L3: 1D Convex Integration

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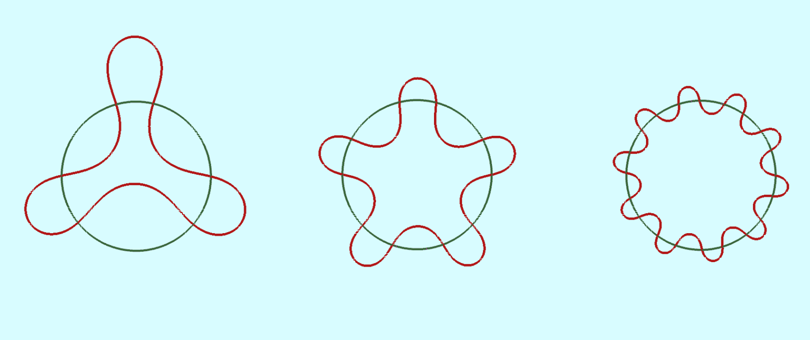


General Approach

Problem.– Let $\mathcal{R} \subset \mathbb{R}^n$ be a path-connected subset (=our differential relation) and $f_0 : [0,1] \xrightarrow{C^1} \mathbb{R}^n$ be a map such

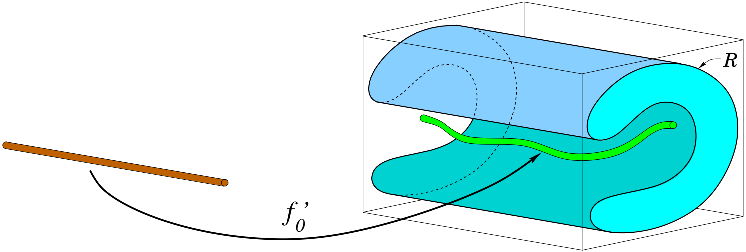
$$\forall t \in [0,1], \quad f_0'(t) \in \ Conv(\mathcal{R}).$$

Find $F : [0,1] \xrightarrow{C^1} \mathbb{R}^n$ such that :

$$\begin{array}{ll} i) & \forall t \in [0,1], \quad F'(t) \in \mathcal{R} \\ ii) & \|F - f_0\|_{C^0} < \delta \end{array}$$

with $\delta > 0$ given.

How to build a solution ?



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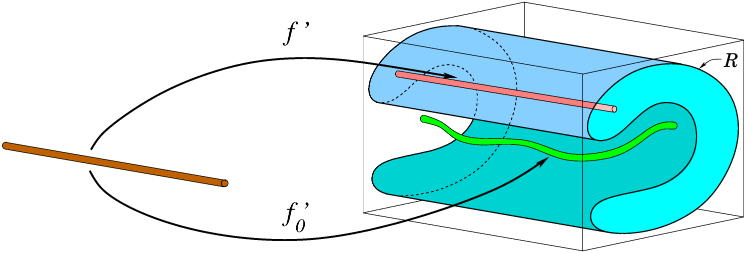

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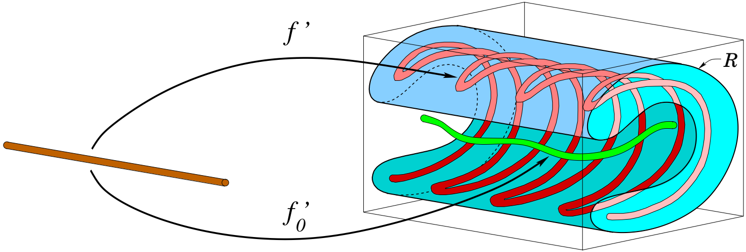


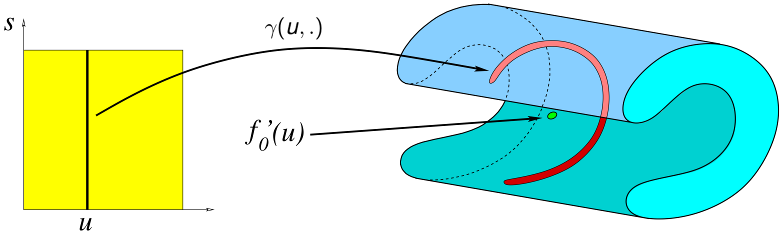
Construction of the solution

Step 1.– Choose a continuous family of loops

$$\begin{array}{rccc} \gamma : & [0,1] & \longrightarrow & C^0(\mathbb{R}/\mathbb{Z}, \mathcal{R}) \\ & u & \longmapsto & \gamma(u,.) \end{array}$$

such that

$$\forall u \in [0,1], \quad \int_{[0,1]} \gamma(u,s)ds = f_0'(u).$$

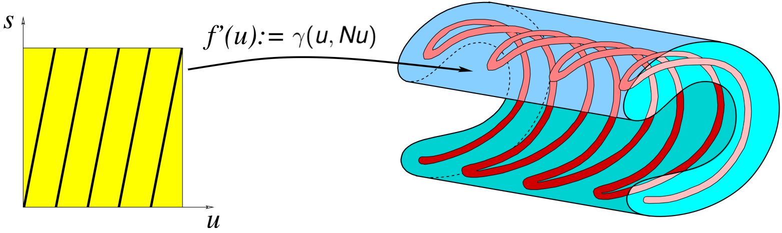


Construction of the solution

Step 2.– We define $F := CI_\gamma(f_0, N)$ to be the map obtained by a **convex integration** from f_0 :

$$F(t) := f_0(0) + \int_0^t \gamma(u, Nu)du$$

where $N \in \mathbb{N}^*$ is a free parameter.



C^0-Density

• Obviously $F = CI_\gamma(f_0, N)$ fulfills condition *i*) since

$$F'(t) = \gamma(t, Nt) \in \mathcal{R}$$

for all $t \in [0, 1]$. The fact that F also fulfills condition *ii*) for N large enough will ensue from the following proposition

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Proposition (C^0-density).– *If γ is of class C^1 and satisfies the average condition*

$$\forall u \in [0,1], \quad \int_{[0,1]} \gamma(u,s)ds = f_0'(u)$$

then

$$\|F - f_0\|_{C^0} \leq \frac{1}{N}\left(2\|\gamma\|_{C^0} + \|\partial_1\gamma\|_{C^0}\right)$$

where $\|\gamma\|_{C^0} = \sup_{(u,s)\in[0,1]^2} \|\gamma(u,s)\|_{\mathbb{E}^3}$.

C^0-Density

Proof : Let $t \in [0,1]$. Put $n = \lfloor Nt \rfloor$, $I_j = [\frac{j}{N}, \frac{j+1}{N}]$ for $0 \leq j \leq n-1$, $I_n = [\frac{n}{N}, t]$. Since

$$\forall t \in I, \qquad F(t) := f_0(0) + \int_0^t \gamma(s, Ns)ds$$

we obviously have

$$F(t) - f_0(0) = \sum_{k=0}^{n} F^{[k]}$$

where

$$F^{[k]} = \int_{I_k} \gamma(s, Ns)\mathrm{d}s.$$

C^0-Density

Since

$$\begin{aligned} f_0(t) &= f_0(0) + \int_{x=0}^{t} \frac{\partial f_0}{\partial x}(x)\mathrm{d}x \\ &= f_0(0) + \int_{x=0}^{t}\int_{u=0}^{1} \gamma(x,u)\mathrm{d}u\mathrm{d}x \end{aligned}$$

we also have

$$f_0(t) - f_0(0) = \sum_{j=0}^{n} f^{[j]}$$

with

$$f^{[j]} = \int_{R_j} \gamma(x,u)\mathrm{d}x\mathrm{d}u$$

and $R_j = I_j \times [0,1]$.

C^0-Density

We consider $j \in [0, n-1]$. By the change of variables $u = Ns - j$, we get

$$F^{[j]} = \int_0^1 \frac{1}{N}\gamma\left(\frac{u+j}{N}, u\right) \mathrm{d}u.$$

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We now define

$$\begin{array}{rccc} H_j : & R_j & \to & \mathbb{R}^n \\ & (x,u) & \mapsto & \gamma(\frac{u+j}{N}, u). \end{array}$$

In particular, H_j is constant over each horizontal segment in R_j. It ensues that

$$F^{[j]} = \int_{R_j} H_j(x,u)\mathrm{d}x\mathrm{d}u$$

implying

$$|F^{[j]} - f_0^{[j]}| \leq \int_{R_j} \|\gamma(\frac{u+j}{N}, u) - \gamma(x,u)\|\mathrm{d}x\mathrm{d}u \leq \frac{1}{N^2}\|\partial_1\gamma\|_\infty.$$

The last inequality follows from the mean value theorem and the fact that the area of $R_j = [\frac{j}{N}, \frac{j+1}{N}] \times [0,1]$ is $1/N$.

C^0-Density

For $j = n$ we can use a simpler upper bound :

$$\|F^{[n]} - f_0^{[n]}\| \leq \|F^{[n]}\| + \|f_0^{[n]}\| \leq \frac{2}{N}\|\gamma\|_\infty.$$

We finally obtain

$$\begin{aligned}\|F(t) - f_0(t)\| &\leq \sum_{j=0}^{n-1} \|F^{[j]} - f_0^{[j]}\| + \|F^{[n]} - f_0^{[n]}\| \\ &\leq \frac{1}{N}\|\partial_1\gamma\|_\infty + \frac{2}{N}\|\gamma\|_\infty.\end{aligned}$$

□

Summing up...

• To sum up, we are able to construct a solution of our initial problem as long as we have found a family of loops

$$\begin{array}{rccc} \gamma: & [0,1] & \longrightarrow & C^0(\mathbb{R}/\mathbb{Z},\mathcal{R}) \\ & u & \longmapsto & \gamma(u,.) \end{array}$$

that satisfies the **average condition** i. e.

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$$\forall u \in\ [0,1], \quad \int_{[0,1]} \gamma(u,s)ds = f_0'(u).$$

• The existence of such a family γ is the **Fundamental Lemma** of Convex Integration Theory.

Fundamental Lemma

Notation.– Let $\mathcal{R} \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$ (not necessarily path connected) and $\sigma \in \mathcal{R}$. We denote by $IntConv(\mathcal{R}, \sigma)$ the interior of the convex hull of the component of $\mathcal{R}$ to which σ belongs.

Fundamental Lemma (Gromov, 1969).– *Let $\mathcal{R} \subset \mathbb{R}^n$ be an open set, $\sigma \in \mathcal{R}$ and $z \in IntConv(\mathcal{R}, \sigma)$ There exists a loop $\gamma : \mathbb{S}^1 \longrightarrow \mathcal{R}$ with base point σ such that :*

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Proof.– On the blackboard.

Fundamental Lemma

Remark.– *A priori* $\gamma \in \Omega_\sigma(\mathcal{R})$, but it is obvious that we can choose γ among back and forth loops *i.e* the space :

$$\Omega_\sigma^{BF}(\mathcal{R}) = \{\gamma \in \Omega_\sigma(\mathcal{R}) \mid \forall s \in [0,1] \;\; \gamma(s) = \gamma(1-s)\}.$$

The point is that the above space is contractible. For every $\tau \in [0,1]$ we then denote by $\gamma^\tau : \mathbb{R}/\mathbb{Z} \longrightarrow \mathcal{R}$ the loop defined by

$$\gamma^\tau(s) = \begin{cases} \gamma(s) & \text{if} \quad s \in \left[0, \frac{\tau}{2}\right] \cup \left[1 - \frac{\tau}{2}\right] \\ \gamma(\tau) & \text{if} \quad s \in \left[\frac{\tau}{2}, 1 - \frac{\tau}{2}\right]. \end{cases}$$

This homotopy induces a deformation retract of $\Omega_\sigma^{BF}(\mathcal{R})$ to the constant loop

$$\begin{array}{rccc} \widetilde{\sigma} : & \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathcal{R} \\ & s & \longmapsto & \sigma. \end{array}$$

Parametric Fundamental Lemma

Parametric version of the Fundamental Lemma. – *Let* $E = [a,b] \times \mathbb{R}^n \stackrel{\pi}{\longrightarrow} [a,b]$ *be a trivial bundle and* $\mathcal{R} \subset E$ *be an open set. Let* $\mathfrak{S} \in \Gamma(\mathcal{R})$ *and* $z \in \Gamma(E)$ *such that :*

$$\forall p \in [a,b],\ z(p) \in IntConv(\mathcal{R}_p, \mathfrak{S}(p))$$

where $\mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R}$*. Then, there exists* $\gamma : [a,b] \times \mathbb{S}^1 \stackrel{C^\infty}{\longrightarrow} \mathcal{R}$ *such that :*

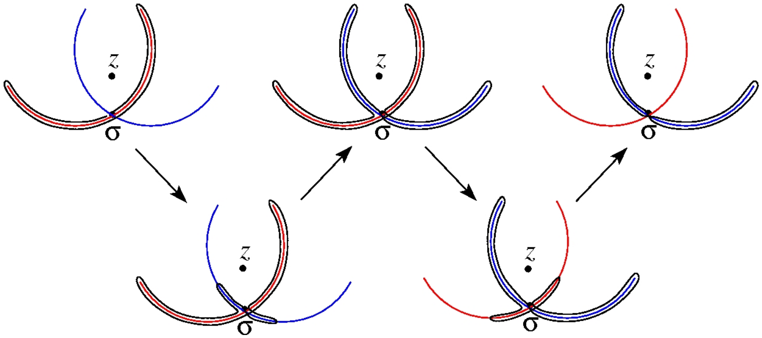
$$\gamma(.,0) = \gamma(.,1) = \mathfrak{S} \in \Gamma(\mathcal{R}),$$

$$\forall p \in [a,b],\ \gamma(p,.) \in Concat(\Omega^{BF}_{\mathfrak{S}(p)}(\mathcal{R}_p))$$

and

$$\forall p \in [a,b],\ z(p) = \int_0^1 \gamma(p,s)ds.$$

Idea of Proof : concatenation of BF loops



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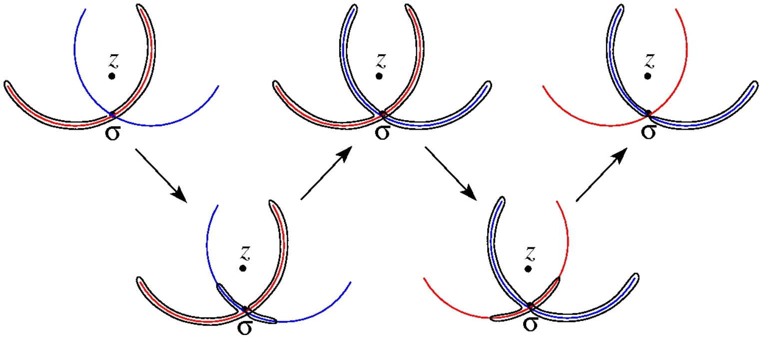

Observation.– The parametric lemma still holds if the parameter space $[a, b]$ is replaced by a compact manifold P.

Relative Parametric Fundamental Lemma

Parametric version of the Fundamental Lemma (continuation). – *Let K be a closed subset of $[a,b]$. If for some open neighborhood $V(K)$ of K we have*

$$\forall p \in V(K), \quad z(p) = \mathfrak{S}(p)$$

then the family of loops $\gamma : [a,b] \times \mathbb{S}^1 \longrightarrow \mathcal{R}$ can be chosen such that

$$\gamma(p,.) \text{ is the constant loop } \mathfrak{S}(p)$$

for all $p \in V_1(K)$ where $V_1(K) \subset V(K)$ is an open neighborhood K.

Summing up...

• If $\mathcal{R}$ is open then the problem of finding a map F solving $\mathcal{R}$ and C^0 close to f_0 can be solved by a convex integration. Indeed

Proposition.– *Let* $f = CI_\gamma(f_0, N)$ *then :*

i) $\forall t \in [0,1], \partial_t f(t) = \gamma(t, Nt) \in \mathcal{R}$

and if γ *satisfy the average condition :*

ii) $\|f - f_0\|_{C^0} = O(\frac{1}{N})$

Corollary.– *If* N *is large enough then* $f = CI_\gamma(f_0, N)$ *is a solution of the above 1D-problem*

Exercise : the case of closed curves

Exercise.– Let $\mathcal{R} \subset \mathbb{R}^n$ be a connected open subset and $f_0 : \mathbb{S}^1 \xrightarrow{C^1} \mathbb{R}^n$ be a closed curve such that

$$\forall t \in \mathbb{S}^1, \quad f_0'(t) \in \ Conv(\mathcal{R}).$$

Find a closed curve $f : \mathbb{S}^1 \xrightarrow{C^1} \mathbb{R}^n$ such that :

$i)$ $\forall t \in \mathbb{S}^1, \quad f'(t) \in \mathcal{R}$
$ii)$ $\|f - f_0\|_{C^0} < \delta$

with $\delta > 0$ given.

Exercise : the case of closed curves (hints)

• Note that, even if f_0 is a closed curve $f_0(0) = f_0(1)$, the map $F = CI_\gamma(f_0, N)$ obtained by a convex integration from f_0 does not satisfy $F(0) = F(1)$ in general.

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• One natural choice for f is to take

$$\forall t \in [0,1], \quad f(t) := F(t) - t\,(F(1) - F(0))\,.$$

Since $f(0) = f(1)$ this defined a closed curve. Observe also that $\gamma(0,.) = \gamma(1,.)$ implies $f'(0) = f'(1)$.

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• From

$$f'(f) = \gamma(t, Nt) - (F(1) - F(0))$$

we deduce

$$\|f'(t) - \gamma(t, Nt)\| = O\left(\frac{1}{N}\right)$$

Since $\mathcal{R}$ is assumed to be open, $f'(t) \in \mathcal{R}$ if N is large enough.

Exercise : the case of closed curves (hints)

• Condition *ii*) will follow from a C^0-density result for the closed curve f.

Proposition (C^0-density).– *If γ is of class C^1 and satisfies the average condition*

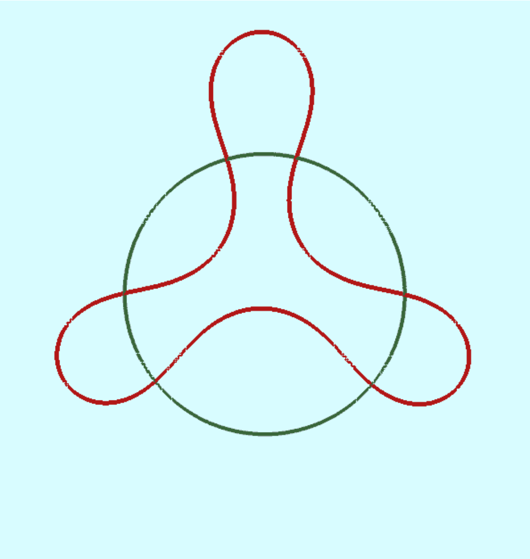
$$\forall u \in \ [0,1], \quad \int_{[0,1]} \gamma(u,s)ds = f_0'(u)$$

then

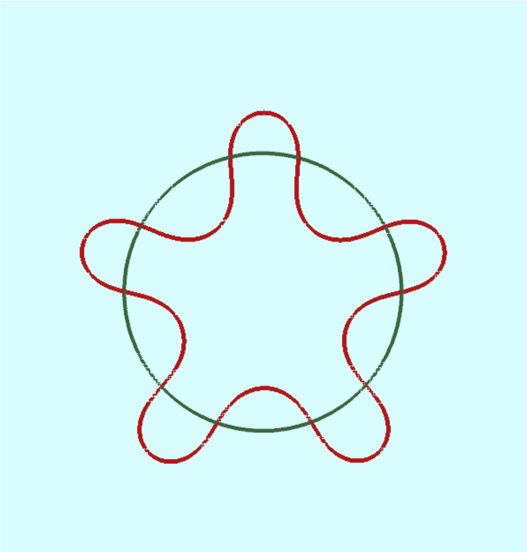
$$\|f - f_0\|_{C^0} \leq \frac{C(\|\gamma\|_{C^0}, \|\partial_1\gamma\|_{C^0})}{N}$$

for some function C (to be determined).

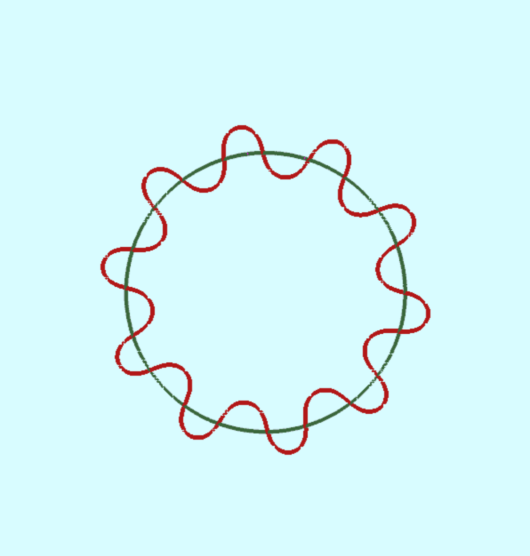
C^0 Density, $N = 3$



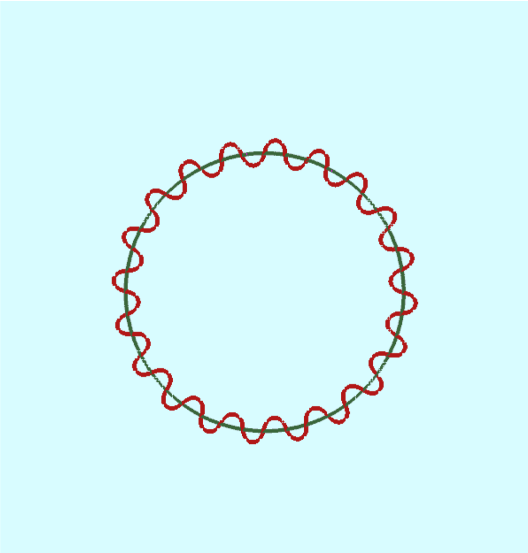
C^0 Density, $N = 5$



C^0 Density, $N = 10$



C^0 Density, $N = 20$

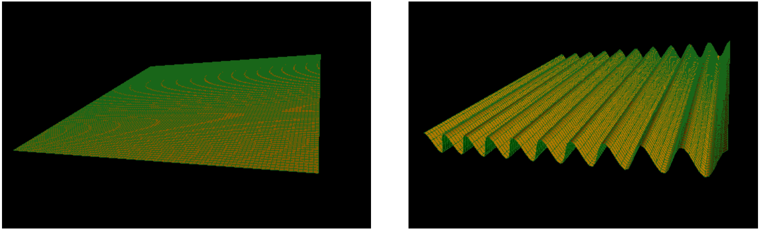


Multi Variables Setting

• In a multi-variable setting, the convex integration formula takes the following form :

$$F(c_1, ..., c_m) := f_0(c_1, ..., c_{m-1}, 0) + \int_0^{c_m} \gamma(c_1, ..., c_{m-1}, s, Ns)ds$$

where $(c_1, ..., c_m) \in [0, 1]^m$.



A corrugated plane

Multi Variables Setting

• The C^0-density property can be enhanced to a $C^{1,\widehat{m}}$-density property where the notation $C^{1,\widehat{m}}$ means that the closeness is measured with the following norm

$$\|F\|_{C^{1,\widehat{m}}} = \max(\|F\|_{C^0}, \|\frac{\partial F}{\partial c_1}\|_{C^0}, ..., \|\frac{\partial F}{\partial c_{m-1}}\|_{C^0}).$$

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Proposition ($C^{1,\widehat{m}}$-density).– *If γ is of class C^1 and satisfies the average condition*

$$\forall u \in [0,1], \quad \int_{[0,1]} \gamma(c_1, ..., c_{m-1}, u, s)ds = f_0'(c_1, ..., c_{m-1}, u)$$

then we have

$$\|F - f_0\|_{C^{1,\widehat{m}}} = O\left(\frac{1}{N}\right).$$

Proof : left as an exercise.

Whitney-Graustein Theorem

Definition.– A C^1 closed curve $f : \mathbb{S}^1 \to \mathbb{R}^2$ is said to be *regular* (or to be an *immersion* of the circle) if for every $t \in \mathbb{S}^1$ we have $f'(t) \neq 0$.

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Definition.– Let $f_0, f_1 : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ be two regular curves. A *regular homotopy* between f_0 and f_1 is a C^1 map

$$\begin{array}{rccc} F: & \mathbb{S}^1 \times [0,1] & \longrightarrow & \mathbb{R}^2 \\ & (x,s) & \longmapsto & F_s(x) = F(x,s) \end{array}$$

such that $F_0 = f_0$, $F_1 = f_1$ and F_s is regular.

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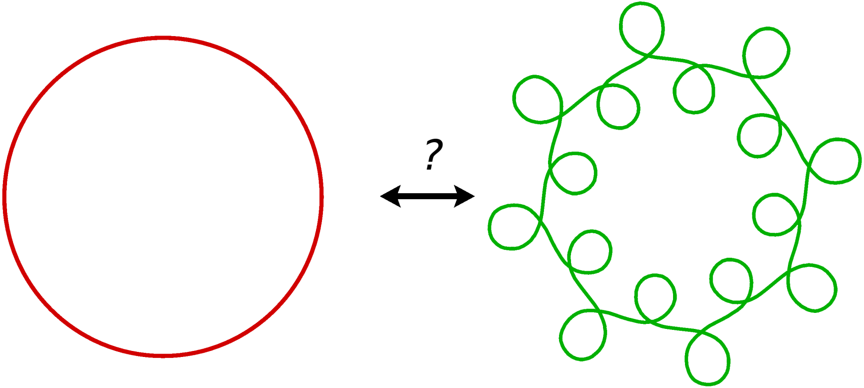
such that $F_0 = f_0$, $F_1 = f_1$ and F_s is regular.

• The relation of regular homotopy is an equivalence relation whose equivalence classes identify with path connected components of the space of immersions $I(\mathbb{S}^1, \mathbb{R}^2)$.

Whitney-Graustein Theorem



Problem.– Classify regular curves up to regular homotopy.

Whitney-Graustein Theorem

We assume that $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} = [0, 1]/\partial[0, 1]$ and $\mathbb{R}^2$ are endowed with an orientation.

Definition.– Let f be a regular closed curve. The *turning number* $TN(f)$ of f is the number of counterclockwise turns of f' around $(0, 0)$.

Whitney-Graustein Theorem

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Definition.– Let f be a regular closed curve. The *turning number* $TN(f)$ of f is the number of counterclockwise turns of f' around $(0,0)$.

• Therefore the turning number of f is given by

$$TN(f) = deg(\mathbf{t}) = \widetilde{\mathbf{t}}(1) - \widetilde{\mathbf{t}}(0) \in \mathbb{Z}$$

where $\widetilde{\mathbf{t}} : [0,1] \longrightarrow \mathbb{R}$ is a lift of the loop

$$\mathbf{t} := \frac{f'}{\|f'\|} : [0,1] \longrightarrow \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}.$$

Whitney-Graustein Theorem

• Recall that

$$\begin{array}{rccc} deg: & \pi_1(\mathbb{S}^1) & \longrightarrow & \mathbb{Z} \\ & [\mathbf{t}] & \longmapsto & deg(\mathbf{t}) \end{array}$$

is a bijection.

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• Any regular homotopy $(f_t)_{t\in[0,1]}$ induces a homotopy of the loops $(\mathbf{t}_t)_{t\in[0,1]}$ in $\mathbb{S}^1$. Thus the turning number

$$t \mapsto TN(f_t)$$

is constant under regular homotopies.

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• It ensues that the turning number induces a map

$$\begin{array}{rccc} TN: & \pi_0(I(\mathbb{S}^1, \mathbb{R}^2)) & \longrightarrow & \mathbb{Z} \\ & [f] & \longmapsto & TN(f). \end{array}$$

Whitney-Graustein Theorem

• As seen in the figures below, this map is onto :

$\cdots$ $TN(\gamma) = -1$ $TN(f) = 0$ $TN(f) = 1$ $TN(f) = 2$ $TN(f) = 3$ $\cdots$

Whitney-Graustein Theorem

- As seen in the figures below, this map is onto :

$TN(\gamma) = -1 \quad TN(f) = 0 \quad TN(f) = 1 \quad TN(f) = 2 \quad TN(f) = 3$

- It turns out that this map is 1-to-1.

Whitney-Graustein Theorem

Hassler Whitney

Whitney-Graustein Theorem (1937). – *The turning number*

$$TN : \pi_0(I(\mathbb{S}^1, \mathbb{R}^2)) \longrightarrow \mathbb{Z}$$

induces a bijective map

Proof of the Whitney-Graustein Theorem

Proof.– It is enough to show the injectivity. Let f_0 and f_1 be two regular closed curves having the same turning number. We consider the linear interpolation between them :

$$f_t := (1-t)f_0 + t\, f_1, \quad t \in \; [0,1]$$

Unless you are extremely lucky, this interpolation will fail to be regular for some t.

Proof of the Whitney-Graustein Theorem

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$$f_t := (1-t)f_0 + t\, f_1, \quad t \in \; [0,1]$$

Unless you are extremely lucky, this interpolation will fail to be regular for some t.

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• Observe that if f_t is singular at some point $x \in \mathbb{S}^1$, i. e ; $f_t'(x) = (0,0)$, we obviously have

$$f_t'(x) \in IntConv(\mathcal{R}) = \mathbb{R}^2$$

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• We use the parametric version of the fundamental lemma with $P = [0,1] \times \mathbb{S}^1$ to build a family of loops $(\gamma_t)_{t\in[0,1]}$ such that for every $p = (t,x) \in P$:

1) the average of the loop $u \mapsto \gamma_t(x,u)$ is $f_t'(x)$ i. e.

$$\int_0^1 \gamma_t(x,u)du = f_t'(x)$$

2) the base point of the loop $u \mapsto \gamma_t(x,u)$ is $\mathfrak{S}_t(x)$.
3) $\gamma_0(x,.) = \mathfrak{S}_0(x)$ and $\gamma_1(x,.) = \mathfrak{S}_1(x)$.

Whitney-Graustein Theorem

• We consider the family of closed curves $g_t : \mathbb{S}^1 \to \mathbb{R}^2$ given by

$$g_t(x) := G_t(x) - x(G_t(1) - G_t(0)) \text{ with } G_t := CI_{\gamma_t}(f_t, N)$$

If N is large enough, g_t is regular for every $t \in [0, 1]$.

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Thus g_t is a regular homotopy joining f_0 and f_1.

□

Whitney-Graustein Theorem

• An observation : assume that we do not use the relative version of the Parametric Fundamental Lemma. Precisely, assume that $u \mapsto \gamma_0(x, u)$ and $u \mapsto \gamma_1(x, u)$ are not constant map. Then the curve g_0 (resp. g_1) is not equal to f_0 (resp. to f_1). An extra regular homotopy is thus needed to join f_0 to g_0 and g_1 to f_1.

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• This extra homotopy is for free by using the fact that each loop is parametrized back and forth. Indeed...

• Let $(g_0^\tau)_{\tau\in\,[0,1]}$ be the homotopy defined by

$$g_0^\tau(x) := G_0^\tau(x) - x(G_0^\tau(1) - G_0^\tau(0)) \;\text{ with }\; G_0^\tau := CI_{\gamma_0^\tau}(f_0, N)$$

where $\tau \mapsto \gamma_0^\tau$ is the retraction of γ_0 to $\gamma_0(0) = \sigma_0(0) = f_0'$ described at the end of the section *Fundamental Lemma*.

Whitney-Graustein Theorem

• For all $x \in \mathbb{S}^1$ we have

$$(g_0^\tau)'(x) := (G_0^\tau)'(x) - (G_0^\tau(1) - G_0^\tau(0))$$

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• Since $\mathcal{R}$ is open, we deduce form the C^0-property that g_0^τ is a regular homotopy joining f_0 to g_0 provided N is large enough.

• Obviously the same process also give a regular homotopy joining f_1 to g_1.

Beyond the Whitney-Graustein Theorem

Definitions.– The subset $\mathcal{R} = \mathbb{R}^2 \setminus \{(0,0\}$ is called the *differential relation* of regular curves.

• The space of all maps $\mathfrak{S} : \mathbb{S}^1 \to \mathcal{R}$ is denoted $\Gamma(\mathcal{R})$.

• A map $\mathfrak{S} \in \Gamma(\mathcal{R})$ is called *holonomic* if there exists $f : \mathbb{S}^1 \to \mathbb{R}^2$ such that $f' = \mathfrak{S}$.

• In that case the map f is called a *solution* of $\mathcal{R}$. The space of all solutions is denoted by $Sol(\mathcal{R})$. Observe that $Sol(\mathcal{R}) = I(\mathbb{S}^1, \mathbb{R}^2)$.

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Whitney-Graustein Theorem (1937). – *The inclusion J induces a bijective map at the π_0-level :*

$$\pi_0(J) : \pi_0(Sol(\mathcal{R})) \longrightarrow \pi_0(\Gamma(\mathcal{R}))$$

Whitney-Graustein Theorem

• In fact more is true. By considering different parametric spaces P in the proof of the Whitney-Graustein Theorem, we can prove the following generalization.

Generalization of the Whitney-Graustein Theorem. – *For every $k \in \mathbb{N}$ the inclusion J induces a bijective map at the π_k-level :*

$$\pi_k(J) : \pi_k(Sol(\mathcal{R})) \longrightarrow \pi_k(\Gamma(\mathcal{R}))$$

In other words, J is a weak homotopy equivalence.

Hassler Whitney