L5 - Constructions of C^1-isometric maps

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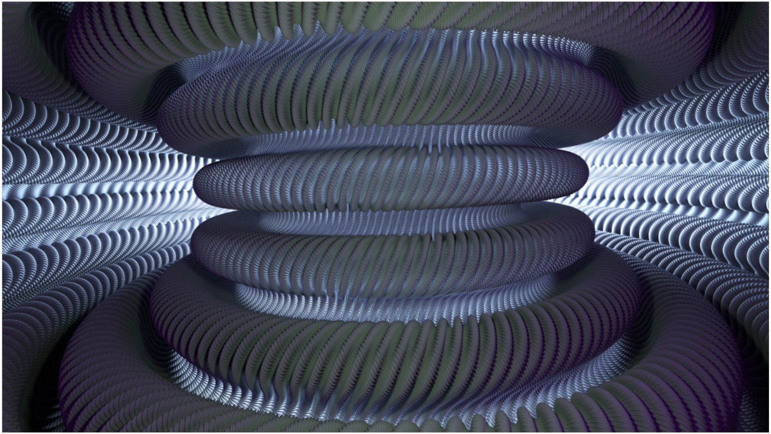

The Hevea Project

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• However a question remained open : what is the geometry of Nash-Kuiper isometric maps ?

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• However a question remained open : what is the geometry of Nash-Kuiper isometric maps ?

• During the period 2007-2012, a multidisciplinary team (the *Hevea Project*) used the Gromov Convex Integration Theory to explicitly construct a C^1 isometric embedding of a Flat Torus inside $\mathbb{E}^3$ in order to answer to this question.

Construction of an Isometric Embedding of the Flat Torus

Definition.– A (two dimensional) *Flat Torus* is the quotient of $\mathbb{E}^2$ by a lattice $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where $\{e_1, e_2\}$ is a basis of $\mathbb{E}^2$. We denote by $\mathbb{T}_\Lambda$ the flat torus $\mathbb{E}^2/\Lambda$.

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• Two flat tori $\mathbb{T}_{\Lambda_1}$ and $\mathbb{T}_{\Lambda_2}$ are isometric if and only if there exists an isometry of $\mathbb{E}^2$ which sends the lattice Λ_1 on the lattice Λ_2.

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• We denote by $Iso(T_\Lambda)$ the isometry group of T_Λ and $Iso_O(\mathbb{T}_\Lambda)$ its isotropy group at the origine O, *i. e.* the subgroup of isometry Φ of $\mathbb{T}_\Lambda$ such that $\Phi(O) = O$. We have the following exact sequence

$$0 \longrightarrow Iso_O(\mathbb{T}_\Lambda) \longrightarrow Iso(\mathbb{T}_\Lambda) \longrightarrow \mathbb{T}_\Lambda \longrightarrow 0$$

where $Iso(\mathbb{T}_\Lambda) \longrightarrow \mathbb{T}_\Lambda$ is $\Phi \longmapsto \Phi(O)$.

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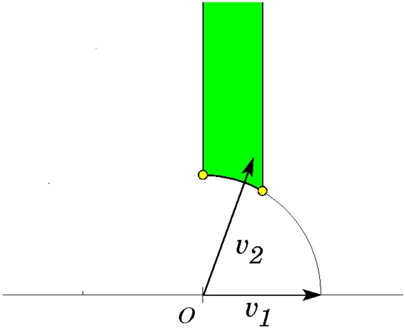
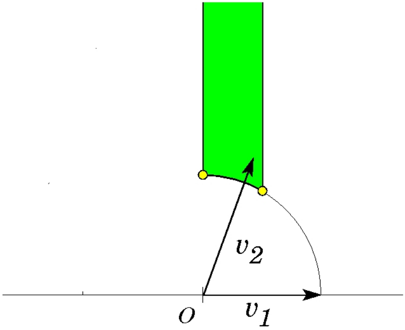


• Let v_1 be a shortest vector of $\Lambda \setminus \{O\}$ and v_2 be a shortest vector of $\Lambda \setminus \mathbb{Z}v_1$. It is easy to see that v_1 and v_2 spans Λ.

Construction of an Isometric Embedding of the Flat Torus



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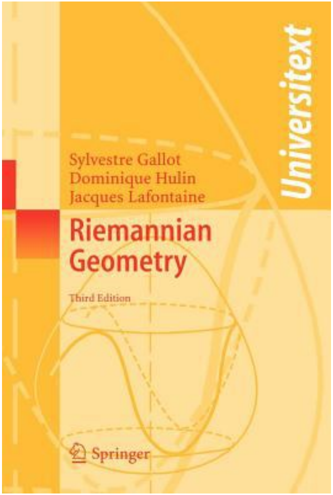
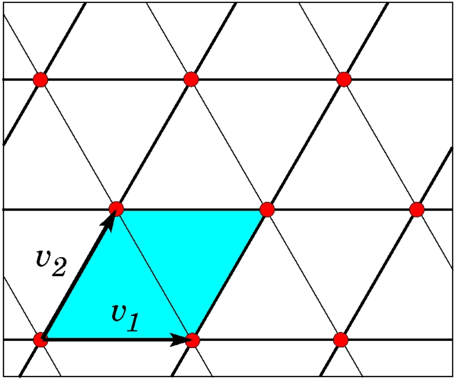
• The green domain is the space of similarity classes of flat tori (=classification of flat tori up to isometries and homotheties).

Construction of an Isometric Embedding of the Flat Torus



See p. 60 to 63 for more details

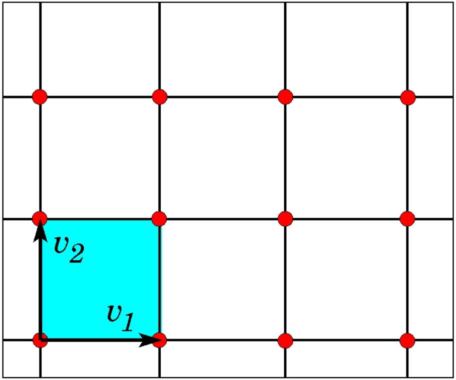
Construction of an Isometric Embedding of the Flat Torus



A hexagonal lattice

If Λ is a **hexagonal lattice** then $Iso_O(\mathbb{T}_\Lambda)$ is generated by a rotation of angle $\frac{\pi}{3}$ and a reflection r along v_1. It has 12 elements.

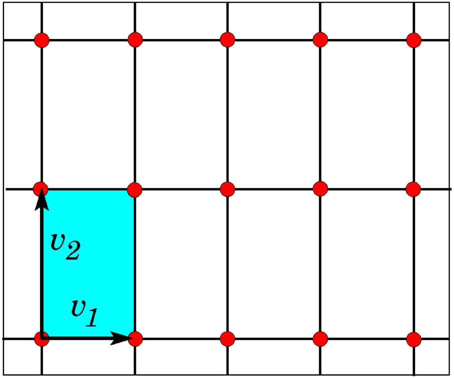
Construction of an Isometric Embedding of the Flat Torus



A square lattice

If Λ is a **square lattice** then $Iso_O(\mathbb{T}_\Lambda)$ is generated by a rotation of angle $\frac{\pi}{2}$ and r. We thus have $|Iso_O(\mathbb{T}_\Lambda)| = 8$.

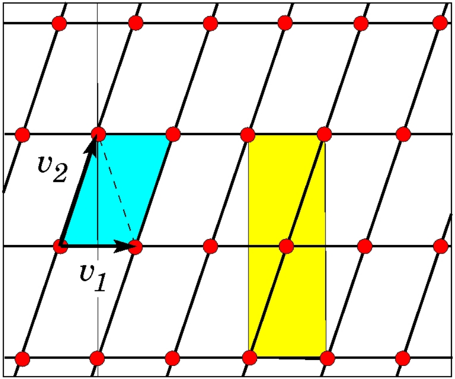
Construction of an Isometric Embedding of the Flat Torus



A rectangular lattice

If Λ is a (proper) **rectangular lattice** then $Iso_O(\mathbb{T}_\Lambda)$ is generated by $-Id$ and r and $|Iso_O(\mathbb{T}_\Lambda)| = 4$.

Construction of an Isometric Embedding of the Flat Torus



A centred rectangular lattice : orthogonal projection of v_2 is $\frac{1}{2}v_1$. This implies that the yellowed region is a rectangle with as center a vertex of of the lattice, hence the name.

If Λ is a (proper) **centred rectangular lattice** then $Iso_O(\mathbb{T}_\Lambda)$ is generated by $-Id$ and r and $|Iso_O(\mathbb{T}_\Lambda)| = 4$. In particular, the isotropy group is the same as the one of a rectangular lattice.

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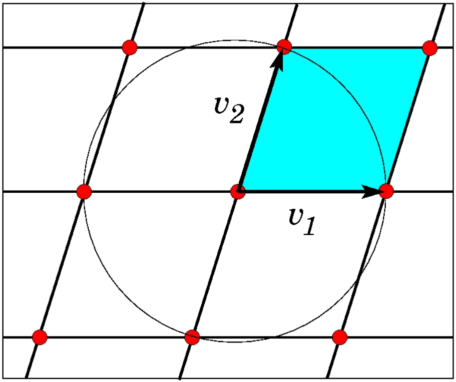
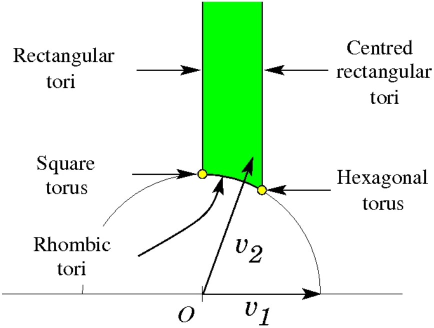


A rhombic lattice : the length of v_1 is the one of v_2 hence the name.

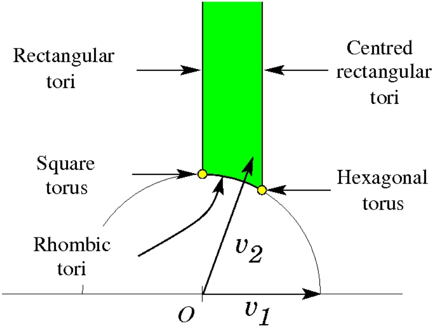
If Λ is a (proper) **rhombic lattice** then $Iso_O(\mathbb{T}_\Lambda)$ is generated by $-Id$ and a reflection s along $v_1 + v_2$ and $|Iso_O(\mathbb{T}_\Lambda)| = 4$.

Construction of an Isometric Embedding of the Flat Torus



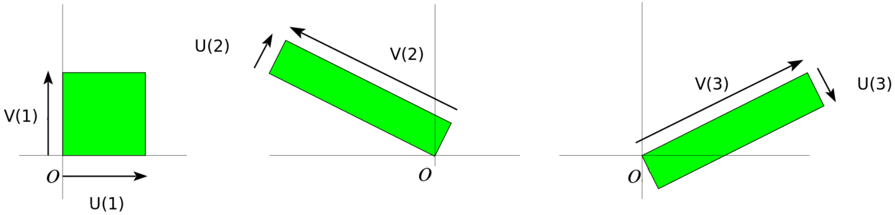
• In all the other cases, that is, when Λ is a (proper) **parallelogrammic lattice**, then the triangle defined by O, $O+v_1$ and $O+v_2$ is scalene and $Iso_O(\mathbb{T}_\Lambda)=\{Id,-Id\}$. In particular, there is no isometry that reverses the orientation.

Construction of an Isometric Embedding of the Flat Torus



• The Hexagonal torus has the largest isotropy group amongs all flat tori, the Square torus has the second largest isotropy group. We are going to build an explicit isometric embedding of the **square flat torus**.

Construction of an Isometric Embedding of the Flat Torus



Three fundamental domains Dom_i, $i \in \{1, 2, 3\}$, of the same square flat torus $\mathbb{T}^2$.

Exercise.– Let (e_1, e_2) be an orthonormal basis of $\mathbb{E}^2$. Show that the quotients $\mathbb{E}^2/\mathbb{Z}U(i) \oplus \mathbb{Z}V(i)$, $i \in \{1, 2, 3\}$ with

$$\begin{array}{lll} U(1) = e_1 & U(2) = \frac{1}{5}(e_1 + 2e_2) & U(3) = \frac{1}{5}(e_1 - 2e_2) \\ V(1) = e_2 & V(2) = -2e_1 + e_2 & V(3) = 2e_1 + e_2 \end{array}$$

define the same square flat torus (up to isometries).

Construction of an Isometric Embedding of the Flat Torus

The initial map.– Let $R > r > 0$. We define

$$
\begin{array}{rccc}
f_0: & \mathbb{T}^2 = \mathbb{E}^2/\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 & \longrightarrow & \mathbb{E}^3 \\
 & (u,v) & \longmapsto & \left\{ \begin{array}{l} \dfrac{1}{2\pi}(R + r\cos 2\pi u)\cos 2\pi v \\ \dfrac{1}{2\pi}(R + r\cos 2\pi u)\sin 2\pi v \\ \dfrac{r}{2\pi}\sin 2\pi u \end{array} \right.
\end{array}
$$

Construction of an Isometric Embedding of the Flat Torus

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$$(u,v) \quad \longmapsto \quad \begin{cases} \dfrac{1}{2\pi}(R + r\cos 2\pi u)\cos 2\pi v \\ \dfrac{1}{2\pi}(R + r\cos 2\pi u)\sin 2\pi v \\ \dfrac{r}{2\pi}\sin 2\pi u \end{cases}$$

- A straightforward computation shows that

$$f_0^*\langle .,.\rangle_{\mathbb{E}^3} = r^2 du^2 + (R + r\cos 2\pi u)^2 dv^2.$$

Therefore $f_0^*\langle .,.\rangle_{\mathbb{E}^3} < du^2 + dv^2$ iff $R + r < 1$.

Construction of an Isometric Embedding of the Flat Torus

• We choose R and r such that $R + r < 1$. This implies that f_0 is a strictly short embedding.

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• Let

$$\Delta := \langle .,. \rangle_{\mathbb{E}^2} - f_0^* \langle .,. \rangle_{\mathbb{E}^3}$$

be the isometric default of f_0.

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be the isometric default of f_0.

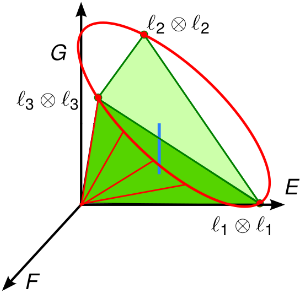
• The image of

$$\begin{array}{rccc} \Delta : & \mathbb{T}^2 & \longrightarrow & \mathcal{S}_2^+(\mathbb{R}^2) \\ & (u,v) & \longmapsto & (1-r^2)du^2 + (1-R-r\cos 2\pi u)^2 dv^2 \end{array}$$

is a segment lying inside the positive cone of inner products of $\mathbb{R}^2$:

$$\mathcal{S}_2^+(\mathbb{R}^2) = \{Edu^2 + 2Fdudv + Gdv^2 \mid E > 0, EG - F^2 > 0\} \subset \mathbb{R}^3.$$

Construction of an Isometric Embedding of the Flat Torus



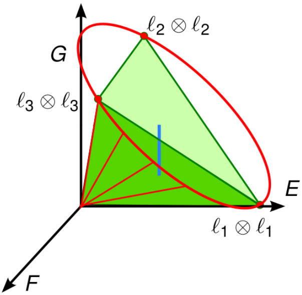
• Let ℓ_1, ℓ_2 and ℓ_3 be the three linear forms of $\mathbb{E}^2$ defined by

$$\forall\, i \in \{1,2,3\}, \quad \ell_i(.) = \langle \frac{U(i)}{\|U(i)\|_{\mathbb{E}^2}}, . \rangle_{\mathbb{E}_2}$$

that is

$$\ell_1 = du, \ \ \ell_2 = \frac{1}{\sqrt{5}}(du + 2dv) \ \text{ and } \ \ell_2 = \frac{1}{\sqrt{5}}(du - 2dv).$$

Construction of an Isometric Embedding of the Flat Torus



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- Observe that

$$\ell_i(U(i)) = \|U(i)\|_{\mathbb{E}_2} \quad (\neq 1 \quad \text{for} \quad i \in \{2,3\}).$$

Construction of an Isometric Embedding of the Flat Torus

- Let $\mathcal{C}$ be the positive cone spanned by the $\ell_i \otimes \ell_i$'s :

$$\mathcal{C} := \{\rho_1 \ell_1 \otimes \ell_1 + \rho_2 \ell_2 \otimes \ell_2 + \rho_3 \ell_3 \otimes \ell_3 \mid \rho_1 > 0, \rho_2 > 0, \rho_3 > 0\},$$

- A straightforward computation shows that if

$$B = B_{xx} \mathrm{d}x \otimes \mathrm{d}x + B_{xy}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + B_{yy} \mathrm{d}y \otimes \mathrm{d}y$$

then

$$\rho_1 = B_{xx} - \frac{1}{4} B_{yy}, \quad \rho_2 = \frac{5}{4}(\frac{1}{2} B_{yy} + B_{xy}), \quad \rho_3 = \frac{5}{4}(\frac{1}{2} B_{yy} - B_{xy}).$$

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then

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• It is then easily checked that $\Delta(\mathbb{T}^2)$ lies inside $\mathcal{C}$. In other words, there exist three positive functions $\rho_1(\Delta_0), \rho_2(\Delta_0)$ and $\rho_3(\Delta)$ such that

$$\Delta = \rho_1(\Delta)\ell_1 \otimes \ell_1 + \rho_2(\Delta)\ell_2 \otimes \ell_2 + \rho_3(\Delta)\ell_3 \otimes \ell_3.$$

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• In order to divide the value of the three coefficients of the isometric default Δ approximately by two, we proceed by three successive convex integrations.

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• More precisely, we first set

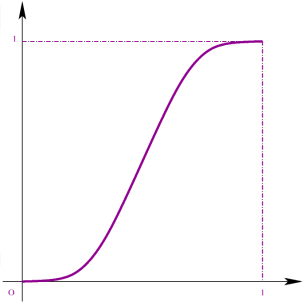
$$\mu_{1,1} := f_0^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} + \frac{3}{4}\rho_1(\Delta_{1,1})\ell_1 \otimes \ell_1 \quad \text{with} \quad \Delta_{1,1} := \Delta,$$

and we define a quasi-isometric map $F_{1,1} : ([0,1] \times \mathbb{S}^1, \mu_{1,1}) \longrightarrow \mathbb{E}^3$ via a convex integration in the direction $U(1) = e_1$:

$$F_{1,1} := CI_\gamma(f, U(1), N_{1,1})$$

and with the family of loops γ used in the Lecture 2 to solve the "Step 2 problem" in codimension 1.

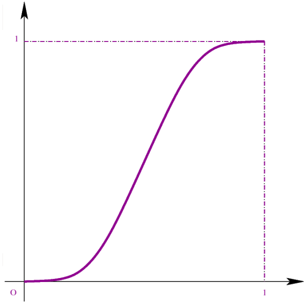
Construction of an Isometric Embedding of the Flat Torus



The function w.

• The map $F_{1,1} : [0,1] \times \mathbb{S}^1 \mapsto \mathbb{E}^3$ does not descend to the quotient $\mathbb{T}^2$ in general.

Construction of an Isometric Embedding of the Flat Torus



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- Let $w : [0,1] \mapsto [0,1]$ be any S-shaped function satisfying

$$w(0) = 0, \quad w(1) = 1 \quad \text{and} \quad \forall k \in \mathbb{N}^*, \quad w^{(k)}(0) = w^{(k)}(1) = 0.$$

Construction of an Isometric Embedding of the Flat Torus

- For all $(u,v) \in [0,1] \times \mathbb{S}^1$, we set

$$f_{1,1}(u,v) := F_{1,1}(u,v) - w(u)\left(F_{1,1}(1,v) - F_{1,1}(0,v)\right).$$

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• The map $f_{1,1}$ descends to the quotient $\mathbb{T}^2$ and induces a quasi isometric map (still denoted by $f_{1,1}$) between $(\mathbb{S}^1 \times \mathbb{S}^1, \mu_{1,1})$ and $\mathbb{E}^3$.

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• We have

$$\begin{aligned} f_{1,1}(u, v) - F_{1,1}(u, v) &= -w(u) \left(F_{1,1}(1, v) - F_{1,1}(0, v) \right) \\ &= -w(u) \left(F_{1,1}(1, v) - f_0(0, v) \right) \\ &= -w(u) \left(F_{1,1}(1, v) - f_0(1, v) \right). \end{aligned}$$

Thus

$$\| f_{1,1} - F_{1,1} \|_{C^0} \leq \| F_{1,1} - f_0 \|_{C^0} = O\left(\frac{1}{N_{1,1}} \right)$$

Construction of an Isometric Embedding of the Flat Torus

- Similarly

$$\partial_v f_{1,1}(u,v) - \partial_v F_{1,1}(u,v) = -w(u)\left(\partial_v F_{1,1}(1,v) - \partial_v f_0(1,v)\right)$$

thus

$$\|\partial_v f_{1,1} - \partial_v F_{1,1}\|_{C^0} \leq \|\partial_v F_{1,1} - \partial_v f_0\|_{C^0} = O\left(\frac{1}{N_{1,1}}\right).$$

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- Regarding the ∂_u derivative we have :

$$\partial_u f_{1,1}(u,v) - \partial_u F_{1,1}(u,v) = -w'(u)\left(F_{1,1}(1,v) - f_0(1,v)\right)$$

thus

$$\|\partial_u f_{1,1} - \partial_u F_{1,1}\|_{C^0} \leq \|w'\|_{C^0}\|F_{1,1} - f_0\|_{C^0} = O\left(\frac{1}{N_{1,1}}\right).$$

Construction of an Isometric Embedding of the Flat Torus

- To sum up, we have

$$\|f_{1,1} - F_{1,1}\|_{C^1} = O\left(\frac{1}{N_{1,1}}\right).$$

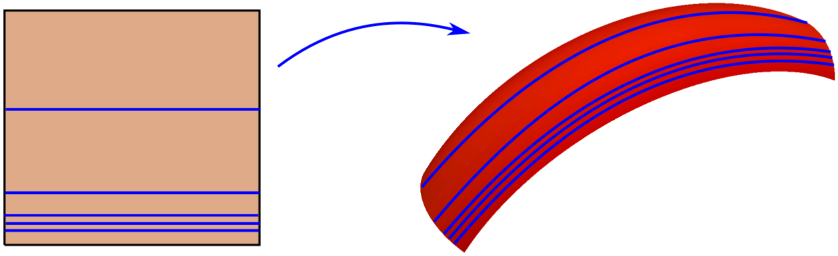
Thus, if $F_{1,1} : ([0,1] \times \mathbb{S}^1, \mu_{1,1}) \longrightarrow \mathbb{E}^3$ is a quasi-isometric map i. e.

$$\mu_{1,1} - F_{1,1}^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} = O\left(\frac{1}{N_{1,1}}\right)$$

then so is $f_{1,1} : (\mathbb{S}^1 \times \mathbb{S}^1, \mu_{1,1}) \longrightarrow \mathbb{E}^3$, i. e.

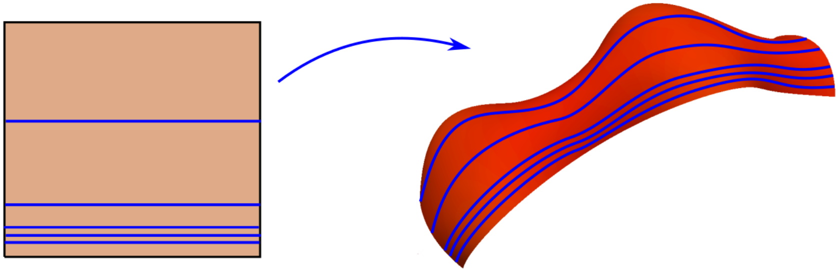
$$\mu_{1,1} - f_{1,1}^*\langle\cdot,\cdot\rangle_{\mathbb{R}^3} = O\left(\frac{1}{N_{1,1}}\right).$$

Construction of an Isometric Embedding of the Flat Torus



The foliation of $\mathbb{T}^2$ in the $U(1)$ direction and the image of a small portion of $\mathbb{T}^2$ by f_0

Construction of an Isometric Embedding of the Flat Torus



The foliation of $\mathbb{T}^2$ in the $U(1)$ direction and the image of a small portion of $\mathbb{T}^2$ by $f_{1,1}$

Construction of an Isometric Embedding of the Flat Torus

- The new isometric default

$$\begin{aligned}\Delta_{1,2} &:= \langle .,. \rangle_{\mathbb{E}^2} - f_{1,1}^*\langle \cdot,\cdot \rangle_{\mathbb{R}^3} \\ &= \rho_1(\Delta_{1,2})\,\ell_1\otimes\ell_1 \;+\; \rho_2(\Delta_{1,2})\,\ell_2\otimes\ell_2 \;+\; \rho_3(\Delta_{1,2})\,\ell_3\otimes\ell_3\end{aligned}$$

satisfies

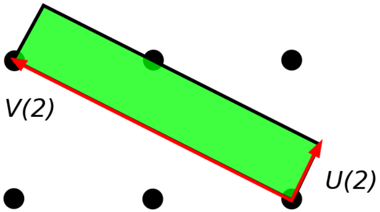
$$\rho_1(\Delta_{1,2}) = \frac{1}{4}\rho(\Delta_{1,1}) + O\left(\frac{1}{N_{1,1}}\right)$$

and

$$\rho_2(\Delta_{1,2}) = \rho_2(\Delta_{1,1}) + O\left(\frac{1}{N_{1,1}}\right), \quad \rho_3(\Delta_{1,2}) = \rho_3(\Delta_{1,1}) + O\left(\frac{1}{N_{1,1}}\right).$$

In particular $\rho_2(\Delta_{1,3}) > 0$ for $N_{1,1}$ large enough.

Construction of an Isometric Embedding of the Flat Torus



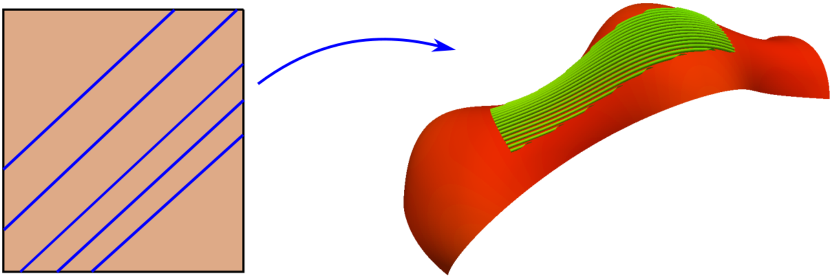
The convex integration is done along parallel curves to the $U(2)$ direction.

- We next set

$$\mu_{1,2} := f_{1,1}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \frac{3}{4} \rho_2(\Delta_{1,2}) \ell_2 \otimes \ell_2,$$

and build a quasi isometry $f_{1,3} : (\mathbb{S}^1 \times \mathbb{S}^1, \mu_{1,3}) \mapsto \mathbb{E}^3$ via a convex integration $CI_\gamma(f_{1,2}, U(3), N_{1,3})$ along the $U(3)$ direction which is then corrected to descend to the quotient.

Construction of an Isometric Embedding of the Flat Torus



The foliation of $\mathbb{T}^2$ in the $U(2)$ direction and the image of a small portion of $\mathbb{T}^2$ by $f_{1,2}$

Construction of an Isometric Embedding of the Flat Torus

- The new isometric default

$$\begin{aligned}\Delta_{1,3} &:= \langle .,. \rangle_{\mathbb{E}^2} - f_{1,2}^*\langle \cdot,\cdot \rangle_{\mathbb{R}^3} \\ &= \rho_1(\Delta_{1,3})\,\ell_1\otimes\ell_1 \;+\; \rho_2(\Delta_{1,3})\,\ell_2\otimes\ell_2 \;+\; \rho_3(\Delta_{1,3})\,\ell_3\otimes\ell_3\end{aligned}$$

satisfies

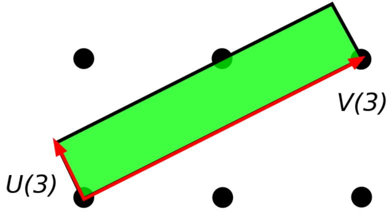
$$\rho_2(\Delta_{1,3}) = \frac{1}{4}\rho_2(\Delta_{1,2}) + O\left(\frac{1}{N_{1,2}}\right)$$

and

$$\rho_1(\Delta_{1,3}) = \rho_1(\Delta_{1,2}) + O\left(\frac{1}{N_{1,2}}\right), \quad \rho_3(\Delta_{1,3}) = \rho_3(\Delta_{1,2}) + O\left(\frac{1}{N_{1,2}}\right).$$

In particular $\rho_3(\Delta_{1,2}) > 0$ for $N_{1,2}$ large enough.

Construction of an Isometric Embedding of the Flat Torus



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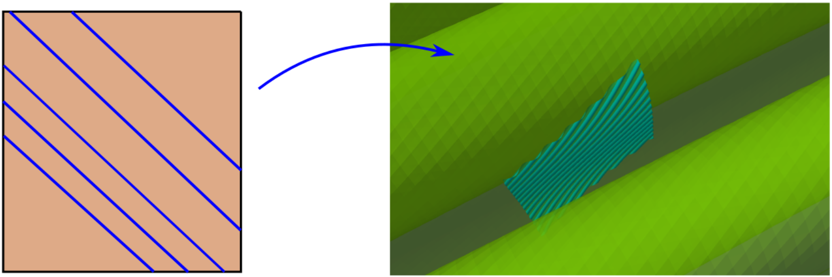
Construction of an Isometric Embedding of the Flat Torus



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Construction of an Isometric Embedding of the Flat Torus

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$$\begin{aligned}\Delta_{2,1} &:= \langle .,. \rangle_{\mathbb{E}^2} - f_{1,3}^*\langle \cdot,\cdot \rangle_{\mathbb{R}^3} \\ &= \rho_1(\Delta_{2,1})\,\ell_1 \otimes \ell_1 \;+\; \rho_2(\Delta_{2,1})\,\ell_2 \otimes \ell_2 \;+\; \rho_3(\Delta_{2,1})\,\ell_3 \otimes \ell_3\end{aligned}$$

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$$\rho_i(\Delta_{2,1}) = \frac{1}{4}\rho_i(\Delta_{1,1}) + O\left(\frac{1}{N_{1,1}}\right) + \left(\frac{1}{N_{1,2}}\right) + \left(\frac{1}{N_{1,3}}\right)$$

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for all $i \in \{1,2,3\}$.

- In particular, if the $N_{1,i}$'s are large enough

$$\|\Delta_{2,1}\| \leq \frac{1}{2}\|\Delta_{1,1}\| = \frac{1}{2}\|\Delta\| \quad \text{and} \quad \Delta_{2,1}(\mathbb{T}^2) \subset \mathcal{C}$$

Construction of an Isometric Embedding of the Flat Torus

- We now iterate the process to obtain a sequence

$$f_0, \quad f_{1,1}, f_{1,2}, f_{1,3}, \quad f_{2,1}, f_{2,2}, f_{2,3}, \quad \cdots$$

such that

$$\|\Delta_{k+1,1}\| \leq \frac{1}{2}\|\Delta_{k,1}\| = \frac{1}{2^k}\|\Delta\| \quad \text{and} \quad \Delta_{k+1,1}(\mathbb{T}^2) \subset \mathcal{C}$$

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• From the Step 4 of the Nash-Kuiper proof (Lecture 1) we know that f_∞ is an embedding provided that the $N_{k,i}$'s are large enough.

Construction of an Isometric Embedding of the Flat Torus

• Thus, the only remaining point is to prove that $(f_{k,3})_{k\in\mathbb{N}^*}$ is C^1-converging.

Construction of an Isometric Embedding of the Flat Torus

• Thus, the only remaining point is to prove that $(f_{k,3})_{k\in\mathbb{N}^*}$ is C^1-converging.

• To do so, we only need to focus on the difference

$$df_{k,i}(U(i)) - df_{k,i-1}(U(i))$$

since the other difference

$$df_{k,i}(V(i)) - df_{k,i-1}(V(i)) = O\left(\frac{1}{N_{k,i}}\right)$$

is controlled by $N_{k,i}$.

Construction of an Isometric Embedding of the Flat Torus

Lemma.– *We have*

$$\|df_{k,i}(U(i)) - df_{k,i-1}(U(i))\|_{C^0} \leq \sqrt{7}\|U(i)\|.\|\frac{3}{4}\rho_i(\Delta_{k,i})\|_{C^0}^{1/2}$$

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• Recall that

$$\rho_i(\Delta_{k,i}) = \frac{1}{4}\rho_i(\Delta_{k,i-1}) + O\left(\frac{1}{N_{1,1}}\right) + \left(\frac{1}{N_{1,2}}\right) + \left(\frac{1}{N_{1,3}}\right)$$

thus, if the $N'_{k,i}$s are large enough, we have

$$\rho_i(\Delta_{k,i}) \leq \frac{1}{2}\rho_i(\Delta_{k,i-1}) \leq \frac{1}{2^k}\rho_i(\Delta)$$

Construction of an Isometric Embedding of the Flat Torus

- As a consequence, if the $N'_{k,i}$s are large enough, we have

$$\|df_{k,i}(U(i)) - df_{k,i-1}(U(i))\|_{C^0} \leq \frac{1}{2^{k/2}}\frac{\sqrt{21}}{2}\|U(i)\|.\|\rho_i(\Delta)\|_{C^0}^{1/2}$$

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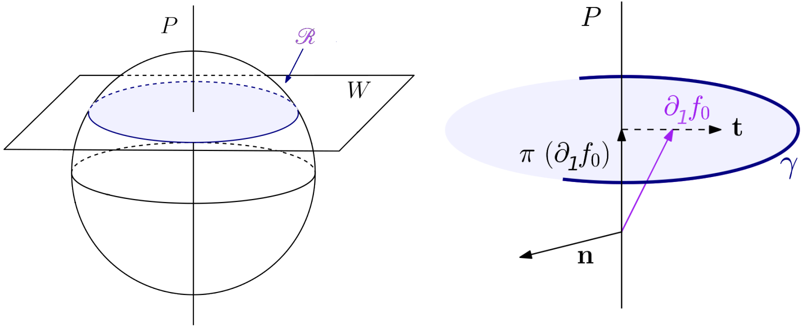
• Let us prove the Lemma !

Construction of an Isometric Embedding of the Flat Torus

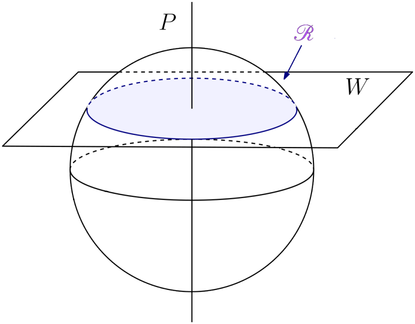


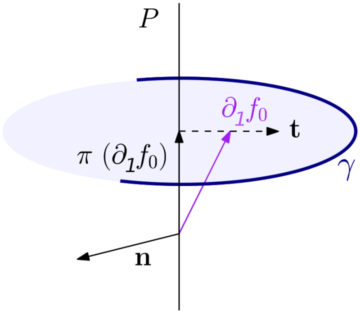
In the picture f_0 stand for $f_{k,i-1}$, ∂_1 for the derivative along $U(i)$ and $P = Span(df_{k,i-1}(V(i)))$.

- First recall our choice of γ (see Lecture 2, "Step 2 problem" in codimension 1) :

$$t \mapsto \gamma(.,t) = r(\cos(\alpha \cos 2\pi t) - J_0(\alpha))\ \mathbf{t} + r \sin(\alpha \cos 2\pi t)\ \mathbf{n} + df_{k,i-1}(U(i))$$

Construction of an Isometric Embedding of the Flat Torus





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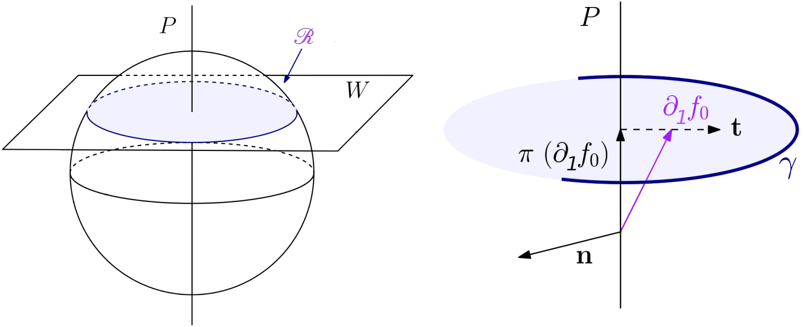
- With

$$\begin{aligned} r^2 &= \mu_{k,i}(U(i), U(i)) - \|\pi(df_{k,i-1}(U(i)))\|^2 \\ &= \|df_{k,i-1}(U(i))\|^2 + \frac{3}{4}\rho_i(\Delta_{k,i})\ell_i(U(i))^2 - \|\pi(df_{k,i-1}(U(i)))\|^2 \\ &= \|df_{k,i-1}(U(i))\|^2 + \frac{3}{4}\rho_i(\Delta_{k,i})\|U(i)\|^2 - \|\pi(df_{k,i-1}(U(i)))\|^2 \end{aligned}$$

Construction of an Isometric Embedding of the Flat Torus



In the picture f_0 stand for $f_{k,i-1}$, ∂_1 for the derivative along $U(i)$ and $P = Span(df_{k,i-1}(V(i)))$.

- And with

$$\alpha = J_0^{-1}\left(\frac{\|df_{k,i-1}(U(i)) - \pi(df_{k,i-1}(U(i)))\|}{r}\right) \in [0, z]$$

where $z \approx 2.40$ is the first positive root of J_0.

Construction of an Isometric Embedding of the Flat Torus

Proof of the Lemma.– Let $p = (u, v)$ be a point of $\mathbb{T}^2$. We denote by $p = (u, v)$ its coordinates in the frame $(O; U(i), V(i))$. We have

$$\begin{aligned} df_{k,i}(U(i))(p) - df_{k,i-1}(U(i))(p) &= \gamma(p, N_{k,i}u) - df_{k,i-1}(U(i))(p) \\ &= r(\cos\theta - J_0(\alpha))\,\mathbf{t} + r\sin(\theta)\,\mathbf{n} \end{aligned}$$

with $\theta = \alpha(p)\cos 2\pi N_{k,i}u$.

Construction of an Isometric Embedding of the Flat Torus

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• Thus

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- To continue, we need a sublemma...

Construction of an Isometric Embedding of the Flat Torus

Sublemma.– *The inequality*

$$1 + J_0^2(\alpha) - 2J_0(\alpha)\cos(\alpha) \leq 7(1 - J_0^2(\alpha))$$

holds for every $\alpha \in [0, z]$ *where* $z \approx 2.40$ *is the first positive root of* J_0.

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Proof of the Sublemma.– Subtracting the right hand side from the left hand side, we rewrite this inequality as

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• By considering the alternating Taylor series of J_0 and $\cos$, we get

$$J_0(\alpha) \leq 1 - \frac{\alpha^2}{4} + \frac{\alpha^4}{64} \quad \text{and} \quad \cos(\alpha) \geq 1 - \frac{\alpha^2}{2}.$$

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• Whence

$$0 \leq 4J_0(\alpha) - \cos(\alpha) \leq 3 - \frac{\alpha^2}{2} + \frac{\alpha^4}{16} \leq 3 + \frac{\alpha^2}{2},$$

where the last inequality follows from $-\frac{\alpha^2}{2} + \frac{\alpha^4}{16} \leq \frac{\alpha^2}{2}$ for all $\alpha \in [0, z]$.

Construction of an Isometric Embedding of the Flat Torus

- We can now write

$$\begin{aligned} 4J_0^2(\alpha) - J_0(\alpha)\cos(\alpha) - 3 &= J_0(\alpha)(4J_0(\alpha) - \cos(\alpha)) - 3 \\ &\leq (1 - \frac{\alpha^2}{4} + \frac{\alpha^4}{64})(3 + \frac{\alpha^2}{2}) - 3. \end{aligned}$$

Putting $x = \alpha^2/4$, this last polynomial can be rewritten

$$(1 - x + \frac{x^2}{4})(3 + 2x) - 3 = \frac{x}{2}(x - x_1)(x - x_2),$$

where $x_1 < 0 < z^2/4 < x_2$. It ensues that this polynomial is negative for $\alpha \in [0, z]$. $\square$

Construction of an Isometric Embedding of the Flat Torus

Back to the proof of the Lemma.– We have stated that

$$\|df_{k,i}(U(i))(p) - df_{k,i-1}(U(i))(p)\|^2 = r^2(1 + J_0(\alpha)^2 - 2J_0(\alpha)\cos\theta)$$

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• Since $\cos\theta = \cos(\alpha\cos 2\pi N_{k,i}u) \geq \cos\alpha$ for $\alpha \in [0, z]$, by the sublemma we have

$$\|df_{k,i}(U(i))(p) - df_{k,i-1}(U(i))(p)\|^2 \leq 7r^2(1 - J_0^2(\alpha))$$

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we have

$$r^2 J_0^2(\alpha) = \|df_{k,i-1}(U(i)) - \pi(df_{k,i-1}(U(i)))\|^2$$

Construction of an Isometric Embedding of the Flat Torus

- By definition of r we have

$$r^2 = \|df_{k,i-1}(U(i))\|^2 - \|\pi(df_{k,i-1}(U(i)))\|^2 + \frac{3}{4}\rho_i(\Delta_{k,i})\|U(i)\|^2$$

thus

$$\begin{aligned} r^2 - r^2 J_0^2(\alpha) = & \|df_{k,i-1}(U(i))\|^2 - \|\pi(df_{k,i-1}(U(i)))\|^2 + \frac{3}{4}\rho_i(\Delta_{k,i})\|U(i)\|^2 \\ & -\|df_{k,i-1}(U(i)) - \pi(df_{k,i-1}(U(i)))\|^2 \end{aligned}$$

- By the Pythagorean theorem we have

$$r^2 - r^2 J_0^2(\alpha) = \frac{3}{4}\rho_i(\Delta_{k,i})\|U(i)\|^2$$

thus

$$\|df_{k,i}(U(i))(p) - df_{k,i-1}(U(i))(p)\|^2 \leq 7\|U(i)\|^2 \frac{3}{4}\rho_i(\Delta_{k,i})(p)$$

$\square$

Construction of an Isometric Embedding of the Flat Torus

• We have constructed a sequence of maps $(f_{k,3})_{k\in\mathbb{N}^*}$ that C^1-converges toward an isometric embedding.

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• The difference $F_{k,i}(1,v) - F_{k,i}(0,v)$ that prevents $F_{k,i}$ to be defined on $\mathbb{S}^1 \times \mathbb{S}^1$ is fixed by spreading the gap on the whole torus. Local constructions are avoided.

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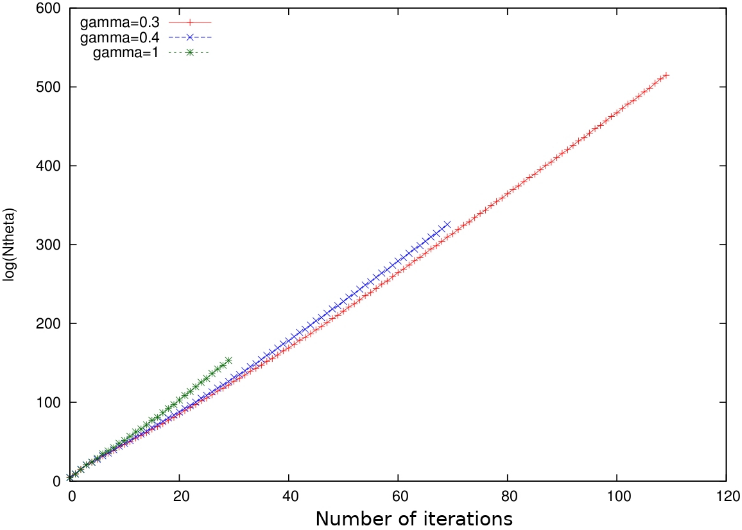
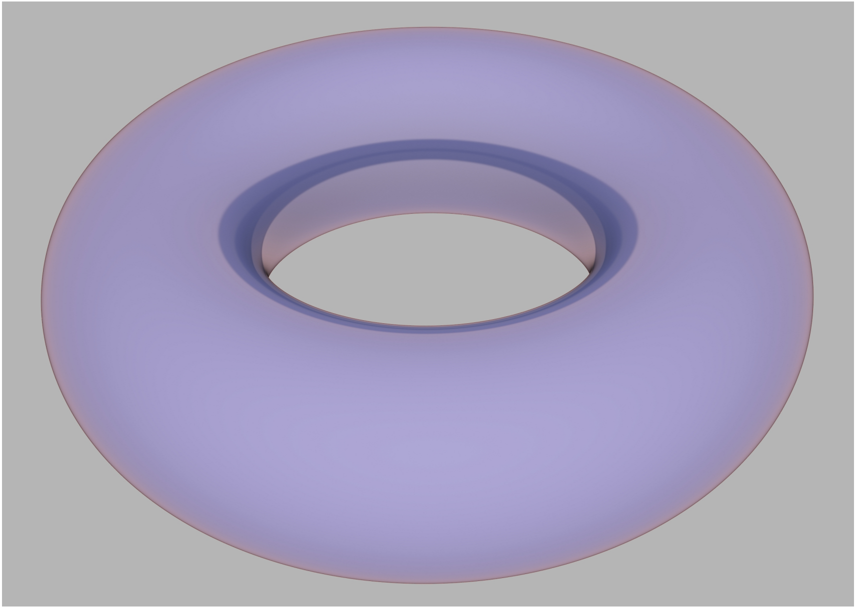
• The numbers $N_{k,i}$ are not given *a priori* but can be computerized. It is observed that the $N_{k,i}$ increase exponentially.

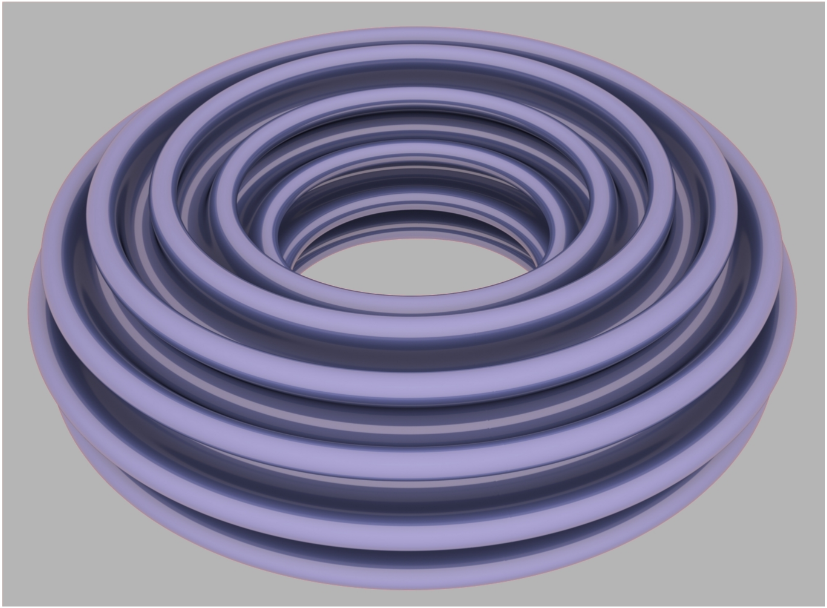
Construction of an Isometric Embedding of the Flat Torus



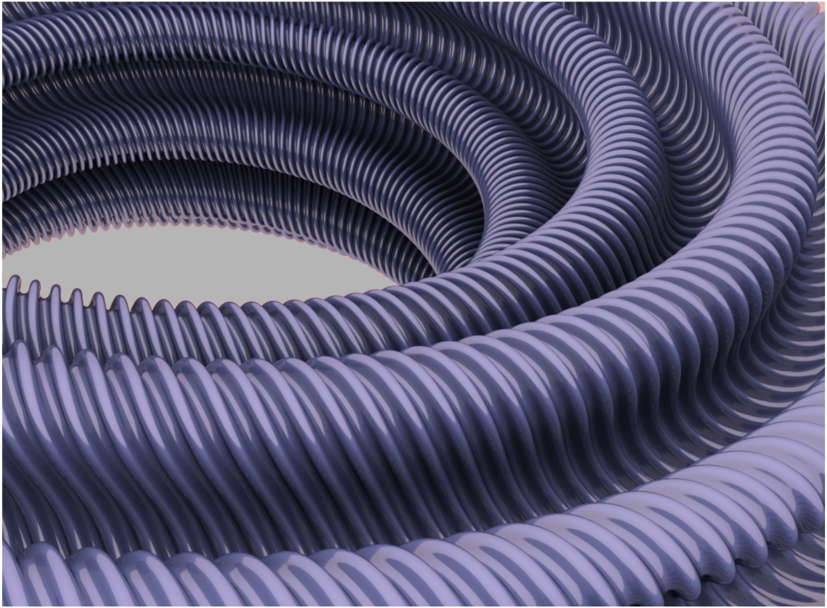
The initial map f_0



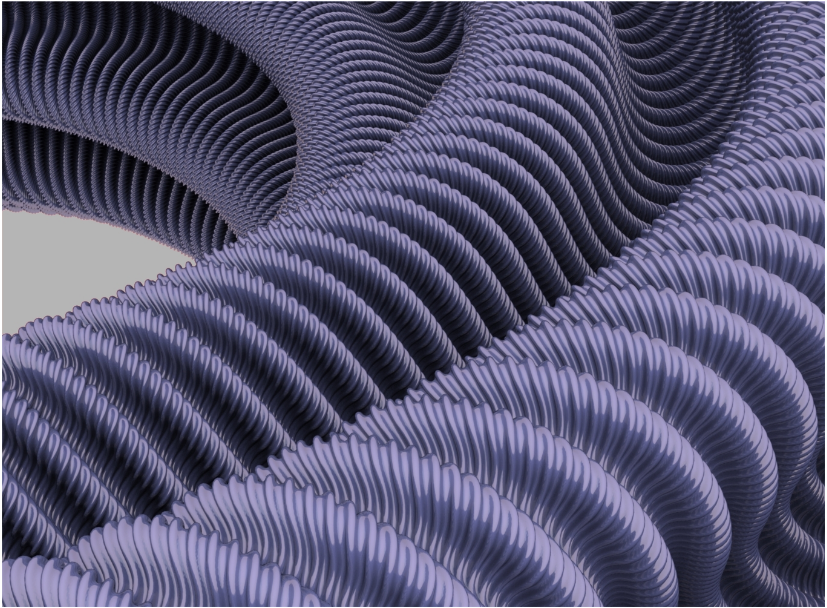
The map $f_{1,1}$ ($N_{1,1} = 12$)



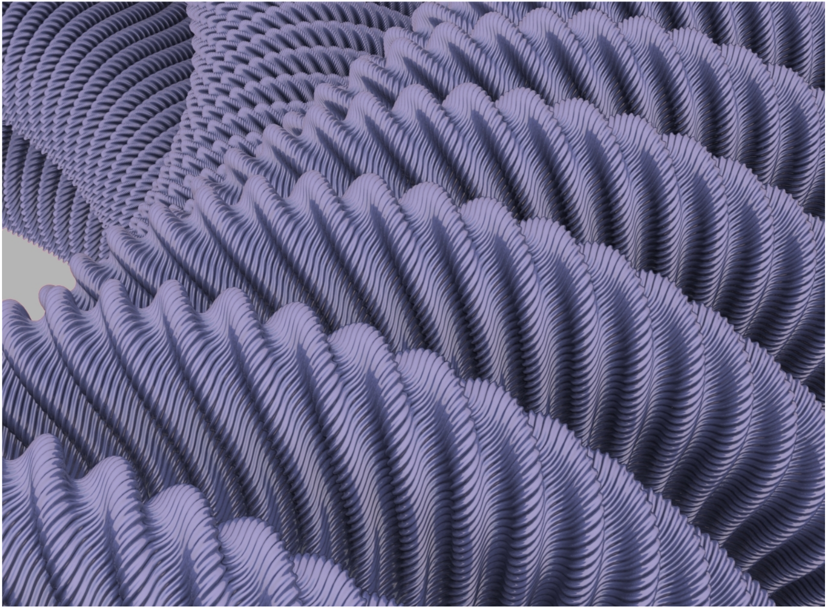
The map $f_{1,2}$ ($N_{1,2} = 80$)



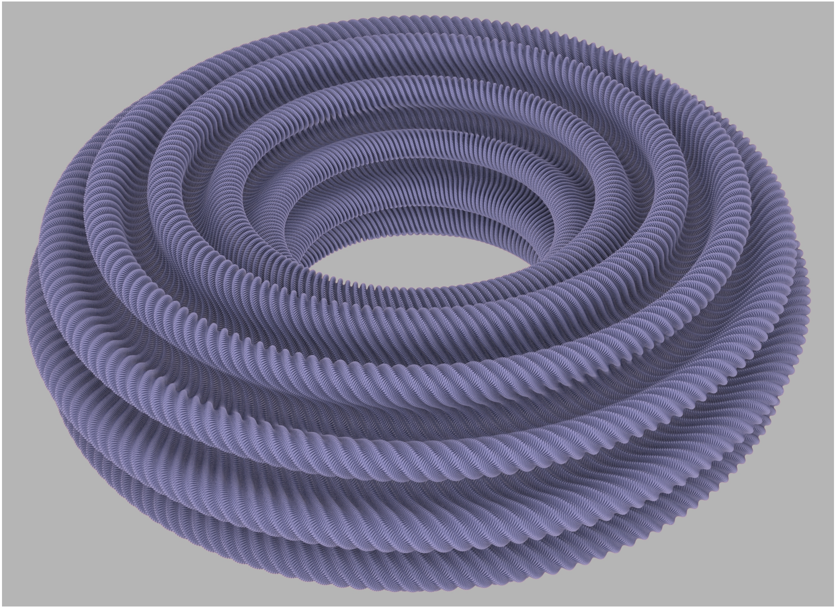
The map $f_{1,3}$ ($N_{1,3} = 500$)



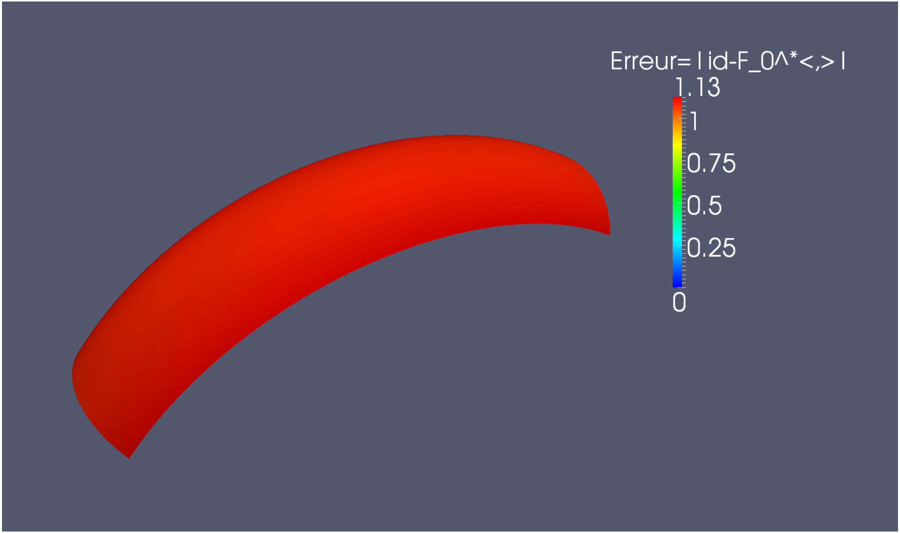
The map $f_{2,1}$ ($N_{2,1} = 9000$)



And so on... up to the Flat Torus

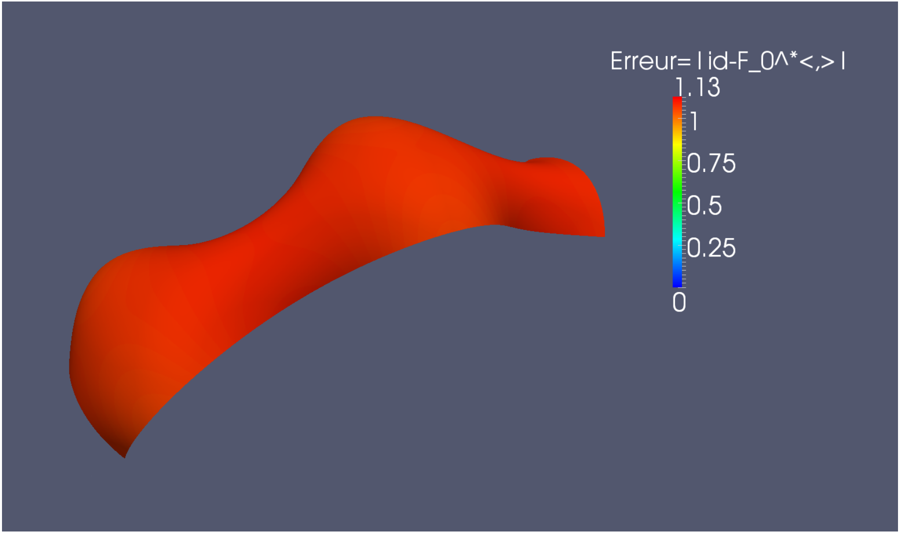


Zoom !



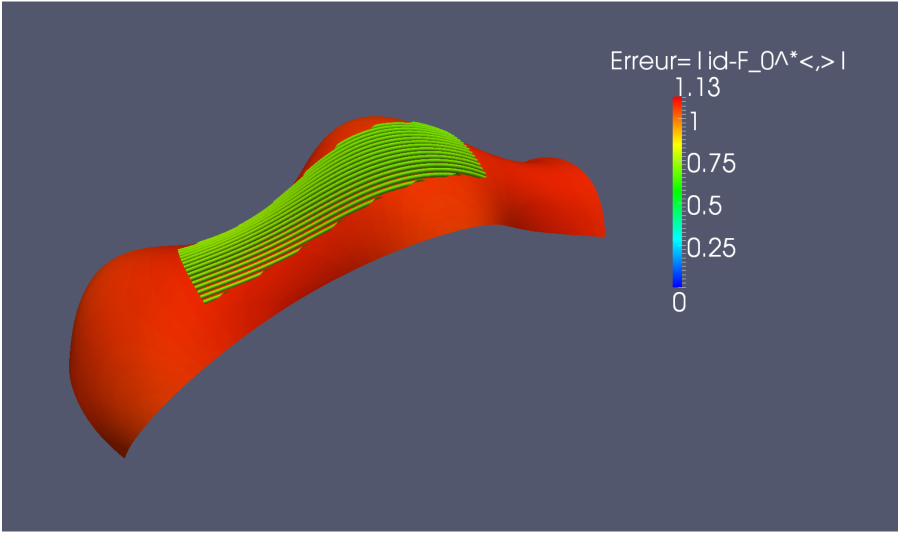
The initial map (detail)

Zoom !



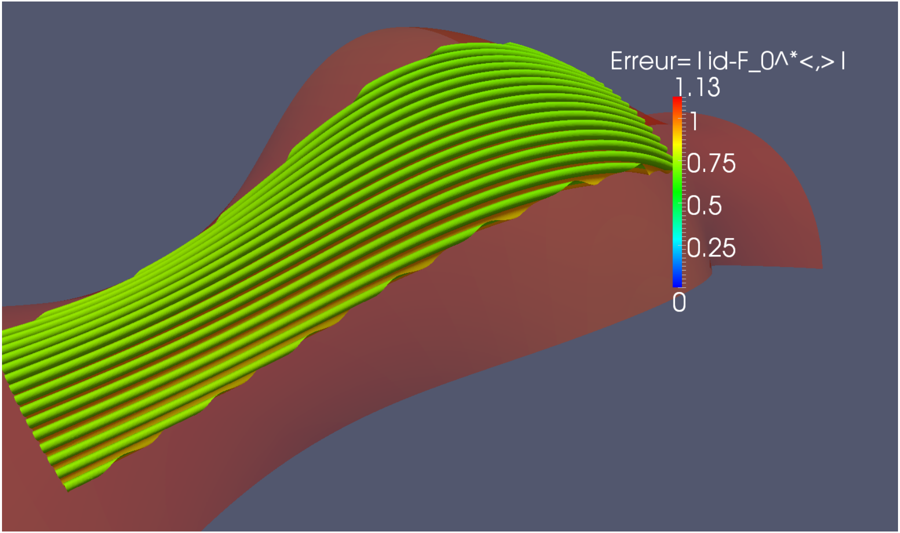
$f_{1,1}$: 8 oscillations

Zoom !



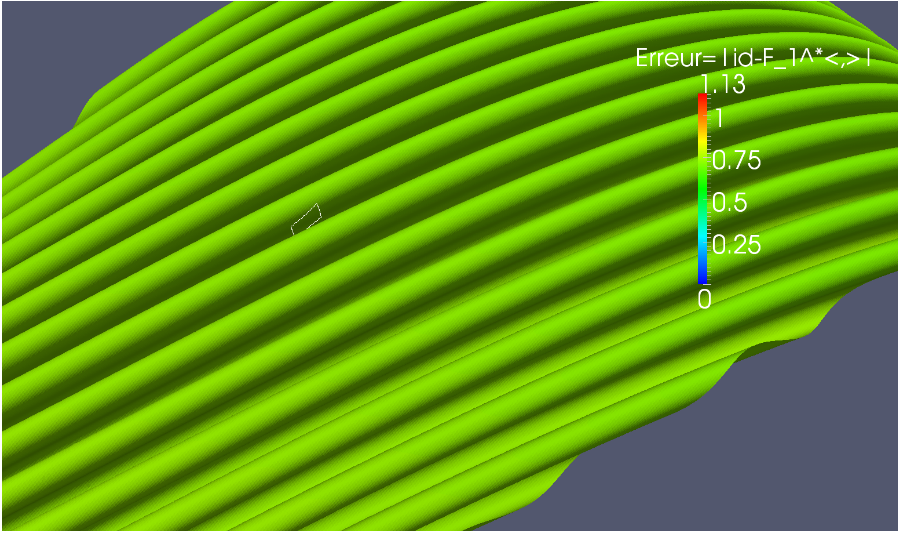
$f_{1,2}$: *64 oscillations*

Zoom !



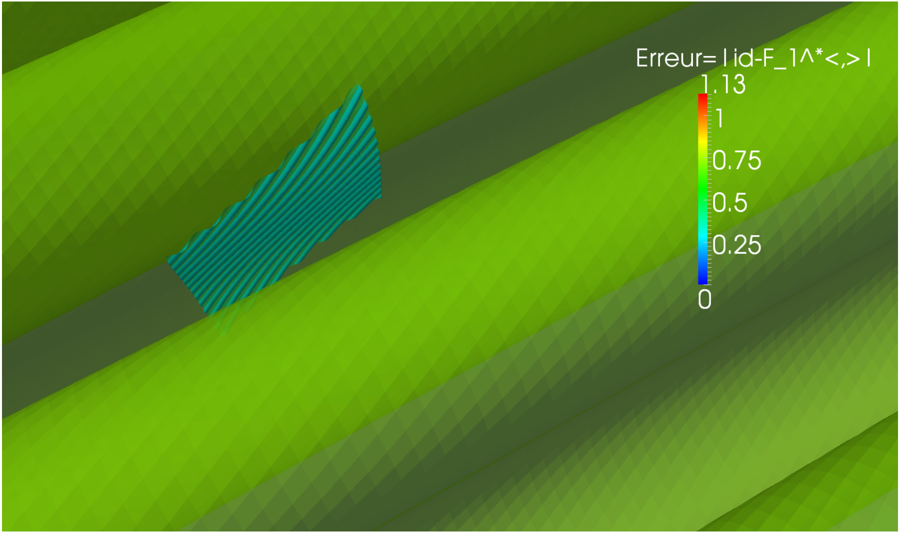
$f_{1,2}$: *64 oscillations*

Zoom !



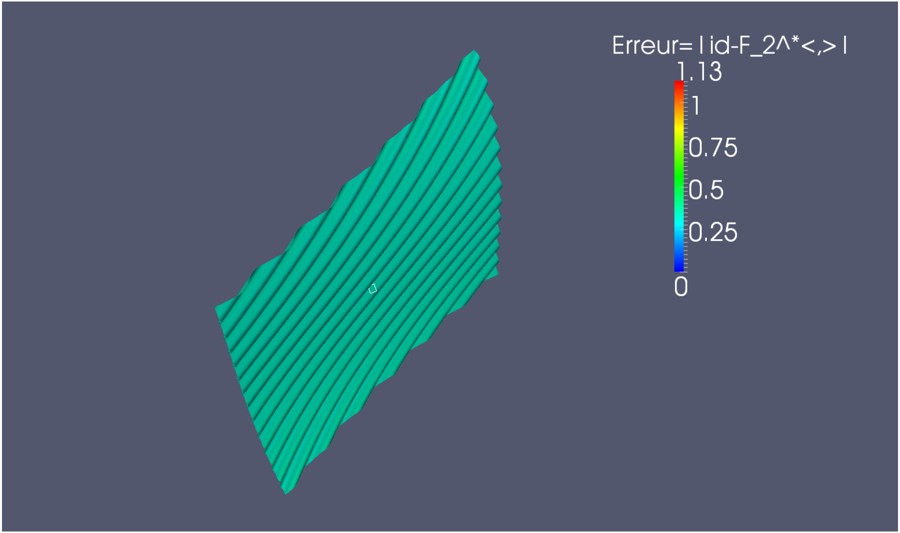
More closely

Zoom !



$f_{1,3}$ *: 4096 oscillations*

Zoom !



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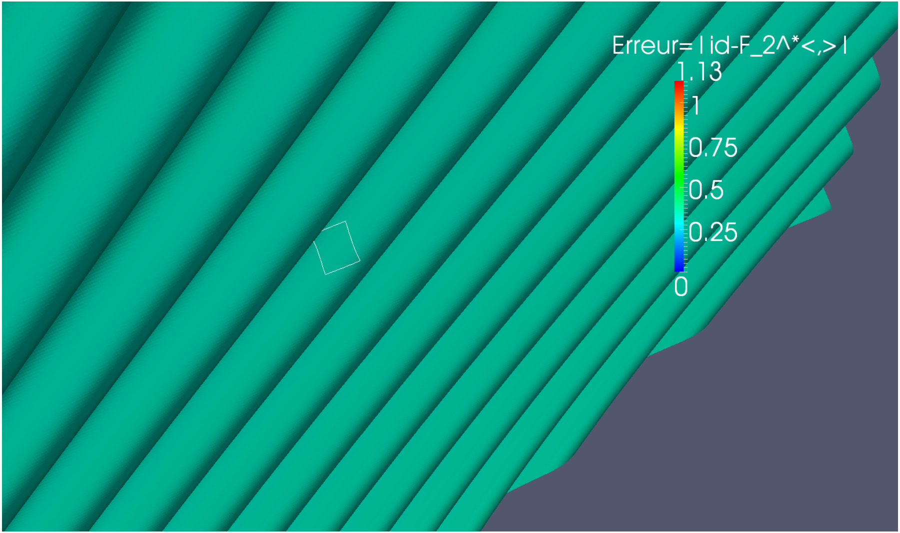
Zoom !



More closely

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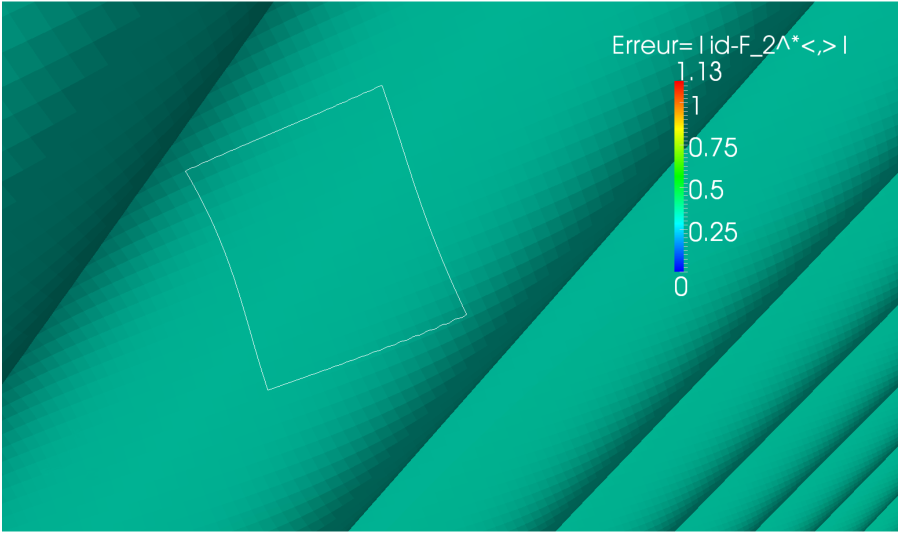
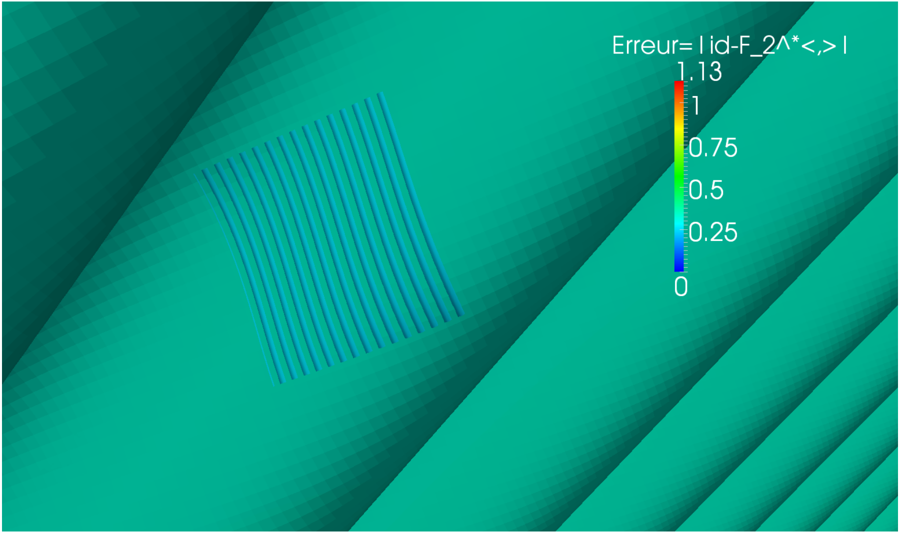


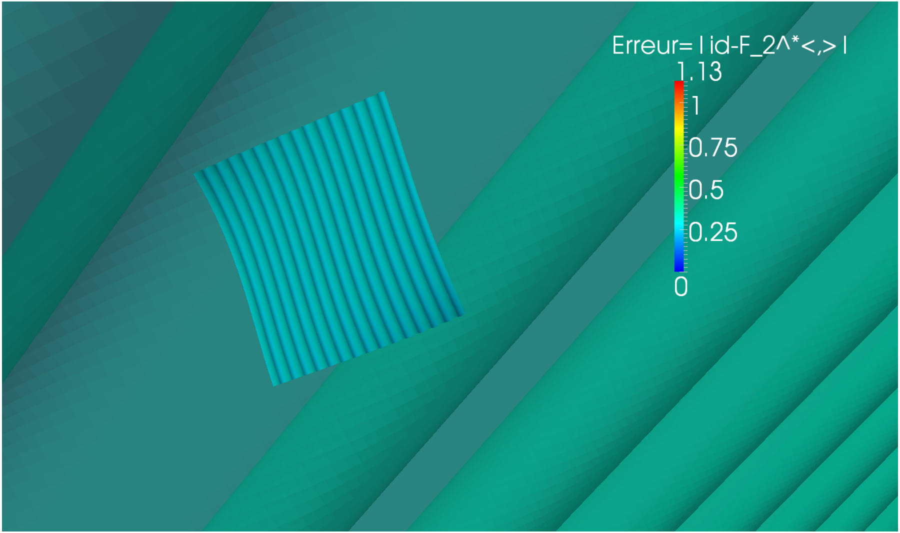
More closely

Zoom !



$f_{2,1}$: *524 288 oscillations*

Zoom !



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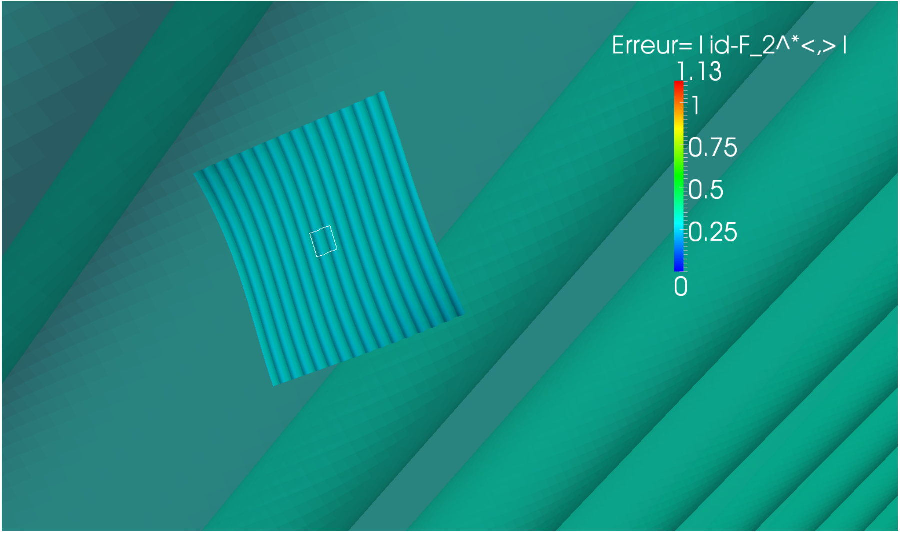

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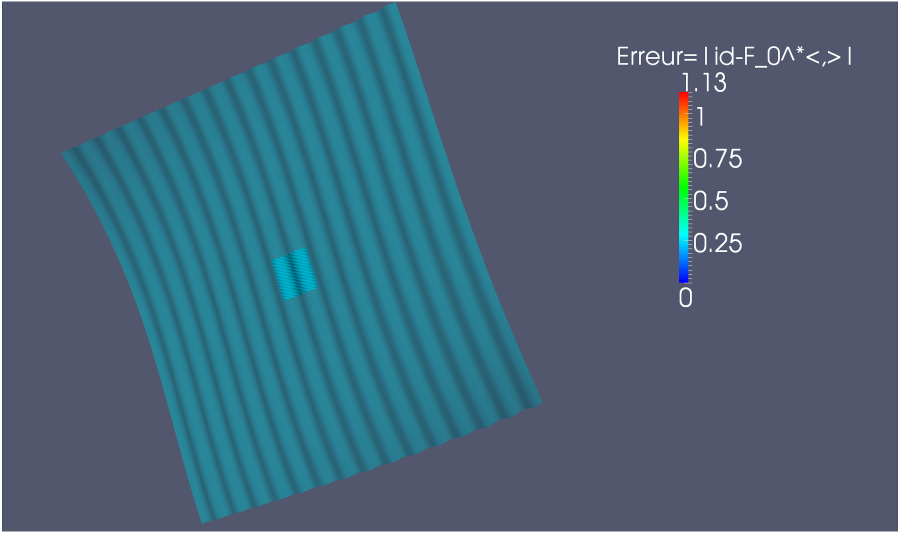
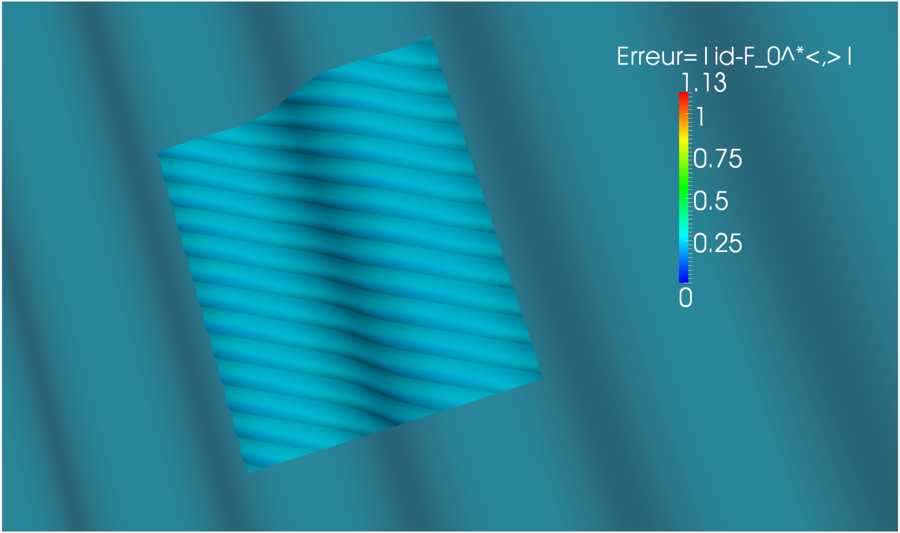


$f_{2,2}$: 2 097 152 oscillations

Zoom !



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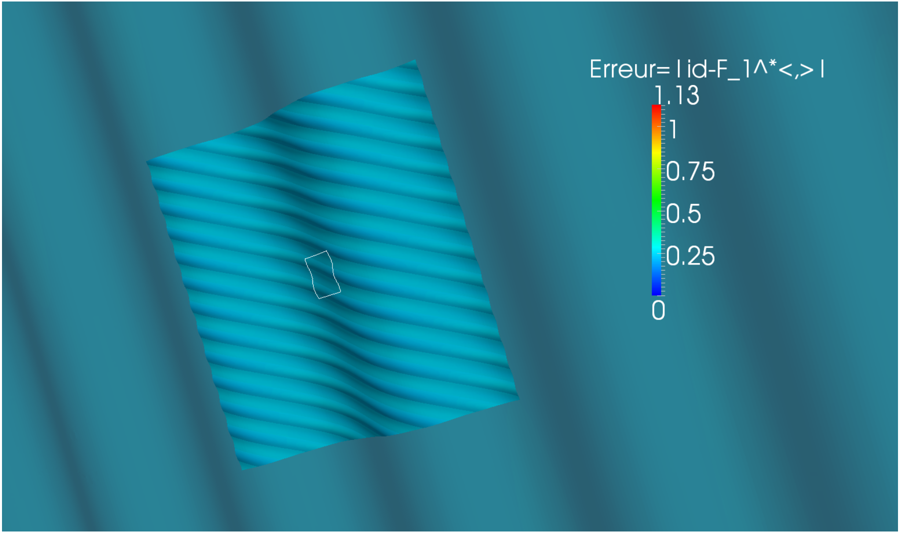

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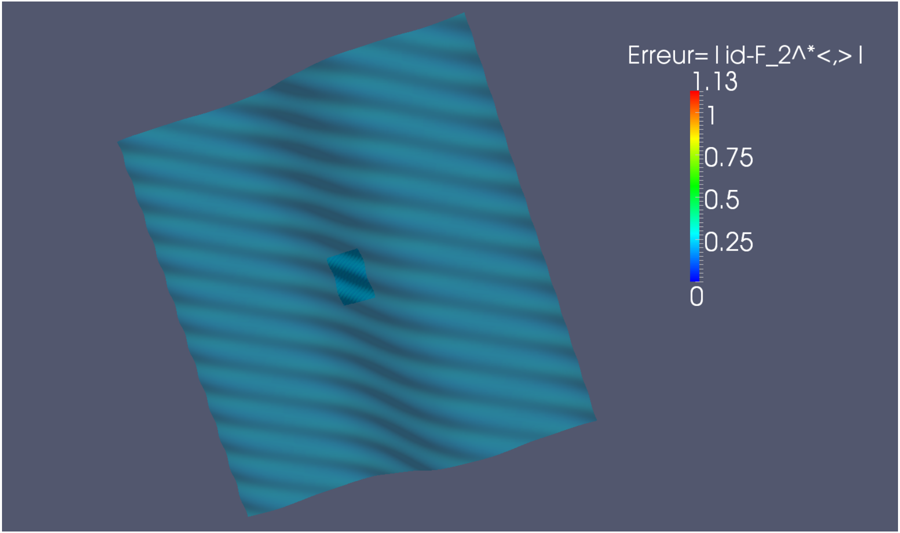
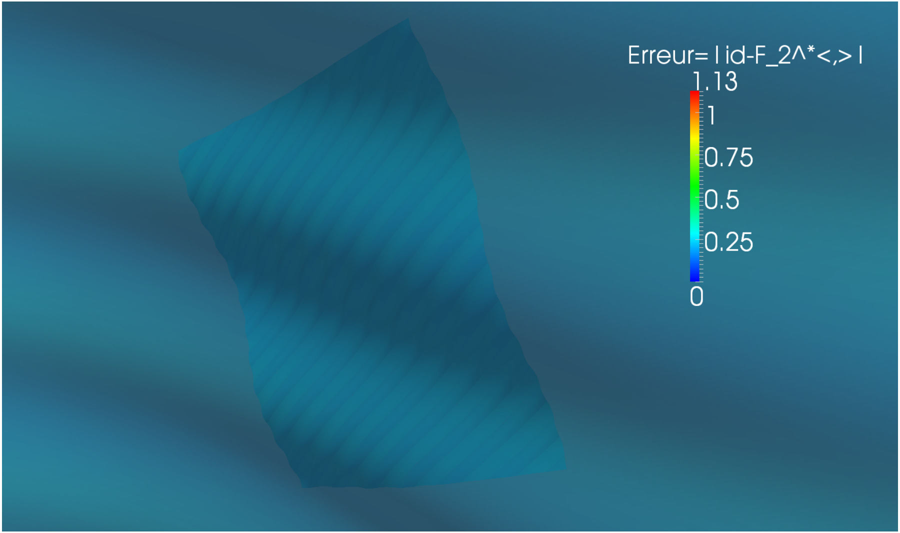


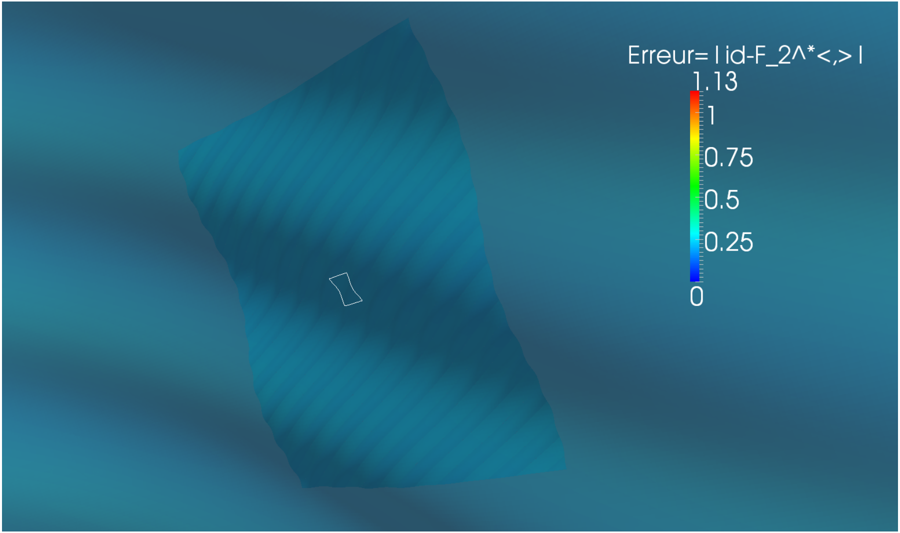
$f_{2,3}$: 16 777 216 oscillations

Zoom !



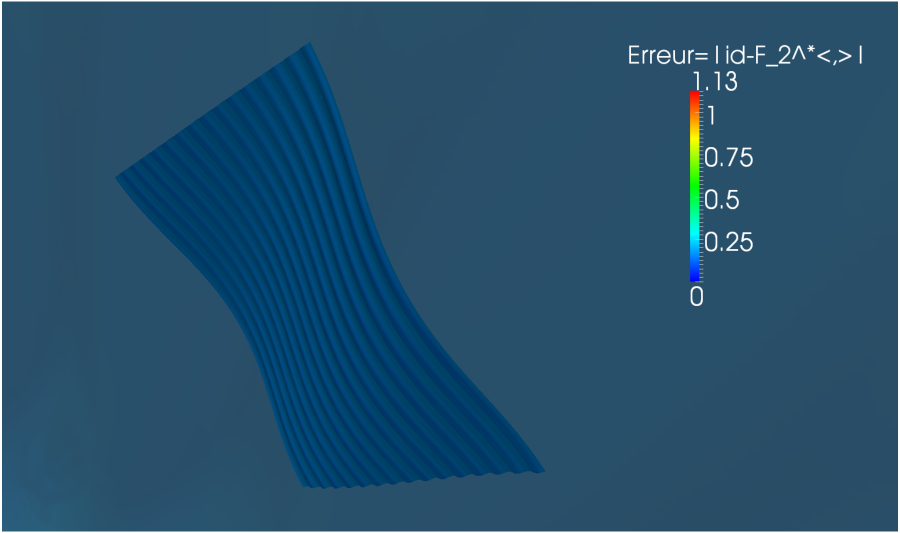
More closely

Zoom !



More closely

Zoom !



$f_{3,1}$: 536 870 912 oscillations

Zoom !

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• We are going to understand this unusual geometry by looking at the situation in a 1-dimensional setting.

Nash-Kuiper in a 1D setting

- We consider an initial short embedding $f_0 : \mathbb{S}^1 = \mathbb{E}/\mathbb{Z} \to \mathbb{C}$ such that :

    (Cond 1) f_0 is of constant speed : $x \mapsto |f_0'(x)| \equiv r_0 < 1$
    (Cond 2) f_0 is radially symmetric that is : $\frac{\partial f_0}{\partial x}(x + \frac{1}{2}) = -\frac{\partial f_0}{\partial x}(x)$.

and the goal is to build an "isometric" map f_∞ :

$$\forall x \in \mathbb{S}^1, \quad |f_\infty(x)| = 1$$

by using the Nash-Kuiper approach.

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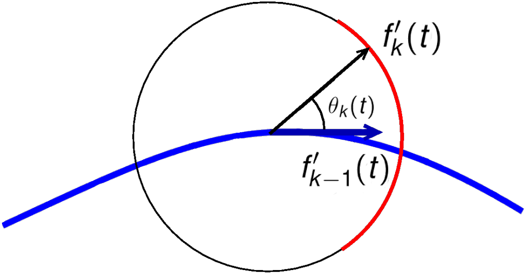
• Precisely, given a sequence of numbers $(r_k)_{k\in\mathbb{N}}$ such that $r_k \uparrow 1$, we would like to build, by using convex integration, a sequence of maps $(f_k)_{k\in\mathbb{N}}$ such that

$$x \mapsto |f_k(x)| \equiv r_k$$

Nash-Kuiper in a 1D setting

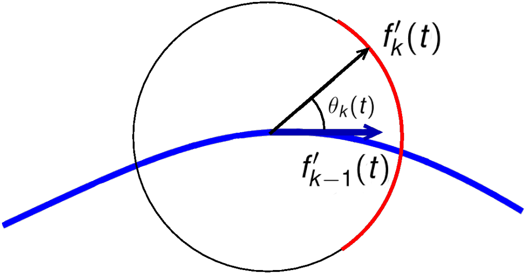


• At the k-th step, the differential relation is a circle of radius r_k. We choose the following family of loops

$$\gamma_k(x,t) := r_k(\cos\theta_k(x,t)\,\mathbf{t}_{k-1}(x) + \sin\theta_k(x,t)\,\mathbf{n}_{k-1}(x))$$

with $\mathbf{t}_{k-1} = \frac{f'_{k-1}}{\|f'_{k-1}\|}$ and $\mathbf{n}_{k-1} = i\mathbf{t}_{k-1}$ and $\theta_k(x,t) = \alpha_k(x)\cos 2\pi t$.

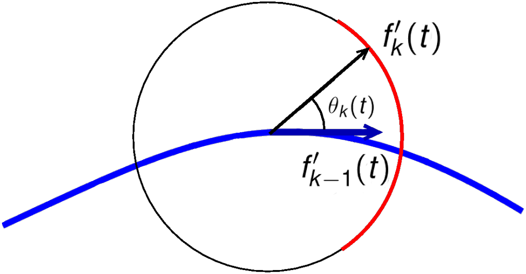
Nash-Kuiper in a 1D setting



• The angle $\alpha(x)$ should be chosen so that the average condition holds :

$$\int_0^1 \gamma_k(x,t)dt = f'_{k-1}(x)$$

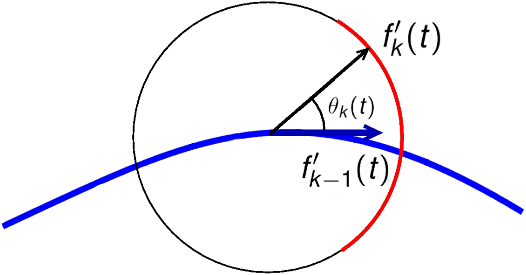
Nash-Kuiper in a 1D setting



- A straightforward computation shows that

$$\int_0^1 \gamma_k(x,t)dt = r_k J_0(\alpha_k(x))\mathbf{t}_{k-1}(x)$$

Nash-Kuiper in a 1D setting

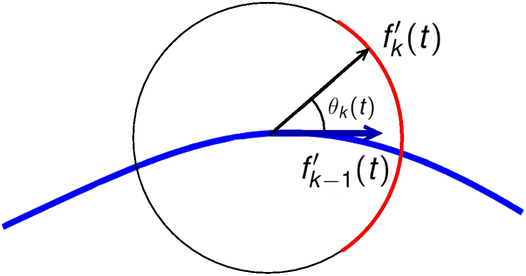


- Thus the average condition is fulfilled iff

$$\frac{r_k J_0(\alpha_k(x))}{|f'_{k-1}(x)|} = 1 \quad \text{i.e.} \quad \alpha_k(x) = J_0^{-1}\left(\frac{r_{k-1}}{r_k}\right)$$

In particular α_k does not depend on x.

Nash-Kuiper in a 1D setting



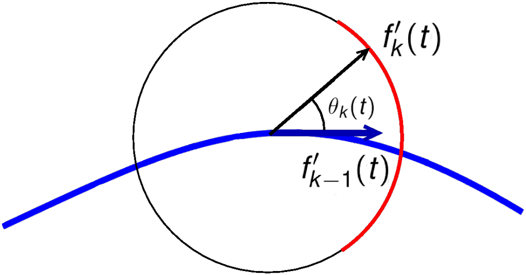
• We introduce the following (abuse of) notation

$$\mathbf{e}^{i\theta} := \cos\theta\,\mathbf{t}_{k-1} + \sin\theta\,\mathbf{n}_{k-1}$$

so that

$$\gamma_k(x,t) := r_k \mathbf{e}^{i\alpha_k \cos 2\pi t} \quad \text{with} \quad \alpha_k = \frac{r_{k-1}}{r_k}$$

Nash-Kuiper in a 1D setting



- Let $f_k = CI_{\gamma_k}(f_{k-1}, \partial_x, N_k)$. We thus have

$$f_k(x) = f_{k-1}(0) + \int_0^x r_k \mathbf{e}^{i\alpha_k \cos 2\pi N_k s} ds$$

Nash-Kuiper in a 1D setting

Lemma.– *Let* $(N_k)_{k\in\mathbb{N}^*}$ *be any sequence of natural even integers and* $f_0 : \mathbb{S}^1 = \mathbb{E}/\mathbb{Z} \to \mathbb{E}^2$ *be a short embedding satisfying conditions* $(Cond\ 1)$ *and* $(Cond\ 2)$*. For every* $k \in \mathbb{N}^*$*, let* $f_k : [0,1] \to \mathbb{E}^2$ *be defined inductively by*

$$f_k(x) := f_{k-1}(0) + \int_0^x r_k \mathbf{e}^{i\alpha_k \cos 2\pi N_k s} \mathrm{d}s.$$

Then f_k *descends to a map* $f_k : \mathbb{E}/\mathbb{Z} \to \mathbb{E}^2$ *which satisfies* $(Cond\ 1)$ *and* $(Cond\ 2)$ *and is "isometric" i. e.*

$$\forall x \in \mathbb{R}/\mathbb{Z}, \quad |f_k'(x)| = r_k.$$

Nash-Kuiper in a 1D setting

Lemma.– *Let $(N_k)_{k\in\mathbb{N}^*}$ be any sequence of natural even integers and $f_0 : \mathbb{S}^1 = \mathbb{E}/\mathbb{Z} \to \mathbb{E}^2$ be a short embedding satisfying conditions $(Cond\ 1)$ and $(Cond\ 2)$. For every $k \in \mathbb{N}^*$, let $f_k : [0,1] \to \mathbb{E}^2$ be defined inductively by*

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$$\forall x \in \mathbb{R}/\mathbb{Z}, \quad |f_k'(x)| = r_k.$$

- The proof is left as an exercise.

Nash-Kuiper in a 1D setting

- Let $(A_k)_{k \in N^*}$ be the sequence of functions defined by

$$\forall x \in \mathbb{S}^1, \quad A_k(x) := \sum_{\ell=1}^{k} \alpha_\ell \cos(2\pi N_\ell x).$$

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$$\frac{\partial f_k}{\partial x}(x) = e^{iA_k(x)} \frac{r_k}{r_0} \frac{\partial f_0}{\partial x}(x).$$

Nash-Kuiper in a 1D setting

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Lemma.– *We have :*

$$\frac{\partial f_k}{\partial x}(x) = e^{iA_k(x)} \frac{r_k}{r_0} \frac{\partial f_0}{\partial x}(x).$$

Proof.– We first observe that

$$\mathbf{e}^{i\theta} = \cos\theta\, \mathbf{t}_{k-1} + \sin\theta\, \mathbf{n}_{k-1} = e^{i\theta} \frac{1}{r_{k-1}} \frac{\partial f_{k-1}}{\partial x}$$

Thus

$$\frac{\partial f_k}{\partial x}(x) \quad = \quad r_k e^{i\alpha_k \cos(2\pi N_k x)} \frac{1}{r_{k-1}} \frac{\partial f_{k-1}}{\partial x}(x)$$

and by induction

$$\frac{\partial f_k}{\partial x}(x) = e^{iA_k(x)} \frac{r_k}{r_0} \frac{\partial f_0}{\partial x}(x) \qquad \square$$

Nash-Kuiper in a 1D setting !

Lemma.– *We have*

$$\alpha_k \sim 2\sqrt{1 - \frac{r_{k-1}}{r_k}}.$$

Proof.– By definition $\alpha_k = J_0^{-1}(\frac{r_{k-1}}{r_k})$. Recall that the Taylor expansion of $J_0(\alpha)$ up to order 2 is

$$\xi = 1 - \frac{\alpha^2}{4} + o(\alpha^2).$$

Let $y = 1 - \xi$ and $X = \alpha^2$, we have $y = \frac{X}{4} + o(X)$ thus $X = 4y + o(y)$ and so $X \sim 4y$. We finally get

$$\alpha \sim 2\sqrt{1 - \xi} \quad \text{and} \quad \alpha_k \sim 2\sqrt{1 - \frac{r_{k-1}}{r_k}}.$$

$\square$

Nash-Kuiper in a 1D setting !

Corollary.– *If*

$$\sum_k \sqrt{1 - \frac{r_{k-1}}{r_k}} < +\infty$$

then $(A_k)_{k\in\mathbb{N}^*}$ *is normally converging toward* $A := \lim_k A_k$ *and* $(f_k)_{k\in\mathbb{N}^*}$ *is* C^1*-converging toward* $f_\infty := \lim_k f_k$ *and*

$$\forall\, x \in \mathbb{S}^1, \qquad \frac{\partial f_\infty}{\partial x}(x) = e^{iA(x)} \frac{1}{r_0} \frac{\partial f_0}{\partial x}(x).$$

Nash-Kuiper in a 1D setting !

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$$\forall\, x \in \mathbb{S}^1, \qquad \frac{\partial f_\infty}{\partial x}(x) = e^{iA(x)} \frac{1}{r_0} \frac{\partial f_0}{\partial x}(x).$$

Remark.– Observe that the sequence $(N_k)_{k\in\mathbb{N}^*}$ plays no role in this corollary. This point is specific to the 1 dimensional setting.

Nash-Kuiper in a 1D setting !

Proof.– From the previous lemma we deduce that

$$\sum \alpha_k < +\infty$$

thus the sequence $(A_k)_{k\in\mathbb{N}}$ is normally converging and

$$A := \lim_{k\to+\infty} A_k$$

is a continuous. Moreover, from the relation

$$\frac{\partial f_k}{\partial x}(x) = e^{iA_k(x)}\frac{r_k}{r_0}\frac{\partial f_0}{\partial x}(x)$$

we also deduce that $(\partial_x f_k)_{k\in\mathbb{N}}$ is normally converging toward

$$e^{iA(x)}\frac{1}{r_0}\frac{\partial f_0}{\partial x}(x).$$

Since $(f_k(0))_{k\in\mathbb{N}}$ obviously converges, we obtain that the sequence $(f_k)_{k\in\mathbb{N}}$ is C^1-converging toward $f_\infty = \lim_{k\to+\infty} f_k$ $\square$

Nash-Kuiper in a 1D setting !

Example 1.– Let $\gamma > 0$ such that $r_0 = 1 - e^{-\gamma}$ and define $(r_k) \uparrow 1$ to be

$$r_k := 1 - e^{-\gamma(k+1)}.$$

Then

$$1 - \frac{r_{k-1}}{r_k} = \frac{r_k - r_{k-1}}{r_k}\frac{r_{k-1}}{r_k} = e^{-\gamma k}\frac{(1 - e^{-\gamma})}{1 - e^{-\gamma(1+k)}}.$$

Thus

$$\sqrt{1 - \frac{r_{k-1}}{r_k}} \sim \sqrt{1 - e^{-\gamma}}\, e^{-\gamma k/2}$$

and

$$\sum_k \sqrt{1 - \frac{r_{k-1}}{r_k}} < +\infty.$$

From the corollary, we deduce that $(f_k)_{k\in\mathbb{N}^*}$ C^1 converges. We also have

$$\alpha_k \sim 2\sqrt{1 - e^{-\gamma}}\, e^{-\gamma k/2}.$$

Nash-Kuiper in a 1D setting !

Example 2.– Let $a \in]0,1[$ and $r_0 := \prod_{k=1}^{\infty} J_0(a^k)$. We define $(r_k)_{k\in\mathbb{N}^*}$ by

$$r_k = \frac{r_{k-1}}{J_0(a^k)} = \frac{r_0}{\prod_{\ell=1}^{k} J_0(a^\ell)} = \frac{\prod_{k=1}^{\infty} J_0(a^k)}{\prod_{\ell=1}^{k} J_0(a^\ell)}.$$

Observe that, since J_0 is decreasing in $[0,z]$ and $J_0(0) = 1$, the sequence $(r_k)_k$ is increasing and bounded from above by 1. The limit of $(r_k)_k$ is obviously 1.

Nash-Kuiper in a 1D setting !

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Observe that, since J_0 is decreasing in $[0,z]$ and $J_0(0) = 1$, the sequence $(r_k)_k$ is increasing and bounded from above by 1. The limit of $(r_k)_k$ is obviously 1.

- We then have

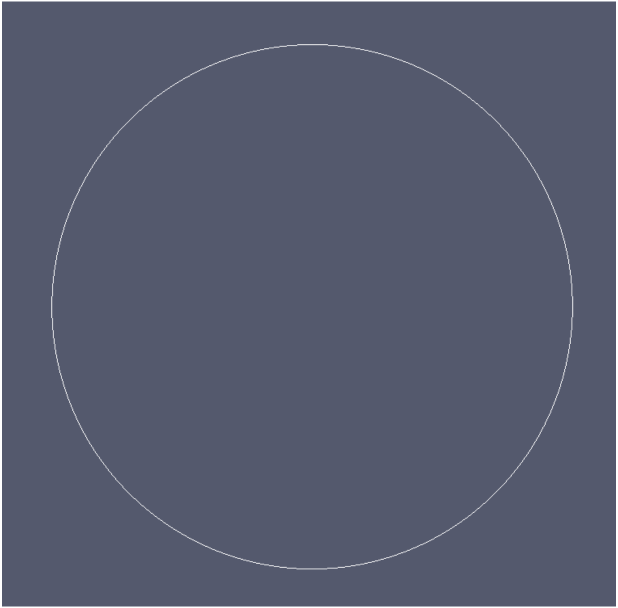
$$\alpha_k = J_0^{-1}\left(\frac{r_{k-1}}{r_k}\right) = a^k$$

and

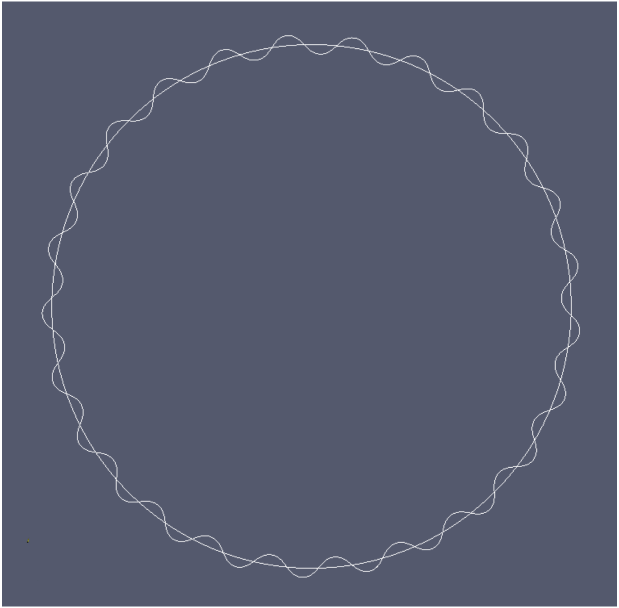
$$A_k(x) = \sum_{\ell=1}^{k} a^\ell \cos(2\pi N_\ell x)$$

for all $x \in \mathbb{S}^1$. Obviously $(A_k)_k$ is normally converging and thus $(f_k)_k$ is C^1-converging.

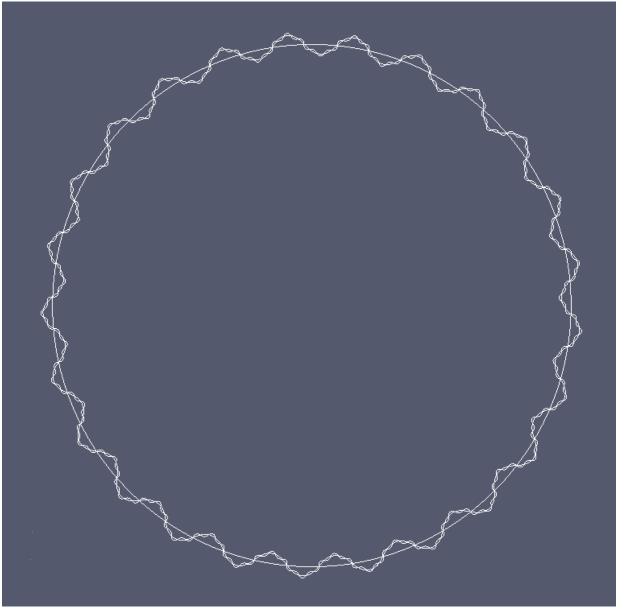
Nash-Kuiper in a 1D setting !



Nash-Kuiper in a 1D setting !



Nash-Kuiper in a 1D setting !



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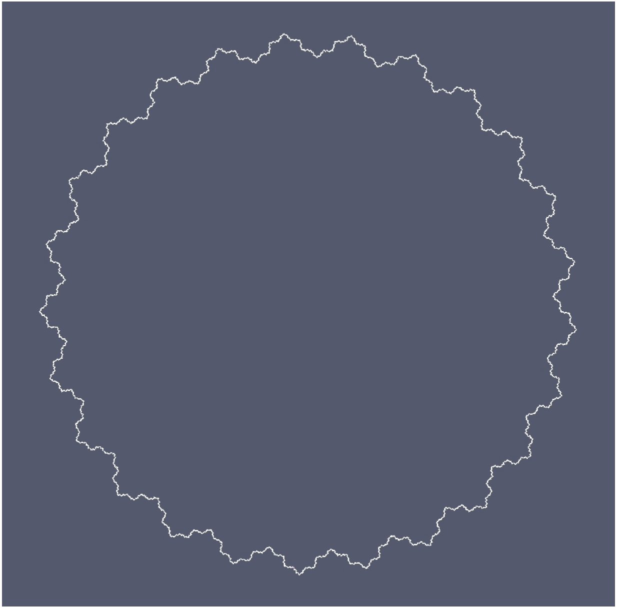

C^1-fractal structure

Definition.– The function $W_{a,b} : \mathbb{R} \to \mathbb{R}$ defined by

$$W_{a,b}(x) = \sum_{k=1}^{\infty} a^k \cos(b^k x), \quad 0 < a < 1 < ab$$

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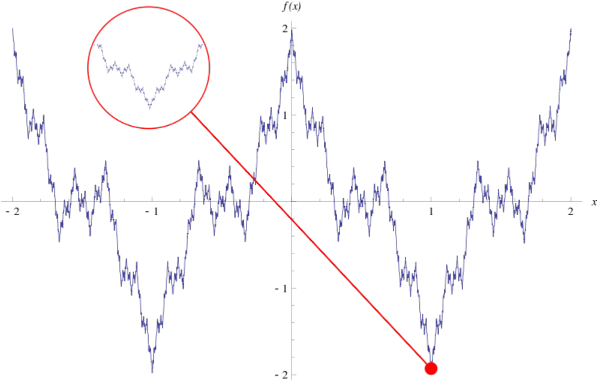

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Theorem (Karl Weierstrass 1872, G. H. Hardy. 1918).– *The function $W_{a,b}$ is continuous on $\mathbb{R}$ but nowhere differentiable.*

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Example 2 (continuing).– If we choose $N_\ell = b^\ell$ for some $b \geq 1/a$ then the map f_∞ obtained by the Nash-Kuiper process is C^1-fractal. Indeed, the function A is the Weierstrass function $W_{a,b}$.

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• We express the C^1 fractality of f_∞ under a form that allows a generalization to surfaces.

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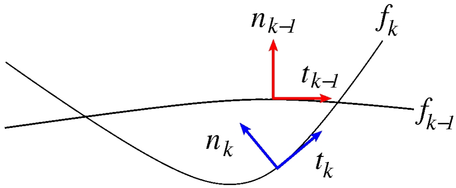
• Let $C_k : \mathbb{S}^1 \longrightarrow SO(2)$ be the matrix valued map such that

$$\forall x \in \mathbb{S}^1, \quad \begin{pmatrix} t_k(x) \\ n_k(x) \end{pmatrix} = \mathcal{C}_k(x) \cdot \begin{pmatrix} t_{k-1}(x) \\ n_{k-1}(x) \end{pmatrix}$$



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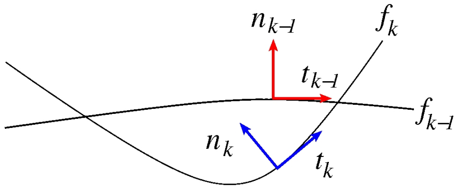

• Such a matrix C_k is called a **corrugation matrix.**

C^1-Fractal Structure

• Corrugation matrices encode the data of the successive convex integrations :

$$\mathcal{C}_k(x) := \begin{pmatrix} \cos\theta_k(x) & \sin\theta_k(x) \\ -\sin\theta_k(x) & \cos\theta_k(x) \end{pmatrix}$$

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$$\begin{pmatrix} t_\infty \\ n_\infty \end{pmatrix} = \left(\prod_{k=1}^{\infty} \mathcal{C}_k \right) \cdot \begin{pmatrix} t_0 \\ n_0 \end{pmatrix}$$

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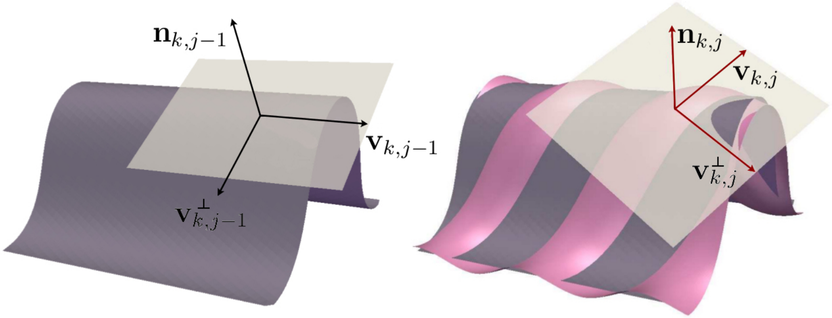
• Identifying $\mathbb{R}^2$ and $\mathbb{C}$, we have

$$\left(\prod_{k=1}^{\infty} \mathcal{C}_k\right) = e^{-iA(x)} \quad \text{with} \quad A(x) = \sum_k \alpha_k \cos(2\pi N_k x).$$

C^1-Fractal Structure



From the circle to the torus.– We denote by $\mathcal{C}_{k,j}$ the $SO(3)$ matrix such that :

$$\begin{pmatrix} v^{\perp}_{k,j} \\ v_{k,j} \\ n_{k,j} \end{pmatrix} = \mathcal{C}_{k,j} \cdot \begin{pmatrix} v^{\perp}_{k,j-1} \\ v_{k,j-1} \\ n_{k,j-1} \end{pmatrix}.$$

C^1-Fractal Structure

Let $f_\infty : \mathbb{T}^2 \longrightarrow \mathbb{E}^3$ be the limit of the maps :

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Beware ! – Unlike the 1-dimensional case, the analytic expressions of the $\mathcal{C}_{k,j}$'s are simply ugly... but fortunately, their asymptotic expressions are nice.

A digression : the loss of derivatives

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$$f(t) := f_0(0) + \int_0^t r(u)\mathbf{e}^{i\alpha(u)\cos 2\pi N u}\,\mathrm{d}u.$$

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• In particular

$$\frac{\partial f}{\partial t}(t) = r(t)\mathbf{e}^{i\alpha(t)\cos 2\pi N t},$$

therefore, if f_0 is C^k then f is C^k also.

A digression : the loss of derivatives

• In the two dimensional setting we have :

$$f(t,s) := f_0(0,s) + \int_0^t r(u,s)\mathbf{e}^{i\alpha(u,s)\cos 2\pi N u}\,\mathrm{d}u + \text{gluing term}$$

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• The integral over the variable t can not recover the loss of derivative due to the presence of the partial derivative $\partial_s f$ in the definition of $\mathbf{n}$. Therefore if f_0 is C^k then, generically, f is C^{k-1} only.

C^1-Fractal Structure

Corrugation Theorem ($\sim$, Jabrane, Lazarus, Thibert, 2012).– *For every $p \in \mathbb{T}^2$, we have*

$$\mathcal{C}_{k,j+1}(p) = \mathcal{L}_{k,j+1}(p) \cdot \mathcal{R}_{k,j}(p)$$

where

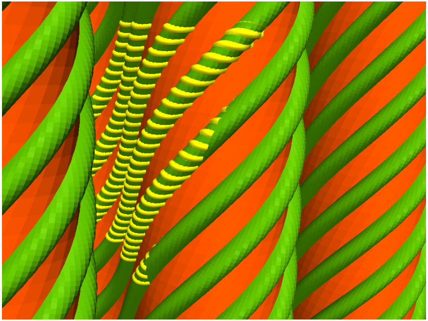
$$\mathcal{L}_{k,j+1} := \begin{pmatrix} \cos\theta_{k,j+1} & 0 & \sin\theta_{k,j+1} \\ 0 & 1 & 0 \\ -\sin\theta_{k,j+1} & 0 & \cos\theta_{k,j+1} \end{pmatrix} + O\left(\frac{1}{N_{k,j+1}}\right)$$

with $\theta_{k,j+1}(p) := \alpha_k(p)\cos(2\pi N_k u)$ *and*

$$\mathcal{R}_{k,j} := \begin{pmatrix} -\sin\beta_j & -\cos\beta_j & 0 \\ \cos\beta_j & -\sin\beta_j & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(\|\Delta_{k,j}\|)$$

where $\Delta_{k,j} = \langle .,. \rangle_{\mathbb{E}^2} - f_{k,j}^* \langle .,. \rangle_{\mathbb{E}^3}$ *is the isometric default.*

C^1-Fractal Structure



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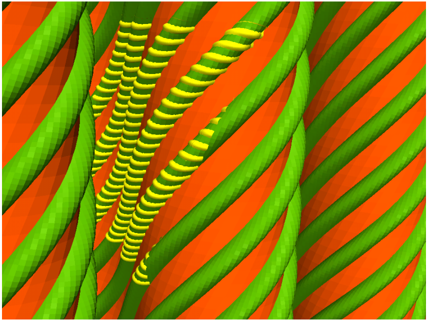

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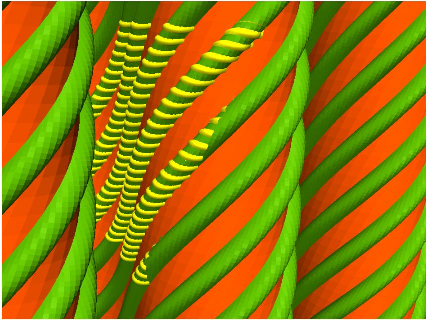

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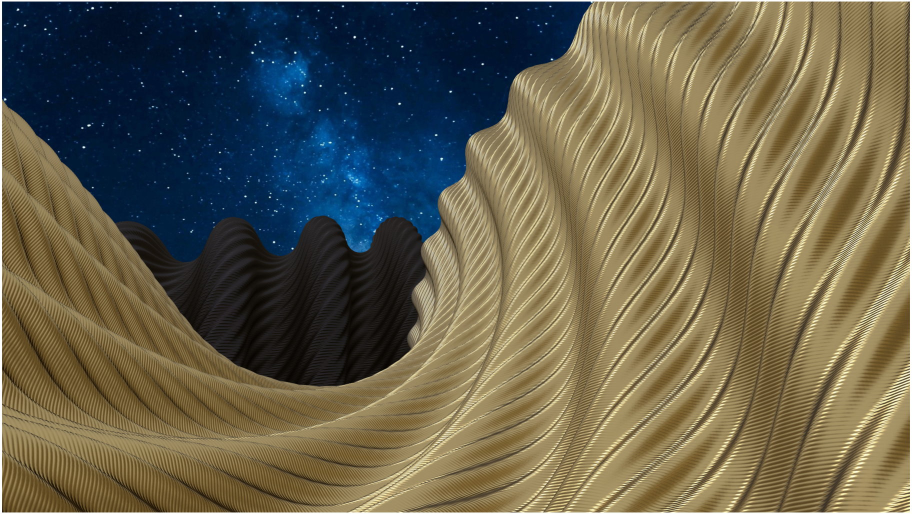
- ... and strongly suggests that f_∞ is C^1-fractal in the following sense : it is C^1 and the graph of its normal map is fractal.

More (pictures, articles) on the Hevea Web Site



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The Hevea Team