L4 - Gromov Theorem for Ample Relations

Vincent Borrelli

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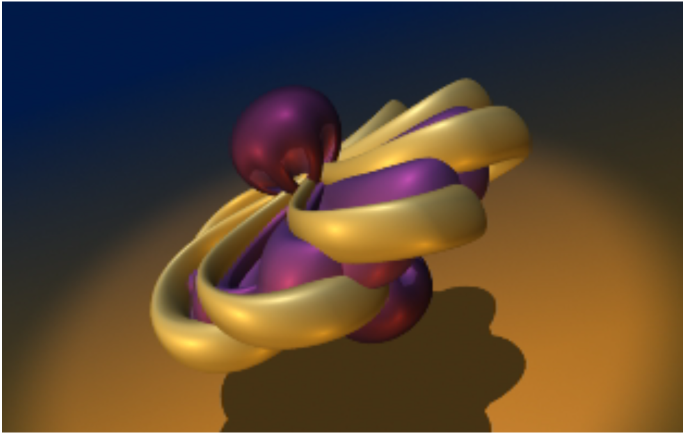
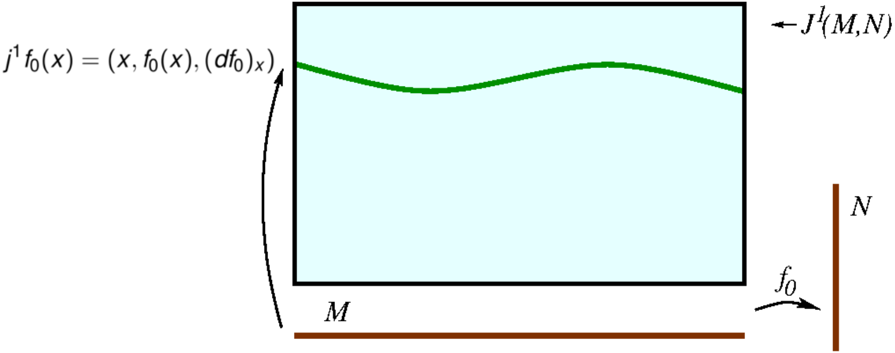


What is the *h*-principle ?



The 1-jet Space.–

$$J^1(M, N) = \{(x, y, L) \mid x \in M, y \in N, L \in \mathcal{L}(T_x M, T_y N)\}.$$

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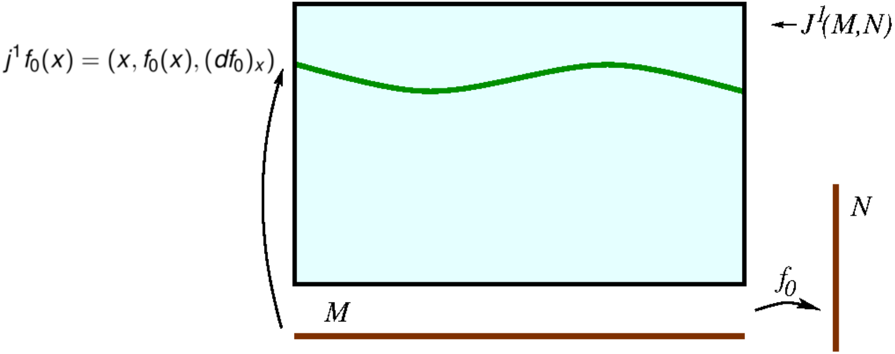

Holonomic section.– Any section $x \mapsto \mathfrak{S}(x) = (x, f_0(x), L(x))$ such that $L(x) = (df_0)_x$, i. e. $\mathfrak{S} = j^1 f_0$.

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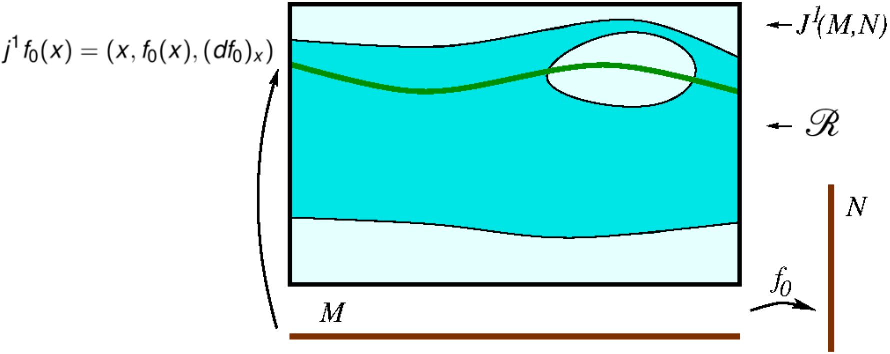

Differential Relation.– Any subset $\mathcal{R}$ of $J^1(M,N)$.

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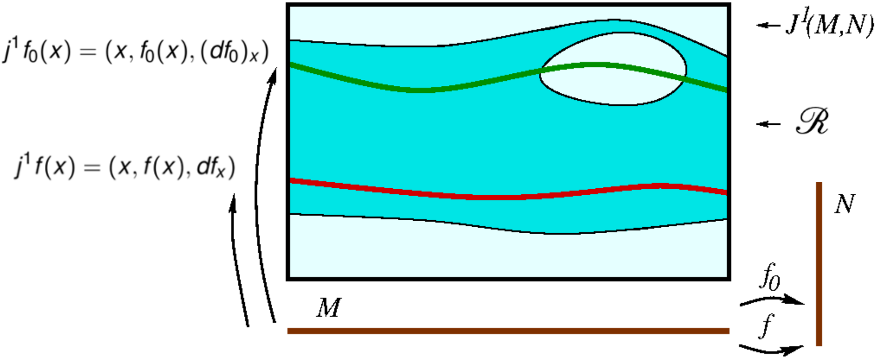

Solution of $\mathcal{R}$.– Any map $f : M \longrightarrow N$ such that $j^1 f(M) \subset \mathcal{R}$. We denote by $\mathcal{S}ol(\mathcal{R})$ the **space of solutions** of $\mathcal{R}$.

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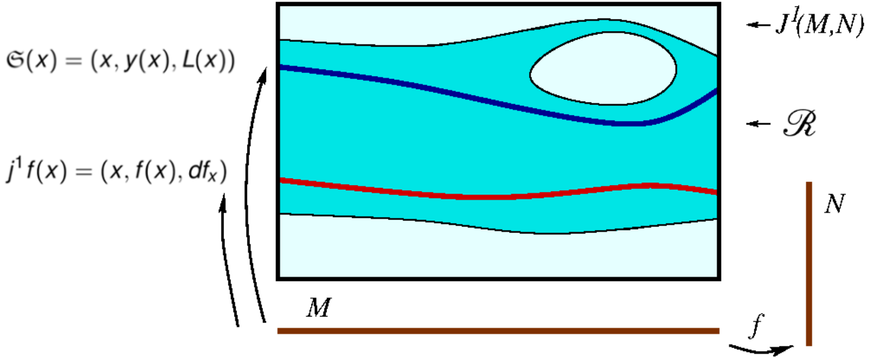

Formal Solution.– Any section $\mathfrak{S} : M \longrightarrow \mathcal{R}$. We denote by $\Gamma(\mathcal{R})$ the **space of formal solutions** of $\mathcal{R}$.

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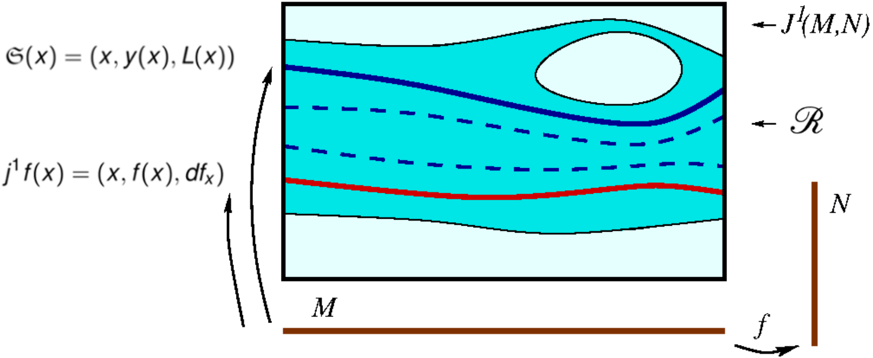

H-Principle.– A differential relation $\mathcal{R}$ satisfies the ***h*-principle** (or **homotopy principle**) if every formal solution $\mathfrak{S} : M \longrightarrow \mathcal{R}$ is homotopic in $\Gamma(\mathcal{R})$ to the 1-jet of a solution of $\mathcal{R}$.

What is the *h*-principle ?

- The natural inclusion

$$J: \quad \begin{array}{ccc} C^1(M,N) & \longrightarrow & J^1(M,N) \\ f & \longmapsto & j^1 f. \end{array}$$

induces a map

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• Note that a differential relation $\mathcal{R}$ satisfies the *h*-principle if and only if the map $\pi_0(J)$ is onto

$$\pi_0(J) : \pi_0(\mathcal{S}ol(\mathcal{R})) \twoheadrightarrow \pi_0(\Gamma(\mathcal{R})).$$

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H-principles

1-parametric *h*-principle.– A differential relation $\mathcal{R}$ satisfies the **1-parametric *h*-principle** it satisfies the *h*-principle and if, for any family of sections $\mathfrak{S}_t \in \Gamma(\mathcal{R})$ such that $\mathfrak{S}_0 = j^1 f_0$ and $\mathfrak{S}_1 = j^1 f_1$, there exists a homotopy $H : [0,1]^2 \to \Gamma(\mathcal{R})$ such that :

$$H(0,t) = \mathfrak{S}_t,\ H(s,0) = \mathfrak{S}_0,\ H(s,1) = \mathfrak{S}_1,\ \text{et } H(1,t) = j^1 f_t.$$

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• Thus, a differential relation $\mathcal{R}$ satisfies the 1-parametric *h*-principle if and only if

$$\pi_0(J) : \pi_0(\mathcal{S}ol(\mathcal{R})) \longrightarrow \pi_0(\Gamma(\mathcal{R})).$$

is a bijective map.

Homotopy Equivalence

Definition.– Let X and Y be two topological spaces. A map $f : X \longrightarrow Y$ is a *homotopy equivalence* if there exists

$$g : Y \longrightarrow X$$

such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X.

• In other words, X and Y are homotopically indistinguishable.

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- Example : $X = \mathbb{S}^{n-1}$ and $Y = \mathbb{R}^n \setminus \{*\}$
- Example : $X = \{x_1, x_2\}$ and $\mathbb{R}^n \setminus H$ where H is a hyperplane.

H-principles

Definition.– A map $f : X \longrightarrow Y$ is a *weak homotopy equivalence* if the map

$$\pi_0(f) : \pi_0(X) \to \pi_0(Y)$$

is bijective and if, for every $k \in \mathbb{N}^*$ and for every $x \in X$, the map f induces an isomorphism

$$\pi_k(f) : \pi_k(X, x) \simeq \pi_k(Y, f(x)).$$

• If X is path-connected then first condition is automatic, and it suffices to state the second condition for a single point x in X.

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• If $f : X \mapsto Y$ is a homotopy equivalence then it is a weak homotopy equivalence.

H-principles

Parametric *h*-principle.– A differential relation $\mathcal{R}$ satisfies **the parametric *h*-principle** if the map

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• It turns out that several differential relations arising from differential geometry satisfy the parametric *h*-principle.

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• An infinite dimensional version of the Whitehead Theorem states that any weak homotopy equivalence between two Fréchet metrizable manifolds is a homotopy equivalence.

• Recall that a Fréchet space is a complete topological vector space which is separated (=is a Hausdorff space) and whose topology is induced by a countable family of seminormes $|.|_n$. Such a space is metrizable by setting $d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x-y|_n}{1+|x-y|_n}$.

• The spaces $\mathcal{S}ol(\mathcal{R})$ and $\Gamma(\mathcal{R})$ are Fréchet metrizable. Consequently, the parametric h-principle for $\mathcal{R}$ implies that $J : \mathcal{S}ol(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$ is a **homotopy equivalence**.

Examples of relations satisfying the *h*-principle

- The Whitney-Graustein Theorem (1937) shows that the relation

$$\mathcal{R} = \{(x, y, v) \in \mathbb{S}^1 \times \mathbb{R}^2 \times \mathbb{R}^2 \mid v \neq (0, 0)\}$$

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• If f is an immersion then $df_p(T_pM)$ is a n-dimensional subspace of $T_{f(p)}N$. The image $f(M)$ has no crease or tip.

Examples of relations satisfying the *h*-principle

• The space of immersions from M to $\mathbb{R}^n$ is denoted by $I(M, \mathbb{R}^n)$.

• Let $(K_n)_{n\in\mathbb{N}}$ be a countable family of compact sets covering M. For every $n \in \mathbb{N}$, we define

$$d_n(f,g) := \sup_{x\in K_n} \|f(x) - g(x)\| + \sup_{x\in K_n} \|df_x - dg_x\|$$

and we endow $I(M, \mathbb{R}^n)$ with the distance

$$d(f,g) := \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n(f,g)}{1 + d_n(f,g)}.$$

Smale Theorem on Sphere Immersions

Stephen Smale

Smale Theorem (1957). – *Let $m < n$. The relation*

$$\mathcal{R} = \{(x, y, L) \in J^1(\mathbb{S}^m, \mathbb{R}^n) \mid \textit{rank } L = m\}$$

of immersions of $\mathbb{S}^m$ into $\mathbb{R}^n$ satisfies the 1-parametric h-principle.

Smale Theorem on Sphere Immersions

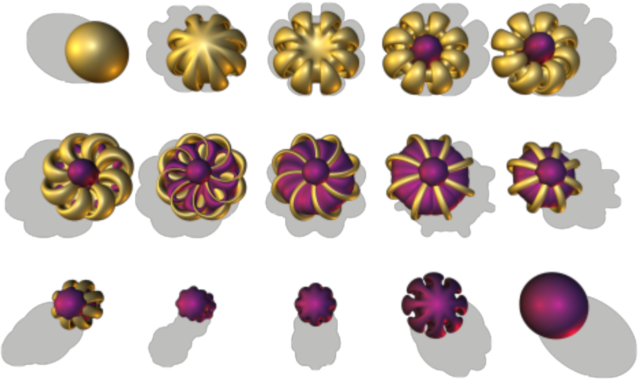

Image : The Geometric Center

Corollary (Smale 1957).– *The space $I(\mathbb{S}^2, \mathbb{R}^3)$ is path-connected. In particular, it is possible to realize an eversion of the 2-sphere among immersions.*

Proof of the Sphere Eversion

Proof of the corollary.– Since $\mathcal{R}$ satisfies the 1-parametric *h*-principle, the map

$$J : \pi_0(I(\mathbb{S}^2, \mathbb{R}^3)) \mapsto \pi_0(\Gamma(\mathcal{R}))$$

is a 1-to-1. The proof of the corollary thus reduces to the computation of $\pi_0(\Gamma(\mathcal{R}))$ with

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- It turns out that $\pi_2(GL_+(\mathbb{R}^3)) = \{0\}$. □

Hirsch Theorem on Immersions

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Hirsch Theorem (1959). – *Let M^m and N^n be two manifolds with $m < n$. The relation of immersions of M^m into N^n :*

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satisfies the parametric h-principle. Precisely, the map

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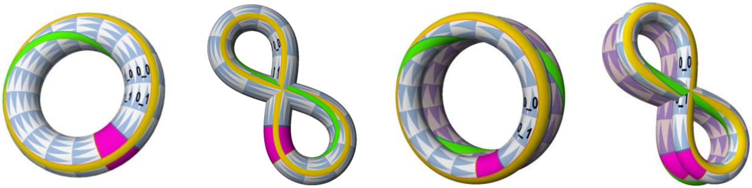
• Recall that an **open manifold** is a manifold without boundary and with no compact component.

Exercice : Immersions of the 2-Torus



Exercice.– Apply the Hirsch Theorem to show that Card $\pi_0(I(\mathbb{T}^2, \mathbb{R}^3)) = 4$.

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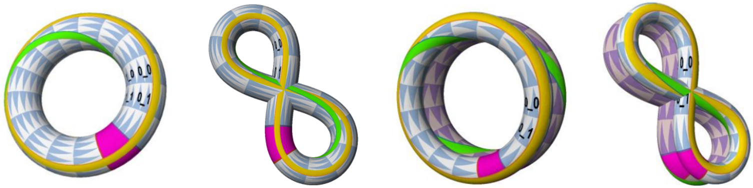


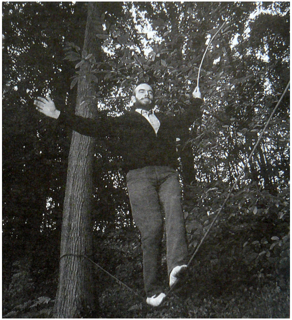
Exercice.– Apply the Hirsch Theorem to show that
Card $\pi_0(I(\mathbb{T}^2, \mathbb{R}^3)) = 4$.

• We recall that $\pi_1(GL_+(\mathbb{R}^3)) = \mathbb{Z}/2\mathbb{Z}$ and we admit that the space $C^0(\mathbb{T}^2, GL_+(\mathbb{R}^3))$ has four components. Precisely,

$$\begin{array}{rccc} \Phi: & \pi_0(C^0(\mathbb{T}^2, GL_+(\mathbb{R}^3))) & \longrightarrow & \pi_1(GL_+(\mathbb{R}^3)) \times \pi_1(GL_+(\mathbb{R}^3)) \\ & [f] & \longmapsto & [f_{|\mathbb{S}^1\times\{*\}}] \times [f_{|\{*\}\times\mathbb{S}^1}] \end{array}$$

is a bijective map.

Isometric Immersions



Mikhail Gromov

Theorem (Nash 1954 - Kuiper 1955 -Gromov 1986). – *Let* (M^m, g) *and* (N^n, h) *be two Riemannian manifold with* $m < n$. *The relation of isometric immersions of* M^m *into* N^n *:*

$$\mathcal{R} = \{(x, y, L) \in J^1(M^m, N^n) \mid L^*h = g\}$$

satisfies the parametric h-principle. The weak homotopy equivalence is given by the map $J : f \longmapsto j^1 f$.

Isometric Immersions

Corollary (Gromov 1986). – *There exists a C^1 isometric eversion of the 2-sphere.*

More examples of relations satisfying the *h*-principle...

... with Jean-Claude Sikorav in the second part of this course.

The *h*-Principe for Ample Relations

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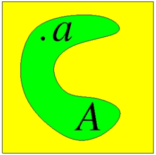
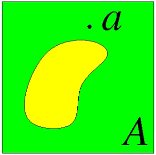
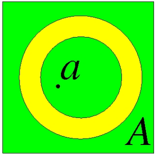
• It remains to define what is an **ample** differential relation and to give a (sketch of the) proof of this theorem.

Ample Relations

Definition.– A subset $A \subset \mathbb{R}^n$ is *ample* if for every $a \in A$ the interior of the convex hull of the connected component to which a belongs is $\mathbb{R}^n$ *i. e.* : $IntConv(A, a) = \mathbb{R}^n$ (in particular $A = \emptyset$ is ample).







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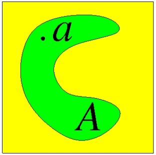

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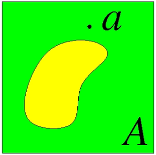

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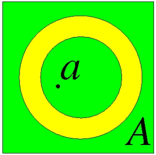

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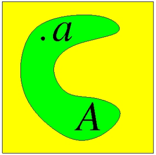

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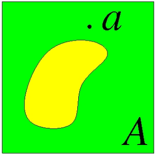

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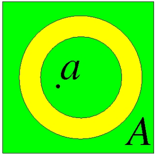

A is not ample.

Example.– The complement of a linear subspace $F \subset \mathbb{R}^n$ is ample if and only if Codim $F \geq 2$.

Ample Relations

Definition.– Let $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a fiber bundle, a subset $\mathcal{R} \subset E$ is said to be *ample* if, for every $p \in P$, $\mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R}$ is ample in $\mathbb{R}^n$.

Remark.– If $\mathcal{R} \subset E$ is ample and $z : P \longrightarrow E$ is a section, then, for every $p \in P$, we have $z(p) \in Conv(\mathcal{R}_p, \sigma(p))$.

Ample Relations in $J^1(M, N)$

- Locally, we identify $J^1(M, N)$ with

$$\begin{array}{rcl} J^1(\mathcal{U}, \mathcal{V}) & = & \mathcal{U} \times \mathcal{V} \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) = \mathcal{U} \times \mathcal{V} \times \prod_{i=1}^m \mathbb{R}^n. \\ & = & \{(x, y, v_1, ..., v_m)\} \end{array}$$

where $\mathcal{U}$ and $\mathcal{V}$ are charts of M and N.

- We set :

$$J^1(\mathcal{U}, \mathcal{V})^\perp := \{(x, y, v_1, ..., v_{m-1})\}.$$

- We have

$$\begin{array}{ccc} \mathcal{R}_{\mathcal{U},\mathcal{V}} & \longrightarrow & J^1(\mathcal{U}, \mathcal{V}) \\ & & \downarrow p^\perp \\ & & J^1(\mathcal{U}, \mathcal{V})^\perp. \end{array}$$

Ample Relations in $J^1(M, N)$

- Let $z \in J^1(\mathcal{U}, \mathcal{V})^\perp$, we set

$$\mathcal{R}_z^\perp = (p^\perp)^{-1}(z) \cap \mathcal{R}_{\mathcal{U},\mathcal{V}}.$$

- $\mathcal{R}^\perp$ is a differential relation of the bundle

$$J^1(\mathcal{U}, \mathcal{V}) \xrightarrow{p^\perp} J^1(\mathcal{U}, \mathcal{V})^\perp.$$

Definition. – A differential relation $\mathcal{R} \subset J^1(M, N)$ is *ample* if for every local identification $J^1(\mathcal{U}, \mathcal{V})$ and for every $z \in J^1(\mathcal{U}, \mathcal{V})^\perp$, the space $\mathcal{R}_z^\perp$ is ample in $(p^\perp)^{-1}(z) \simeq \mathbb{R}^n$.

Ample Relations in $J^1(M, N)$

Proposition. – *The differential relation $\mathcal{R}$ of immersions of M^m into N^n is ample if $n > m$.*

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Proof.– Let $J^1(\mathcal{U}, \mathcal{V})$ be any local identification and let $z = (x, y, v_1, ..., v_{m-1}) \in J^1(\mathcal{U}, \mathcal{V})^\perp$. We have

$$(p^\perp)^{-1}(z) \cap \mathcal{R} \simeq \{v_m \in \mathbb{R}^n \mid \{v_1, ..., v_m\} \text{ are linearly independent in } \mathbb{R}^n\}$$

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• If $\{v_1, ..., v_{m-1}\}$ are linearly independent then

$$\begin{aligned} v_m \in (p^\perp)^{-1}(z) \text{ lies inside } \mathcal{R}_{\mathcal{U},\mathcal{V}} \quad &\iff \quad v_m \notin Span(v_1, ..., v_{m-1}) =: \Pi \\ &\iff \quad v_m \in \mathbb{R}^n \setminus \Pi. \end{aligned}$$

Therefore $\mathcal{R}_z^\perp = \mathcal{R}_{\mathcal{U},\mathcal{V}} \cap (p^\perp)^{-1}(z) = \mathbb{R}^n \setminus \Pi$. Since the codimension of Π is $n - (m - 1) \geq 2$, it ensues that $\mathcal{R}_p^\perp$ is ample.

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• If $\{v_1, ..., v_{m-1}\}$ are linearly dependent then$\mathcal{R}_p^\perp = \emptyset$ and thus $\mathcal{R}_p^\perp$ is ample. $\square$

Sketch of the Proof of Gromov Theorem

• We first work locally over a cubic chart $C = [0,1]^m$ of M and an open $\mathcal{V} \approx \mathbb{R}^n$ of N.

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• Let $\mathfrak{S} \in \Gamma(\mathcal{R}_{C,\mathbb{R}^n})$ be a section :

$$\mathfrak{S} : c \longmapsto (c, f_0(c), v_1(c), ..., v_m(c)) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

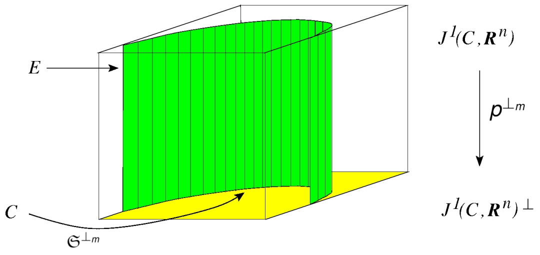
and let $p^{\perp m}$ be the projection

$$(c, y, v_1, ..., v_m) \longmapsto (c, y, v_1, ..., v_{m-1})$$

and

$$\mathcal{R}_z^{\perp m} := \mathcal{R}_{C,\mathbb{R}^n} \cap (p^{\perp m})^{-1}(z)$$

Sketch of the Proof of Gromov Theorem



- We set

$$\begin{array}{rccc} \mathfrak{S}^{\perp m}: & C & \longrightarrow & J^1(C, \mathbb{R}^n)^{\perp m} \\ & c & \longmapsto & (c, f_0(c), v_1(c), ..., v_{m-1}(c)) \end{array}$$

and we denote by E the pull-back bundle :

$$\begin{array}{ccc} E & \longrightarrow & J^1(C, \mathbb{R}^n) \\ \pi \downarrow & & \downarrow p^{\perp m} \\ C & \stackrel{\mathfrak{S}^{\perp m}}{\longrightarrow} & J^1(C, \mathbb{R}^n)^{\perp m} \end{array}$$

Sketch of the Proof of Gromov Theorem

• Let $\mathcal{S}^m \subset E$ be the pull-back of the relation $\mathcal{R}^{\perp_m}$. The relation $\mathcal{S}^m$ is obviously open and ample and $v_m : C \longrightarrow \mathbb{R}^n$ provides a section of $\mathcal{S}^m$ over C.

• We use the parametric version of the Fundamental Lemma with $C := [0,1]^m$ as parameter space and with $\mathcal{S}^m$ as differential relation. There exists $\gamma : C \times [0,1] \longrightarrow \mathcal{S}^m$ such that

$$\gamma(.,0) = \gamma(.,1) = v_m \in \Gamma(\mathcal{S}^m)$$

and

$$\forall c \in C, \quad \gamma(c,.) \in Concat(\Omega^{BF}_{v_m(c)}(\mathcal{S}^m_c))$$

and

$$\forall c \in C, \int_0^1 \gamma(c,s)ds = \frac{\partial f_0}{\partial c_m}(c).$$

Sketch of the Proof of Gromov Theorem

- We set

$$F_1(c) := f_0(c_1, ..., c_{m-1}, 0) + \int_0^{c_m} \gamma(c_1, ..., c_{m-1}, s, N_1 s) ds.$$

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- We then have

$$\|F_1 - f_0\| = O(\frac{1}{N_1})$$

and even more,

$$\|F_1 - f_0\|_{C^{1,\widehat{m}}} = O(\frac{1}{N_1})$$

where

$$\|f\|_{C^{1,\widehat{m}}} = \max(\|f\|_{C^0}, \|\frac{\partial f}{\partial c_1}\|_{C^0}, ..., \|\frac{\partial f}{\partial c_{m-1}}\|_{C^0})$$

is the C^1 norm without the $\|\frac{\partial f}{\partial c_m}\|_{C^0}$ term.

Sketch of the Proof of Gromov Theorem

- By the very definition of $\mathcal{S}^m$, the section

$$c \mapsto (c, f_0(c), v_1(c), ..., v_{m-1}(c), \frac{\partial F_1}{\partial c_m}(c))$$

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• Since $\mathcal{R}_{C,\mathbb{R}^n}$ is open and F_1 is C^0-close to f_0, even if it means to increase N_1, we can assume that

$$c \mapsto (c, F_1(c), v_1(c), ..., v_{m-1}(c), \frac{\partial F_1}{\partial c_m}(c))$$

is a section of $\mathcal{R}_{C,\mathbb{R}^n}$.

Sketch of the Proof of Gromov Theorem

• We then repeat the same process with respect to the variable c_{m-1} to obtain

$$c \mapsto (c, F_1(c), v_1(c), ..., v_{m-2}(c), \frac{\partial F_2}{\partial c_{m-1}}(c), \frac{\partial F_1}{\partial c_m}(c)) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

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• Noticing that $\mathcal{R}_{C,\mathbb{R}^n}$ is open and that F_2 and F_1 are $C^{1,\widehat{c_{m-1}}}$-close, we have if N_2 is large enough :

$$c \mapsto \left(c, F_2(c), v_1(c), ..., v_{m-2}(c), \frac{\partial F_2}{\partial c_{m-1}}(c), \frac{\partial F_2}{\partial c_m}(c)\right) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

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• Iterating over the other variables $v_1, ..., v_{m-2}$ we eventually obtain a holonomic section over C. Moreover $F := F_m$ and f_0 are C^0-close :

$$\|F - f_0\|_{C^0} = O(\frac{1}{N_1} + ... + \frac{1}{N_m}).$$

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• The real problem is the matching the solutions together. Precisely if C is an open cube, K a compact subset of C and f_0 a solution over an open neighborhood $Op(K)$ of K, the point is to construct a solution f such that $f = f_0$ on some $Op_2(K) \subset Op(K)$.

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• To achieve this goal, we need to modify every convex integrations defining $F_1, ..., F_m$. Let $\lambda_1 : C \longrightarrow [0, 1]$ be a compactly supported C^∞ function such that

$$\lambda_1(c) = \begin{cases} 1 & \text{if } c \in Op_2(K) \\ 0 & \text{if } c \in C \setminus Op_1(K). \end{cases}$$

where $Op_2(K) \subset Op_1(K) \subset Op(K)$.

Sketch of the Proof of Gromov Theorem

- Let F_1 be the preceding solution over C obtained from the section

$$\mathfrak{S} : c \longmapsto (c, f_0(c), v_1(c), ..., v_m(c)) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

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$$f_1 := F_1 + \lambda_1(f_0 - F_1).$$

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We set

$$f_1 := F_1 + \lambda_1(f_0 - F_1).$$

• Let $j \in \{1, ..., m\}$, we have

$$\frac{\partial f_1}{\partial c_j} = \frac{\partial F_1}{\partial c_j} + \lambda_1.\left(\frac{\partial f_0}{\partial c_j} - \frac{\partial F_1}{\partial c_j}\right) + \frac{\partial \lambda_1}{\partial c_j}.(f_0 - F_1).$$

Since λ_1 is compactly supported, the $\dfrac{\partial \lambda_1}{\partial c_j}$'s are bounded for every $j \in \{1, ..., m\}$.

Sketch of the Proof of Gromov Theorem

- Let $j \in \{1, ..., m-1\}$. Since F_1 and f_0 are $(C^1, \widehat{m})$-close, we have

$$\|\frac{\partial f_1}{\partial c_j} - \frac{\partial F_1}{\partial c_j}\|_{C^0} = O(\frac{1}{N_1}).$$

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- Let $j = m$. In general,

$$\frac{\partial f_1}{\partial c_m} - \frac{\partial F_1}{\partial c_m}$$

is not small and therefore

$$c \longmapsto \left(c, \frac{\partial f_1}{\partial c_m}(c)\right)$$

should not be a section of $\mathcal{S}^m$.

Sketch of the Proof of Gromov Theorem

• Since λ_1 is 0 over $C \setminus Op_1(K)$, for every $c \in C \setminus Op_1(K)$, we have $F_1 = f_1$ and thus

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• Over $Op(K)$, we admit that it is possible to choose the family of loops $\gamma : C \times [0,1] \to \mathcal{S}^m$ such that, for all $c \in Op_1(K)$, we have

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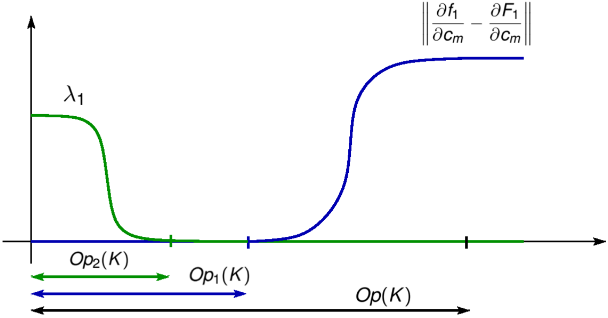
• Thus, for all $c \in Op_1(K)$ we have

$$\frac{\partial F_1}{\partial c_m}(c) = \gamma(c_1, ..., c_{m-1}, c_m, N_1 c_m) = \frac{\partial f_0}{\partial c_m}(c)$$

and the difference $\dfrac{\partial f_0}{\partial c_m} - \dfrac{\partial F_1}{\partial c_m}$ vanishes over $Op_1(K)$.

Sketch of the Proof of Gromov Theorem



- It follows that

$$\lambda_1(c)\left(\frac{\partial f_1}{\partial c_m}(c) - \frac{\partial F_1}{\partial c_m}(c)\right)$$

vanishes for all $c \in Op(K)$ and thus

$$\mathfrak{S}_1 : c \longmapsto (c, f_1(c), v_1(c), ..., v_{m-1}(c), \frac{\partial f_1}{\partial c_m}(c)) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

□

Morris Hirsch