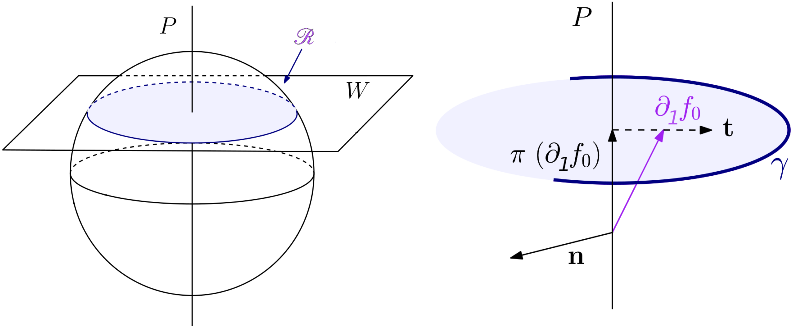
Improving the Kuiper Formula



• It is easily seen that $\mathcal{R}_x$ is a circle whose center is given by the projection $\pi(\partial_1 f_0(x))$ of $\partial_1 f_0(x)$ on $P = Span(\partial_2 f_0(x), ..., \partial_n f_0(x))$ and whose radius is

$$r(x) = \sqrt{\|\partial_1 f_0(x)\|^2 + \rho(x) - \|\pi(\partial_1 f_0(x))\|^2}$$

Improving the Kuiper Formula

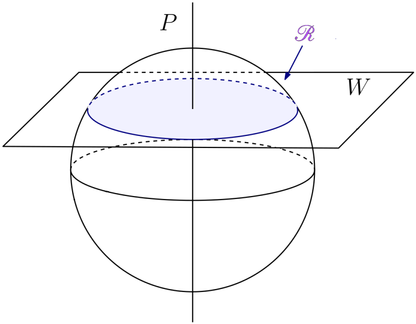


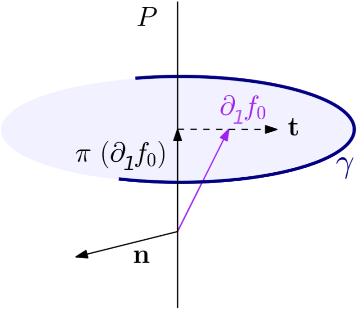
• We have to choose a family of loops $\gamma : [0,1]^n \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^{n+1}$ such that

1) $t \mapsto \gamma(x,t) \in \mathcal{R}_x$

2) $\int_0^1 \gamma(x,t)dt = \partial_1 f_0(x)$

Improving the Kuiper Formula

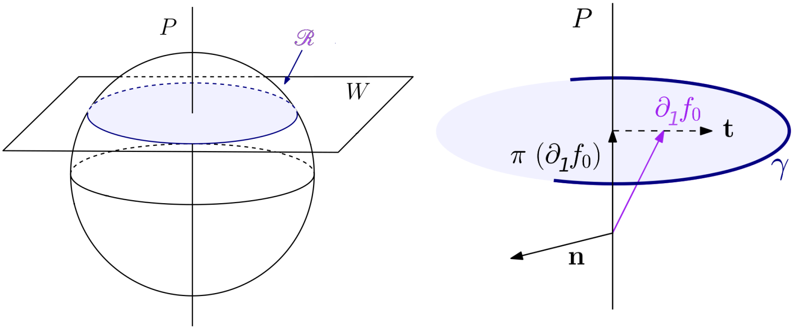




- We set

$$\mathbf{t} = \frac{\partial_1 f_0 - \pi(\partial_1 f_0)}{\|\partial_1 f_0 - \pi(\partial_1 f_0)\|} \quad \text{and} \quad \mathbf{n} = \frac{\partial_1 f_0 \wedge \ldots \wedge \partial_n f_0}{\|\partial_1 f_0 \wedge \ldots \wedge \partial_n f_0\|}$$

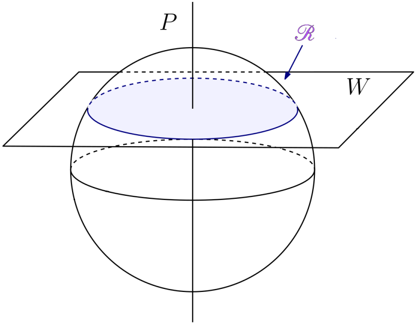
Improving the Kuiper Formula

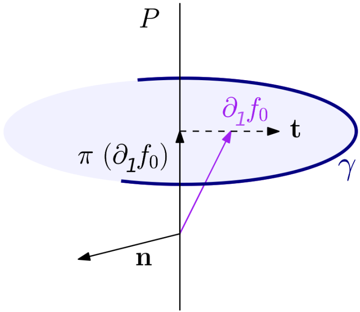


- We define γ to be

$$\gamma(x,t) = \pi(\partial_1 f_0(x)) + r(x)(\cos\theta\ \mathbf{t} + \sin\theta\ \mathbf{n})$$

Improving the Kuiper Formula





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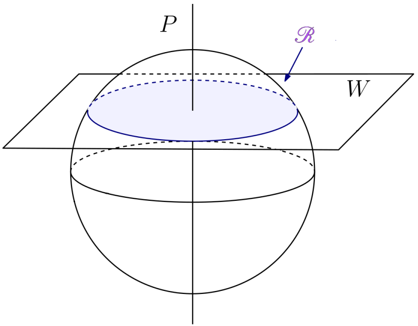
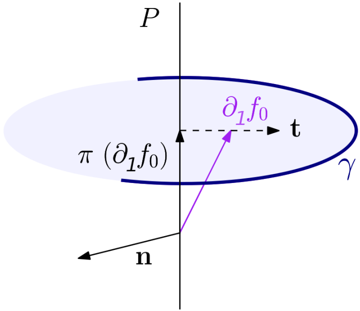
$$\gamma(x,t) = \pi(\partial_1 f_0(x)) + r(x)(\cos\theta\ \mathbf{t} + \sin\theta\ \mathbf{n})$$

with $\theta(x,t) = \alpha(x)\cos 2\pi t$ and $\alpha(x)$ is to be determined.

Improving the Kuiper Formula



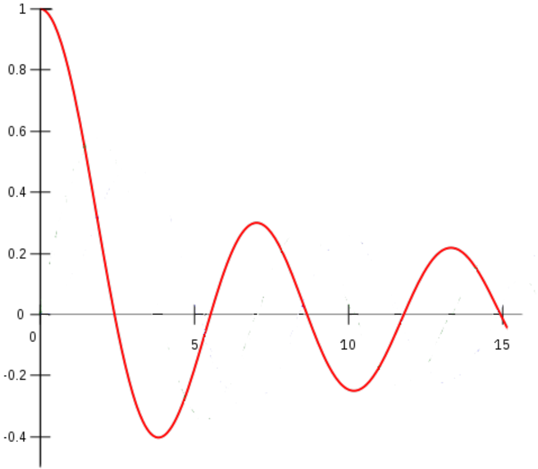


- We then have

$$\int_0^1 \gamma(x,t)dt = r(x)J_0(\alpha(x))\mathbf{t} + \pi(\partial_1 f_0(x))$$

where J_0 the Bessel function.

The Bessel Function J_0



$$J_0(\alpha) = \frac{1}{\pi}\int_0^{\pi} \cos(\alpha \sin u)du$$

Improving the Kuiper Formula

- We then have

$$\int_0^1 \gamma(x,t)dt = r(x)J_0(\alpha(x))\mathbf{t} + \pi(\partial_1 f_0(x))$$

where J_0 the Bessel function.

- To ensure the average to be equal to $\partial_1 f_0$, it is enough to choose

$$\alpha(x) = J_0^{-1}\left(\frac{\|\partial_1 f_0(x) - \pi(\partial_1 f_0(x))\|}{r(x)}\right)$$

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- Since $rJ_0(\alpha)\mathbf{t} + \pi(\partial_1 f_0) = \partial_1 f_0$, we can write

$$\gamma(x,t) = r(\cos(\alpha\cos 2\pi t) - J_0(\alpha))\,\mathbf{t} + r\sin(\alpha\cos 2\pi t)\,\mathbf{n} + \partial_1 f_0$$

Improving the Kuiper Formula

To sum up.– The map $f = CI_\gamma(f_0, \partial_1, N)$ with

$$\gamma(x,t) = r(\cos(\alpha\cos 2\pi t) - J_0(\alpha))\,\mathbf{t} + r\sin(\alpha\cos 2\pi t)\,\mathbf{n} + \partial_1 f_0$$

and

$$r = \sqrt{\|\partial_1 f_0\|^2 + \rho - \|\pi(\partial_1 f_0)\|^2}, \quad \alpha = J_0^{-1}\left(\frac{\|\partial_1 f_0 - \pi(\partial_1 f_0)\|}{r}\right)$$

satisfies the following properties

$i)$ $f^*\langle .,.\rangle = f_0^*\langle .,.\rangle + \rho dx_1 \otimes dx_1 + O(1/N)$

$ii)$ $\|f - f_0\|_{C^0} = O(1/N)$

$iii)$ $\|\partial_i f - \partial_i f_0\|_{C^0} = O(1/N)$ for every $i \neq 1$.

Improving the Kuiper Formula

Analytical expression.– The map $f = CI_\gamma(f_0, \partial_1, N)$ has the following expression

$$f(x) = f_0(0, x_2, ..., x_m) + \int_0^{x_1} \gamma(u, x_2, ..., x_m; Nu)du$$

with

$$\gamma(x,t) = \pi(\partial_1 f_0(x)) + r(x)(\cos(\alpha(x)\cos 2\pi t)\,\mathbf{t}(x) + \sin(\alpha(x)\cos 2\pi t)\,\mathbf{n}(x)).$$

Improving the Kuiper Formula

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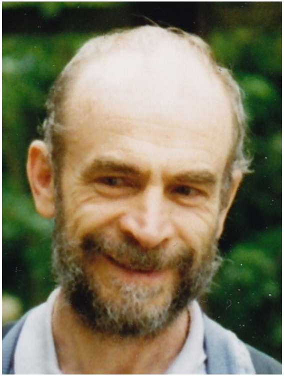
- By comparison the Kuiper formula is :

$$f(x) = f_0(x) - \frac{3\rho(x)}{16N}\sin(2Nx_1)\mathbf{t}(x) + \frac{\sqrt{3\rho(x)}}{\sqrt{2}N}\sin\left(Nx_1 - \frac{3\rho(x)}{16}\sin(2Nx_1)\right)\mathbf{n}(x)$$

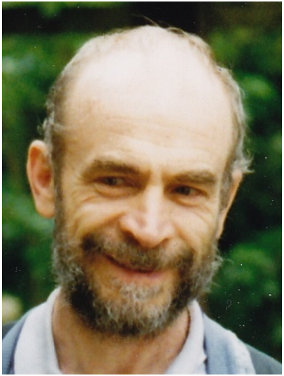
The outrageously simple idea



Mikhaïl Gromov

- The $O(\rho^2)$ default in the Kuiper process deserves to be corrected

The outrageously simple idea



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- The $O(\rho^2)$ default in the Kuiper process deserves to be corrected

- This can be done by combining a geometrical approach with a simple integral formula.

Mikhaïl Gromov