The *H*-principle for Isometric Embeddings

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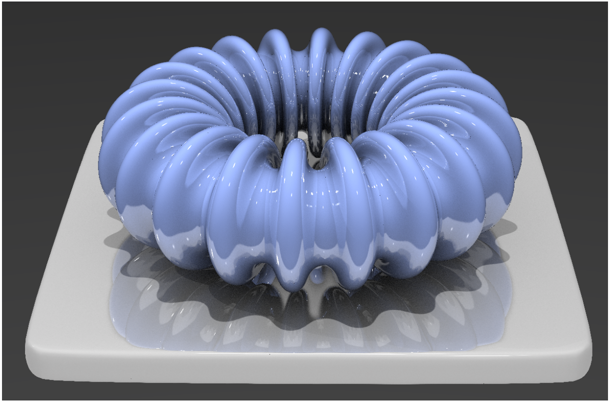

Isometric Maps

Definition.– A map $f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ is an *isometry* if $f^*\langle ., . \rangle = g$.

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The Janet dimension :

$$s_n = \frac{n(n+1)}{2}.$$

Nash-Kuiper Theorem

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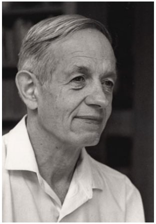
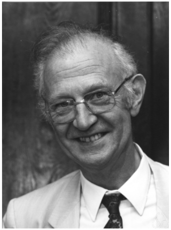


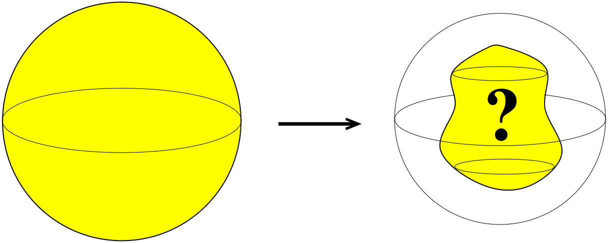
John Forbes Nash



Nicolaas Kuiper

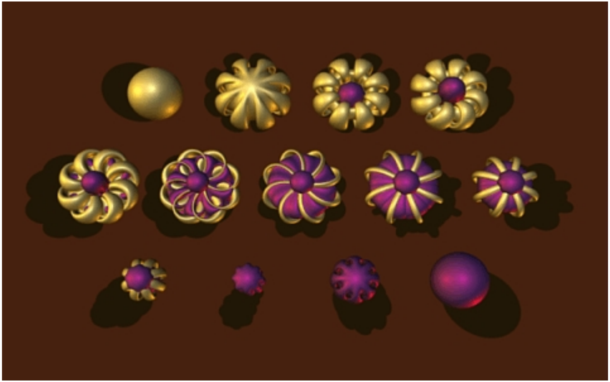
Theorem (1954-55-86).– *Let M^n be a compact Riemannian manifold and $f_0 : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ be a strictly short embedding. Then, for every $\epsilon > 0$, there exists a C^1 isometric embedding $f : (M^n, g) \longrightarrow \mathbb{E}^q$ such that $\|f - f_0\|_{C^0} \leq \epsilon$.*

Nash-Kuiper Spheres



Nash-Kuiper Spheres.– *Let* $0 < r < 1$. *There exists a* C^1*-isometric embedding of the unit sphere of* $\mathbb{E}^3$ *in a ball of radius* r.

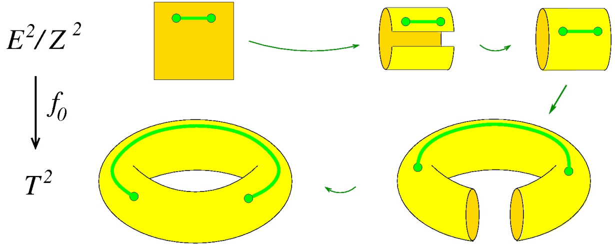
Sphere Eversion



Sphere Eversion.– *The sphere $\mathbb{S}^2 \subset \mathbb{E}^3$ can be turned inside out by a regular homotopy of isometric C^1 immersions.*

Flat Tori

Definition.– Any quotient $\mathbb{E}^2/\Lambda$ of the Euclidean 2-space by a lattice $\Lambda \subset \mathbb{E}^2$ is called a *flat torus*



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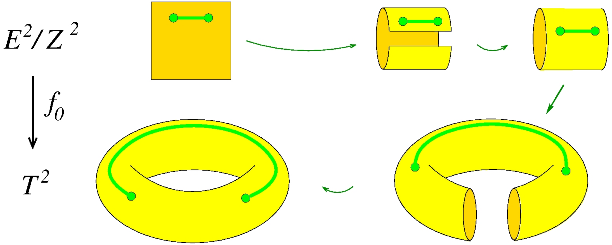


Flat Tori.– *Any flat torus $\mathbb{E}^2/\Lambda$ admits a C^1 isometric embedding in $\mathbb{E}^3$.*

Our Goal

Goal of this talk.– To recover the Nash-Kuiper result on C^1 isometric embeddings from the machinery of the Gromov Integration Theory.

Our main ingredient.– The Gromov Theorem :

Let $\mathcal{R} \subset J^1(M, N)$ be an open and ample differential relation. Then $\mathcal{R}$ satisfies the parametric h-principle i. e.

$$J : \mathcal{S}ol(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$$

is a weak homotopy equivalence.

Our main obstacles.–

- The isometric relation is not ample
- The isometric relation is closed

Decomposition

• For simplicity $M^n = [0,1]^n$.

• The image of the metric distorsion

$$\Delta := g - f_0^*\langle .,.\rangle_{\mathbb{E}^q}$$

lies inside the positive cone $\mathcal{M}$ of inner products of $\mathbb{E}^n$.

• There exist $S \geq \frac{n(n+1)}{2}$ linear forms $\ell_1, \ldots \ell_S$ of $\mathbb{E}^n$ such that

$$g - f_0^*\langle .,.\rangle_{\mathbb{E}^q} = \sum_{j=1}^{S} \rho_j \ell_j \otimes \ell_j$$

where $\rho_j > 0$.

Adapting the Gromov machinery

The strategy is to do the successive convex integrations along the S directions corresponding to the S linear forms $\ell_1, ..., \ell_S$.

rather than

along the n directions of the coordinates in $[0, 1]^n$.

Adapting the Gromov machinery

- This will produce S intermediary maps

$$f_1, \ldots, f_S$$

such that

$$\begin{array}{rcl} g - f_1^*\langle .,. \rangle_{\mathbb{E}^q} & \approx & \rho_2\ell_2^2 + \rho_3\ell_3^2 + \ldots + \rho_S\ell_S^2 \\ g - f_2^*\langle .,. \rangle_{\mathbb{E}^q} & \approx & \rho_3\ell_3^2 + \ldots + \rho_S\ell_S^2 \\ \vdots & & \vdots \\ g - f_{S-1}^*\langle .,. \rangle_{\mathbb{E}^q} & \approx & \rho_S\ell_S^2 \\ g - f_S^*\langle .,. \rangle_{\mathbb{E}^q} & \approx & 0. \end{array}$$

- The map $f := f_S$ is then a solution of $\widetilde{\mathcal{R}} = Op(\mathcal{R})$.

Adapting the Gromov machinery

We have

$$f_j^*\langle .,.\rangle_{\mathbb{E}^q} - f_{j-1}^*\langle .,.\rangle_{\mathbb{E}^q} = (g - f_{j-1}^*\langle .,.\rangle_{\mathbb{E}^q}) - (g - f_j^*\langle .,.\rangle_{\mathbb{E}^q}) \approx \rho_j \ell_j \otimes \ell_j.$$

Hence, the fundamental problem is the following :

Fundamental Problem.– *Given a positive function* ρ*, a linear form* $\ell \neq 0$ *and an embedding* f_0 *how to build an other embedding* f *such that*

$$f^*\langle .,.\rangle_{\mathbb{E}^q} \approx \mu$$

where $\mu := f_0^*\langle .,.\rangle_{\mathbb{E}^q} + \rho\,\ell \otimes \ell$ *?*

The one dimensional case

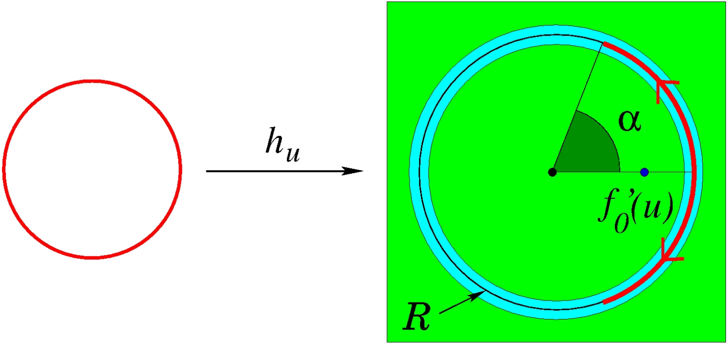
One dimensional fundamental problem.– Let $f_0 : [0,1] \longrightarrow \mathbb{E}^q$ be an embedding, ρ a positive function, $\ell \neq 0$ a linear form on $\mathbb{R}$, how to build an other embedding $f : [0,1] \longrightarrow \mathbb{E}^q$ such that

$$\forall u \in [0,1], \quad \|f'(u)\|^2 \approx \|f_0'(u)\|^2 + \rho(u)\ell^2(\partial_u) \quad ?$$

- For short, we set

$$r(u) := \sqrt{\|f_0'(u)\|^2 + \rho(u)\ell^2(\partial_u)}.$$

Choosing the Loops



with

$$\forall s \in \mathbb{R}/\mathbb{Z}, \qquad h_u(s) := r(u)\mathbf{e}^{i\alpha(u)\cos(2\pi s)}$$

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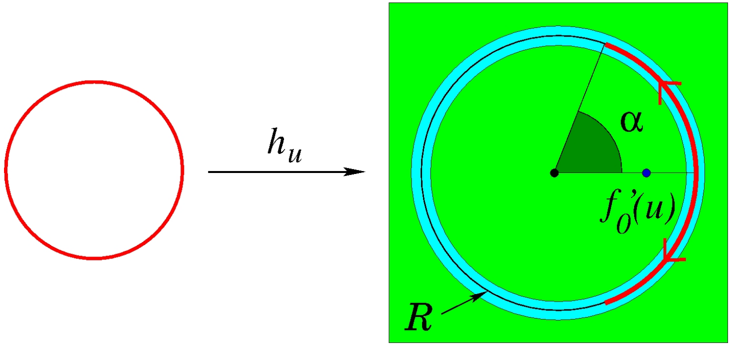

with

$$\forall s \in \mathbb{R}/\mathbb{Z}, \qquad h_u(s) := r(u)\mathbf{e}^{i\alpha(u)\cos(2\pi s)}$$

and $\alpha(u) > 0$ is such that

$$\int_0^1 r(u)\mathbf{e}^{i\alpha(u)\cos(2\pi s)}\,\mathrm{d}s = f_0'(u).$$

Our Convex Integration Process

- The *convex integration formula* :

$$f(t) := f_0(0) + \int_0^t r(u)\mathbf{e}^{i\alpha(u)\cos 2\pi N u}\,\mathrm{d}u.$$

where $\mathbf{e}^{i\theta} := \cos\theta\,\mathbf{t} + \sin\theta\,\mathbf{n}$ with $\mathbf{t} := \frac{f_0'}{\|f_0'\|}$ and $\mathbf{n}$ is a unit normal to the curve f_0.

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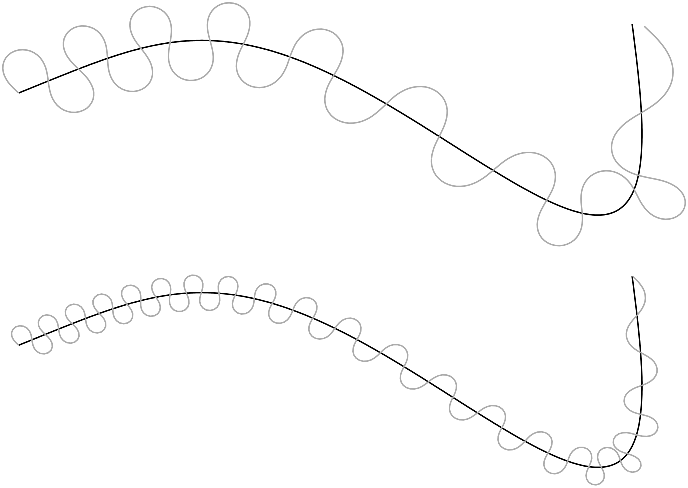
Lemma.– *The map f solves the one dimensional fundamental problem. Its speed $\|f'\|$ is equal to the given function $r = (\|f_0'\|^2 + \rho\ell^2(\partial_c))^{\frac{1}{2}}$. Moreover*

$$\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right)$$

and if N is large enough f is an embedding.

Our Convex Integration Process



A short curve f_0 (black) and the curve f obtained with the one dimensional convex integration formula (grey, $N = 9$ and $N = 20$).

A technical difficulty

- We assume for simplicity that $\ker\ \ell = Span(e_2, ..., e_n)$ and $\ell(e_1) = 1$.

- Let $s \in [0,1]$ and $c = (c_2, ..., c_n) \in [0,1]^{n-1}$, we set

$$f(s,c) := f_0(0,c) + \int_0^s r(u,c)\mathbf{e}^{i\alpha(u,c)\cos 2\pi N u}\,\mathrm{d}u$$

with $r = \sqrt{\mu(e_1, e_1)} = \sqrt{\|df_0(e_1)\|^2 + \rho}$.

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• The map f is **not** a solution of our Fundamental Problem. We do not have

$$\|f^*\langle ., .\rangle_{\mathbb{E}^q} - \mu\|_{C^0} = O\left(\frac{1}{N}\right)$$

with $\mu := f_0^*\langle ., .\rangle_{\mathbb{E}^q} + \rho\,\ell \otimes \ell$.

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- Thus, for all $1 < i$, $1 < j$, we have

$$\begin{aligned}
(f^*\langle .,.\rangle_{\mathbb{E}^q})(e_i,e_j) &= \langle df(e_i), df(e_j)\rangle_{\mathbb{E}^q} \\
&= \langle df_0(e_i), df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right) \\
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- Therefore

$$\|(f^*\langle.,.\rangle_{\mathbb{E}^q} - \mu)(e_i,e_j)\|_{C^0} = O\left(\frac{1}{N}\right)$$

for all $1 < i$, $1 < j$.

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$$df_{(s,c)}(e_1) = r(s,c)\mathbf{e}^{i\alpha(s,c)\cos 2\pi N s}.$$

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- Therefore

$$\|(f^*\langle .,.\rangle_{\mathbb{E}^q} - \mu)(e_1,e_1)\|_{C^0} = 0.$$

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- Indeed, in the one hand

$$
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(f^*\langle .,.\rangle_{\mathbb{E}^q})(e_1,e_j) &= \langle df(e_1), df(e_j)\rangle_{\mathbb{E}^q}\\
&= \langle df(e_1), df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right)\\
&= \langle r\mathbf{e}^{i\bigstar}, df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right)\\
&= \langle r\cos(\bigstar)\mathbf{t}, df_0(e_j)\rangle_{\mathbb{E}^q}\\
&\quad +\langle r\sin(\bigstar)\mathbf{n}, df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right)\\
&= \frac{r\cos(\bigstar)}{\|df_0(e_1)\|}\langle df_0(e_1), df_0(e_j)\rangle_{\mathbb{E}^q}\\
&\quad +O\left(\frac{1}{N}\right)
\end{aligned}
$$

A technical difficulty

- In the other hand, since $\ell(e_j) = 0$, we have

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- Therefore

$$\|(f^*\langle ., .\rangle_{\mathbb{E}^q} - \mu)(e_1, e_j)\|_{C^0} \neq O\left(\frac{1}{N}\right)$$

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unless $\langle df_0(e_i), df_0(e_j)\rangle_{\mathbb{E}^q} \equiv 0$.

Claim.– This difficulty vanishes if the convex integration is done along the integral lines of a vector field W such that

$$\forall j \in \{2, ..., m\}, \quad \mu(W, e_j) = 0$$

i. e. W is μ-orthogonal to ker ℓ.

Adjusting the convex integration formula

Proposition.– *The resulting map f solves the fundamental problem. Precisely*

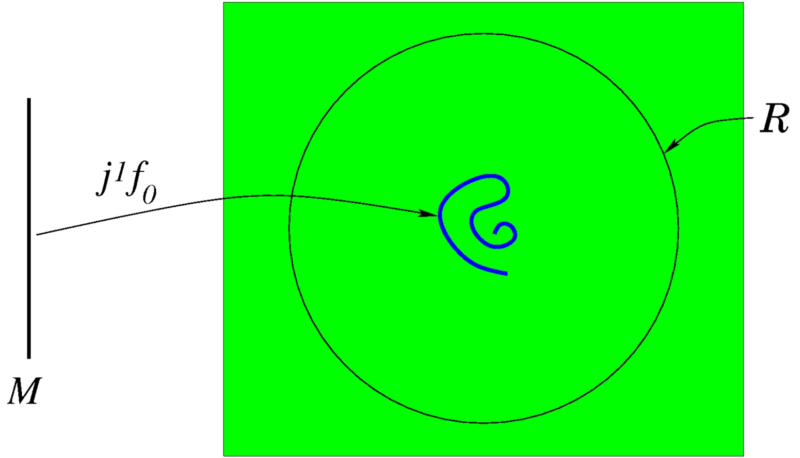
$$\|f^*\langle .,.\rangle_{\mathbb{E}^q}-\mu\| = O\left(\frac{1}{N}\right)$$

where $\mu = f_0^*\langle .,.\rangle_{\mathbb{E}^q}+\rho\,\ell\otimes\ell$. *Moreover*

1) $\|f-f_0\|_{C^0} = O\left(\frac{1}{N}\right)$,
2) $\|df-df_0\|_{C^0} \leq \frac{Cte}{N}+\sqrt{7}\rho^{\frac{1}{2}}|\ell(W)|$,

and if N is large enough, f is an embedding.

Thickening the Differential Relation



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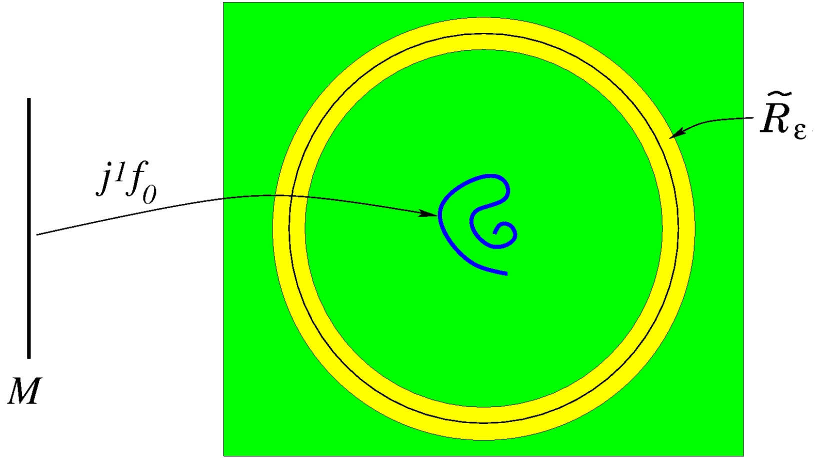

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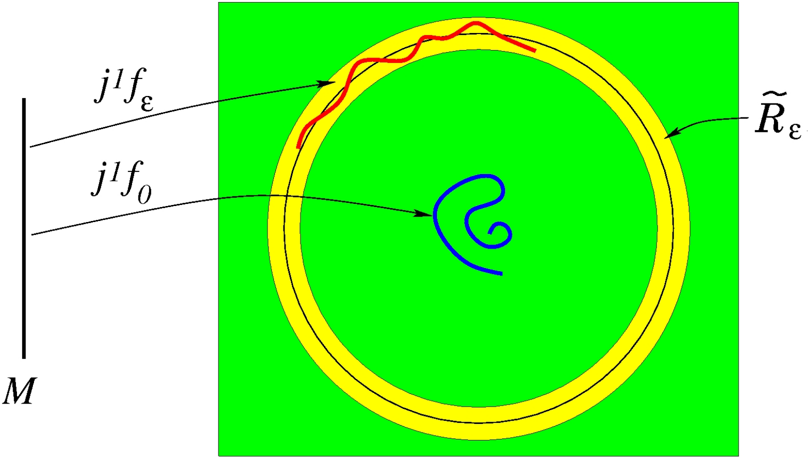

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Question.– Is the limit

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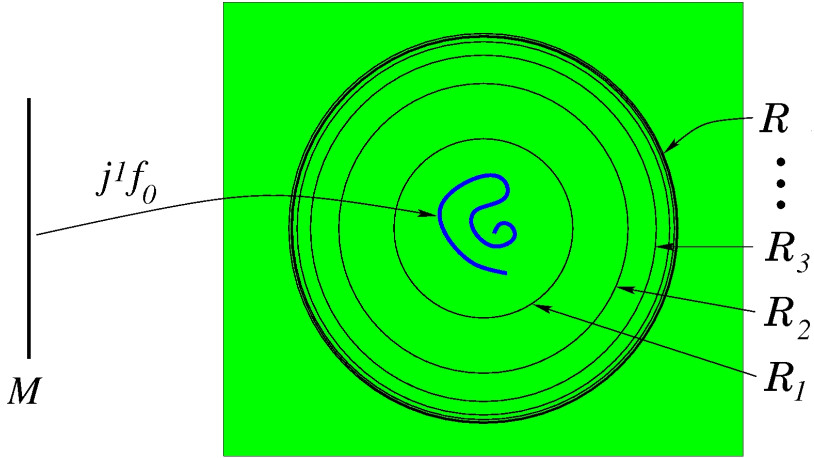
Answer.– No !

$$\lim_{\epsilon \longrightarrow 0} f_\epsilon = f_0.$$

Approximating the Differential Relation



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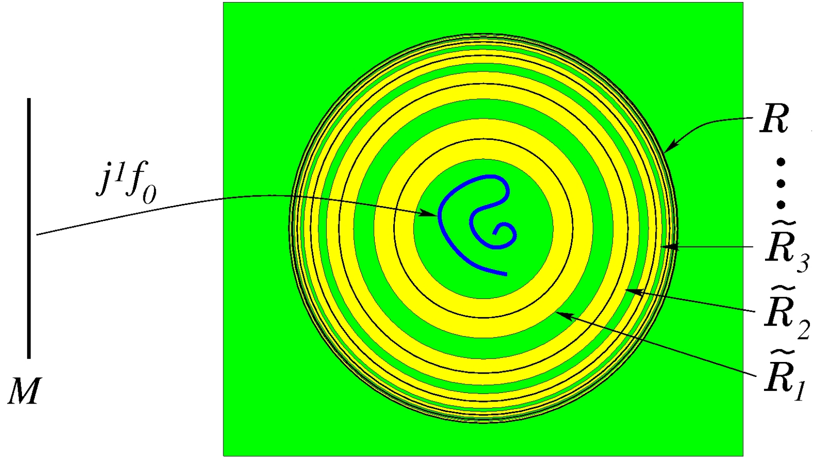
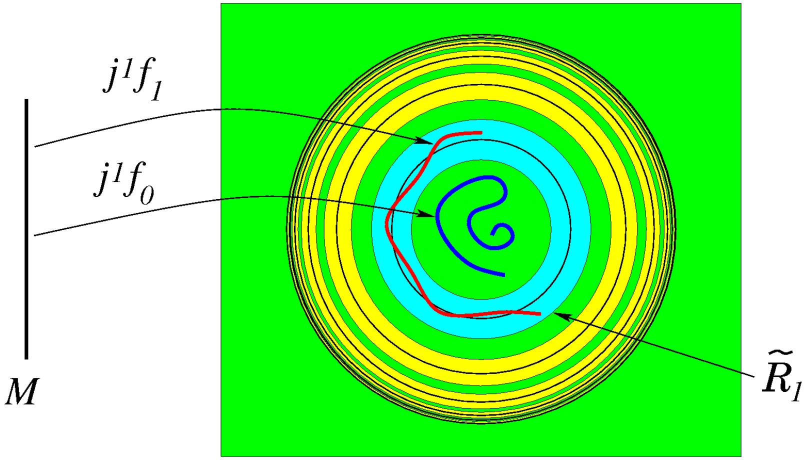


Iterated Convex Integrations



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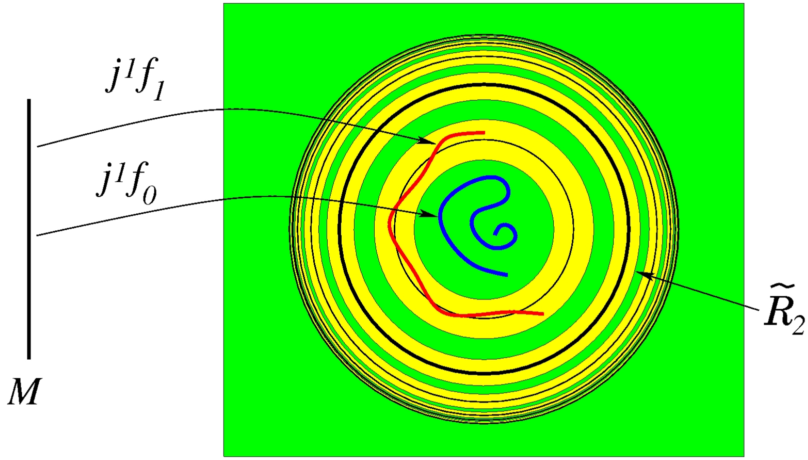

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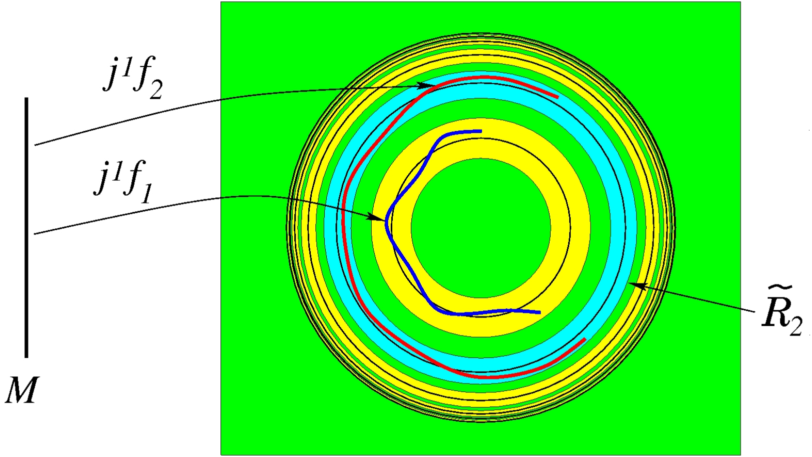

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Answer.– Yes !

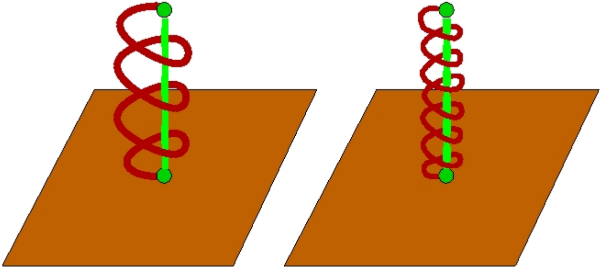
Let us see why
i) it is C^0 converging,
ii) it is C^1 converging.

Consequently

$$f_\infty := \lim_{k\longrightarrow+\infty} f_k \text{ is a } C^1 \text{ isometric map.}$$

C^0 Convergence

It is enough to control the difference $\|f_k - f_{k-1}\|_{C^0}$.



We set

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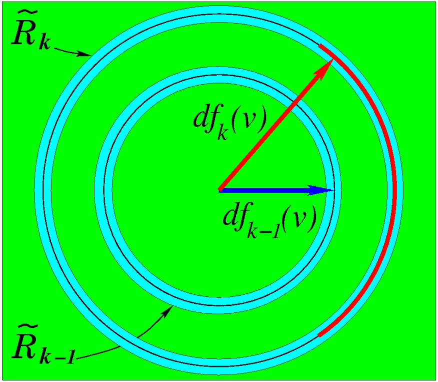

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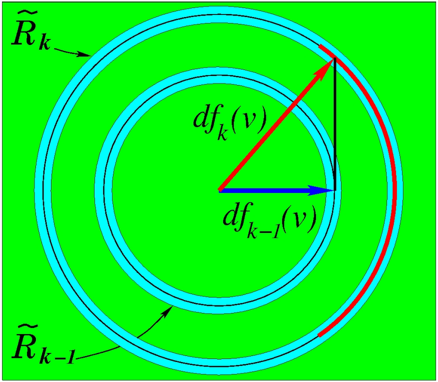

$$\|df_k - df_{k-1}\|_{C^0} \leq C^{te}\sqrt{dist(\widetilde{\mathcal{R}}_{k-1}, \widetilde{\mathcal{R}}_k)}$$

John Nash