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Isometric Maps

How to deal with the non-ampleness?

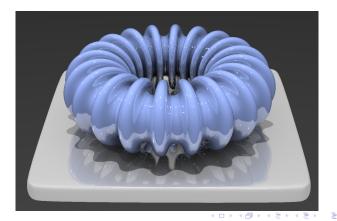
The one dimensiona case

How to deal with a closed relation?

# The H-principle for Isometric Embeddings

Vincent Borrelli

Université Lyon 1



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## **Isometric Maps**

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**Definition.** A map  $f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$  is an *isometry* if  $f^* \langle ., . \rangle = g$ .

The length of curves is preserved by an isometric map.

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# Isometric Maps

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The length of curves is preserved by an isometric map. In a local coordinate system  $x = (x_1, ..., x_n)$ :

$$1 \leq i \leq j \leq n, \quad \langle \frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x) \rangle = g_{ij}(x)$$

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## Isometric Maps

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Isometric Maps

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The length of curves is preserved by an isometric map. In a local coordinate system  $x = (x_1, ..., x_n)$ :

$$1 \leq i \leq j \leq n$$
,  $\langle \frac{\partial f}{\partial x_i}(x), \frac{\partial f}{\partial x_j}(x) \rangle = g_{ij}(x)$ 

The Janet dimension :

$$\mathbf{s}_n = \frac{n(n+1)}{2}.$$

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# **Definition.**– A map $f : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ is called *strictly short*

if  $f^*\langle .., . \rangle < g$ .



# Nash-Kuiper Theorem

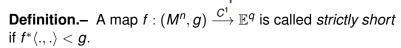
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John Forbes Nash

Nicolaas Kuiper

**Theorem (1954-55-86).**— Let  $M^n$  be a compact Riemannian manifold and  $f_0 : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$  be a strictly short embedding. Then, for every  $\epsilon > 0$ , there exists a  $C^1$  isometric embedding  $f : (M^n, g) \longrightarrow \mathbb{E}^q$  such that  $\|f - f_0\|_{C^0} \le \epsilon$ .

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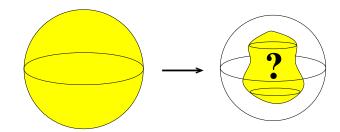
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## Nash-Kuiper Spheres



**Nash-Kuiper Spheres.**– Let 0 < r < 1. There exists a  $C^1$ -isometric embedding of the unit sphere of  $\mathbb{E}^3$  in a ball of radius r.

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## Sphere Eversion



**Sphere Eversion.**– The sphere  $\mathbb{S}^2 \subset \mathbb{E}^3$  can be turned inside out by a regular homotopy of isometric  $C^1$  immersions.

## Flat Tori

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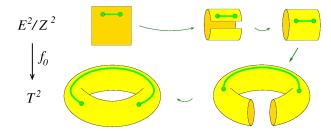
### Isometric Maps

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# **Definition.**– Any quotient $\mathbb{E}^2/\Lambda$ of the Euclidean 2-space by a lattice $\Lambda \subset \mathbb{E}^2$ is called a *flat torus*



## Flat Tori

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## The H-principle for Isometric Embeddings

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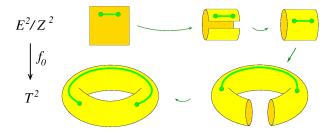
## Isometric Maps

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# **Definition.**– Any quotient $\mathbb{E}^2/\Lambda$ of the Euclidean 2-space by a lattice $\Lambda \subset \mathbb{E}^2$ is called a *flat torus*



# **Flat Tori.**– Any flat torus $\mathbb{E}^2/\Lambda$ admits a $C^1$ isometric embedding in $\mathbb{E}^3$ .

## Our Goal

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**Goal of this talk.**– To recover the Nash-Kuiper result on  $C^1$  isometric embeddings from the machinery of the Gromov Integration Theory.

Our main ingredient.- The Gromov Theorem :

Let  $\mathcal{R} \subset J^1(M, N)$  be an open and ample differential relation. Then  $\mathcal{R}$  satisfies the parametric h-principle i. e.

 $J:\mathcal{Sol}(\mathcal{R})\longrightarrow \Gamma(\mathcal{R})$ 

is a weak homotopy equivalence.

## Our main obstacles.-

- The isometric relation is not ample
- The isometric relation is closed

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# Decomposition

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- For simplicity  $M^n = [0, 1]^n$ .
- The image of the metric distorsion

$$\Delta:=g-\mathit{f}_{0}^{*}\langle.,.
angle_{\mathbb{E}^{q}}$$

lies inside the positive cone  $\mathcal{M}$  of inner products of  $\mathbb{E}^n$ .

• There exist  $S \geq \frac{n(n+1)}{2}$  linear forms  $\ell_1, \ldots \ell_S$  of  $\mathbb{E}^n$  such that

$$g-f_0^*\langle .,.
angle_{\mathbb{R}^q}=\sum_{j=1}^{S}
ho_j\ell_j\otimes\ell_j$$

where  $\rho_i > 0$ .

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# Adapting the Gromov machinery

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**The strategy** is to do the successive convex integrations along the *S* directions corresponding to the *S* linear forms  $\ell_1, ..., \ell_S$ .

rather than

along the *n* directions of the coordinates in  $[0, 1]^n$ .

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# Adapting the Gromov machinery

• This will produce S intermediary maps

$$f_1, ..., f_S$$

## such that

$$\begin{array}{ll} g-f_1^*\langle .,.\rangle_{\mathbb{E}^q} &\approx \rho_2 \ell_2^2 + \rho_3 \ell_3^2 + \ldots + \rho_S \ell_S^2 \\ g-f_2^*\langle .,.\rangle_{\mathbb{E}^q} &\approx \qquad \rho_3 \ell_3^2 + \ldots + \rho_S \ell_S^2 \\ &\vdots &\vdots \\ g-f_{S-1}^*\langle .,.\rangle_{\mathbb{E}^q} &\approx \qquad \rho_S \ell_S^2 \\ g-f_S^*\langle .,.\rangle_{\mathbb{E}^q} &\approx \qquad 0. \end{array}$$

• The map  $f := f_S$  is then a solution of  $\widetilde{\mathcal{R}} = Op(\mathcal{R})$ .

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# Adapting the Gromov machinery

## We have

$$egin{aligned} & f_j^*\langle.,.
angle_{\mathbb{E}^q}-f_{j-1}^*\langle.,.
angle_{\mathbb{E}^q}=&(oldsymbol{g}-f_{j-1}^*\langle.,.
angle_{\mathbb{E}^q})-(oldsymbol{g}-f_j^*\langle.,.
angle_{\mathbb{E}^q})\ &pprox\ &
ho_j\ell_j\otimes\ell_j. \end{aligned}$$

Hence, the fundamental problem is the following :

**Fundamental Problem.**– Given a positive function  $\rho$ , a linear form  $\ell \neq 0$  and an embedding  $f_0$  how to build an other embedding f such that

$$f^*\langle .,.
angle_{\mathbb{E}^q}pprox \mu$$

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where  $\mu := f_0^* \langle ., . \rangle_{\mathbb{E}^q} + \rho \, \ell \otimes \ell$  ?

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# The one dimensional case

## One dimensional fundamental problem.- Let

 $f_0: [0, 1] \longrightarrow \mathbb{E}^q$  be an embedding,  $\rho$  a positive function,  $\ell \neq 0$  a linear form on  $\mathbb{R}$ , how to build an other embedding  $f: [0, 1] \longrightarrow \mathbb{E}^q$  such that

$$\forall u \in [0, 1], \quad \|f'(u)\|^2 \approx \|f'_0(u)\|^2 + \rho(u)\ell^2(\partial_u)$$
 ?

• For short, we set

$$r(u) := \sqrt{\|f'_0(u)\|^2 + \rho(u)\ell^2(\partial_u)}.$$

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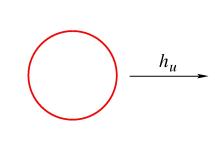
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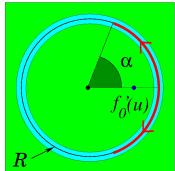
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# Choosing the Loops



with

 $orall oldsymbol{s} \in \mathbb{R}/\mathbb{Z}, \qquad h_u(oldsymbol{s}) := r(u) \mathbf{e}^{i lpha(u) \cos(2\pi s)}$ 

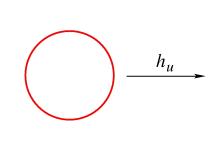
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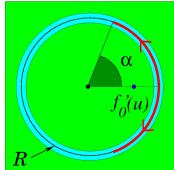
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# Choosing the Loops



## with

$$orall oldsymbol{s} \in \mathbb{R}/\mathbb{Z}, \qquad h_u(oldsymbol{s}) := r(u) oldsymbol{e}^{i lpha(u) \cos(2\pi s)}$$

and  $\alpha(u) > 0$  is such that

$$\int_0^1 r(u) \mathbf{e}^{i\alpha(u)\cos(2\pi s)} \,\mathrm{d}s = f_0'(u).$$

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## **Our Convex Integration Process**

• The convex integration formula :

$$f(t) := f_0(0) + \int_0^t r(u) \mathbf{e}^{i\alpha(u)\cos 2\pi \mathbf{N} u} \,\mathrm{d} u.$$

where  $\mathbf{e}^{i\theta} := \cos \theta \mathbf{t} + \sin \theta \mathbf{n}$  with  $\mathbf{t} := \frac{f'_0}{\|f'_0\|}$  and  $\mathbf{n}$  is a unit normal to the curve  $f_0$ .

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# **Our Convex Integration Process**

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where  $\mathbf{e}^{i\theta} := \cos \theta \mathbf{t} + \sin \theta \mathbf{n}$  with  $\mathbf{t} := \frac{t'_0}{\|t'_0\|}$  and  $\mathbf{n}$  is a unit normal to the curve  $f_0$ .

**Lemma.**– The map *f* solves the one dimensional fundamental problem. Its speed ||f'|| is equal to the given function  $r = (||f'_0||^2 + \rho \ell^2(\partial_c))^{\frac{1}{2}}$ . Moreover

$$\|f-f_0\|_{C^0}=O\left(\frac{1}{N}\right)$$

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and if N is large enough f is an embedding.

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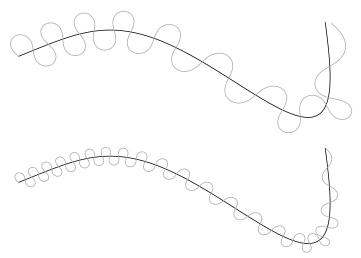
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# **Our Convex Integration Process**



A short curve  $f_0$  (black) and the curve f obtained with the one dimensional convex integration formula (grey, N = 9 and N = 20).

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# A technical difficulty

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• We assume for simplicity that ker  $\ell = Span(e_2, ..., e_n)$  and  $\ell(e_1) = 1$ .

• Let 
$$s \in [0,1]$$
 and  $c = (c_2,...,c_n) \in [0,1]^{n-1},$  we set

$$f(s,c) := f_0(0,c) + \int_0^s r(u,c) \mathbf{e}^{ilpha(u,c)\cos 2\pi N u} \, \mathrm{d} u$$

with 
$$r = \sqrt{\mu(e_1, e_1)} = \sqrt{\|df_0(e_1)\|^2 + \rho}$$
.

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# A technical difficulty

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with 
$$r = \sqrt{\mu(e_1, e_1)} = \sqrt{\|df_0(e_1)\|^2 + \rho}$$
.

## • The map *f* is **not** a solution of our Fundamental Problem. We do not have

$$\|f^*\langle .,.\rangle_{\mathbb{E}^q}-\mu\|_{\mathcal{C}^0}=O\left(rac{1}{N}
ight)$$

with  $\mu := f_0^* \langle ., . \rangle_{\mathbb{E}^q} + \rho \, \ell \otimes \ell$ .

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# A technical difficulty

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• By  $C^{1,\hat{1}}$ -density we have

$$df(e_j) = df_0(e_j) + O(\frac{1}{N})$$

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# A technical difficulty

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• By  $C^{1,\hat{1}}$ -density we have

$$df(e_j) = df_0(e_j) + O(\frac{1}{N})$$

• Thus, for all 1 < i, 1 < j, we have  $(f^* \langle ... \rangle_{\mathbb{R}^q})(e_i, e_i) = \langle df(e_i), df(e_i) \rangle$ 

$$egin{aligned} & {}^*\langle .,.
angle_{\mathbb{E}^q})(m{e}_i,m{e}_j) &= \langle df(m{e}_i),df(m{e}_j)
angle_{\mathbb{E}^q} \ &= \langle df_0(m{e}_i),df_0(m{e}_j)
angle_{\mathbb{E}^q} + O\left(rac{1}{N}
ight) \ &= \mu(m{e}_i,m{e}_j) + O\left(rac{1}{N}
ight) \end{aligned}$$

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• By  $C^{1,\hat{1}}$ -density we have

$$df(e_j)=df_0(e_j)+O(rac{1}{N})$$

• Thus, for all 1 < i, 1 < j, we have

$$\begin{aligned} (f^* \langle ., . \rangle_{\mathbb{E}^q})(\boldsymbol{e}_i, \boldsymbol{e}_j) &= \langle df(\boldsymbol{e}_i), df(\boldsymbol{e}_j) \rangle_{\mathbb{E}^q} \\ &= \langle df_0(\boldsymbol{e}_i), df_0(\boldsymbol{e}_j) \rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right) \\ &= \mu(\boldsymbol{e}_i, \boldsymbol{e}_j) + O\left(\frac{1}{N}\right) \end{aligned}$$

• Therefore

$$\|(f^*\langle .,.\rangle_{\mathbb{E}^q}-\mu)(\boldsymbol{e}_i,\boldsymbol{e}_j)\|_{\boldsymbol{C}^0}=O\left(rac{1}{N}
ight)$$

for all 1 < i, 1 < j.

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• By definition of f we have

$$df_{(s,c)}(e_1) = r(s,c) \mathbf{e}^{i\alpha(s,c)\cos 2\pi Ns}.$$

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# A technical difficulty

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• By definition of f we have

$$df_{(s,c)}(e_1) = r(s,c) \mathbf{e}^{i\alpha(s,c)\cos 2\pi Ns}.$$

• Thus

$$egin{array}{rll} (f^*\langle.,.
angle_{\mathbb{E}^q})(e_1,e_1)&=&\langle df(e_1),df(e_1)
angle_{\mathbb{E}^q}\ &=&r^2(s,c)\ &=&\mu(e_1,e_1) \end{array}$$

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# A technical difficulty

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## • By definition of f we have

$$df_{(s,c)}(e_1) = r(s,c)\mathbf{e}^{i\alpha(s,c)\cos 2\pi Ns}.$$

Thus

$$(f^*\langle .,. 
angle_{\mathbb{R}^q})(e_1,e_1) = \langle df(e_1), df(e_1) 
angle_{\mathbb{R}^q}$$
  
 $= r^2(s,c)$   
 $= \mu(e_1,e_1)$ 

## • Therefore

$$\|(f^*\langle .,.\rangle_{\mathbb{E}^q}-\mu)(\boldsymbol{e}_1,\boldsymbol{e}_1)\|_{\boldsymbol{C}^0}=0.$$

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# • The problem arises with the mixted term $\langle df(e_1), df(e_j) \rangle_{\mathbb{E}^q}$ , j > 1.

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# A technical difficulty

- The problem arises with the mixted term  $\langle df(e_1), df(e_j) \rangle_{\mathbb{E}^q}$ , j > 1.
- Indeed, in the one hand

 $(f^*\langle .,$ 

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^{q}}(e_{1}, e_{j}) &= \langle df(e_{1}), df(e_{j}) \rangle_{\mathbb{E}^{q}} \\ &= \langle df(e_{1}), df_{0}(e_{j}) \rangle_{\mathbb{E}^{q}} + O\left(\frac{1}{N}\right) \\ &= \langle r \mathbf{e}^{j \bigstar}, df_{0}(e_{j}) \rangle_{\mathbb{E}^{q}} + O\left(\frac{1}{N}\right) \\ &= \langle r \cos(\bigstar) \mathbf{t}, df_{0}(e_{j}) \rangle_{\mathbb{E}^{q}} \\ &+ \langle r \sin(\bigstar) \mathbf{n}, df_{0}(e_{j}) \rangle_{\mathbb{E}^{q}} + O\left(\frac{1}{N}\right) \\ &= \frac{r \cos(\bigstar)}{\|df_{0}(e_{1})\|} \langle df_{0}(e_{1}), df_{0}(e_{j}) \rangle_{\mathbb{E}^{q}} \\ &+ O\left(\frac{1}{N}\right) \end{aligned}$$

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• In the other hand, since  $\ell(e_j) = 0$ , we have

$$\mu(\boldsymbol{e}_1, \boldsymbol{e}_j) = \langle df_0(\boldsymbol{e}_1), df_0(\boldsymbol{e}_j) 
angle_{\mathbb{E}^q}.$$

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• In the other hand, since  $\ell(e_j) = 0$ , we have

$$\mu(\boldsymbol{e}_1, \boldsymbol{e}_j) = \langle \textit{df}_0(\boldsymbol{e}_1), \textit{df}_0(\boldsymbol{e}_j) 
angle_{\mathbb{R}^q}.$$

## • Therefore

$$\|(f^*\langle .,.
angle_{\mathbb{E}^q}-\mu)(oldsymbol{e}_1,oldsymbol{e}_j)\|_{oldsymbol{C}^0}
eq O\left(rac{1}{N}
ight)$$

unless  $\langle df_0(e_i), df_0(e_j) \rangle_{\mathbb{E}^q} \equiv 0.$ 

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# A technical difficulty

• In the other hand, since  $\ell(e_j) = 0$ , we have

$$\mu(oldsymbol{e}_1,oldsymbol{e}_j)=\langle d\mathit{f}_0(oldsymbol{e}_1), d\mathit{f}_0(oldsymbol{e}_j) 
angle_{\mathbb{E}^q}.$$

## • Therefore

$$\|(f^*\langle .,.
angle_{\mathbb{E}^q}-\mu)(oldsymbol{e}_1,oldsymbol{e}_j)\|_{C^0}
eq O\left(rac{1}{N}
ight)$$

unless  $\langle df_0(e_i), df_0(e_j) \rangle_{\mathbb{E}^q} \equiv 0.$ 

**Claim.**– This difficulty vanishes if the convex integration is done along the integral lines of a vector field *W* such that

$$\forall j \in \{2,...,m\}, \quad \mu(\boldsymbol{W},\boldsymbol{e}_j) = 0$$

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i. e. *W* is  $\mu$ -orthogonal to ker  $\ell$ .

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# Adjusting the convex integration formula

**Proposition.**– The resulting map f solves the fundamental problem. Precisely

$$\|f^*\langle .,.\rangle_{\mathbb{E}^q}-\mu\|=O\left(rac{1}{N}
ight)$$

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where 
$$\mu = f_0^* \langle ., . \rangle_{\mathbb{E}^q} + \rho \, \ell \otimes \ell$$
. Moreover  
1)  $\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right)$ ,  
2)  $\|df - df_0\|_{C^0} \le \frac{Cte}{N} + \sqrt{7}\rho^{\frac{1}{2}}|\ell(W)|$ ,

and if N is large enough, f is an embedding.

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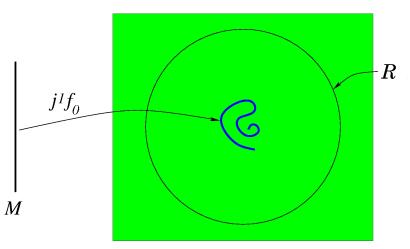
Isometric Maps

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How to deal with a closed relation?

# Thickening the Differential Relation



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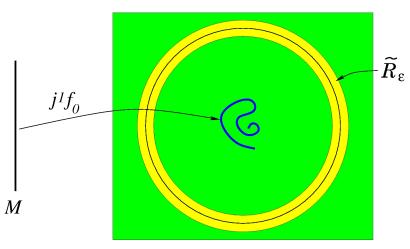
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# Thickening the Differential Relation



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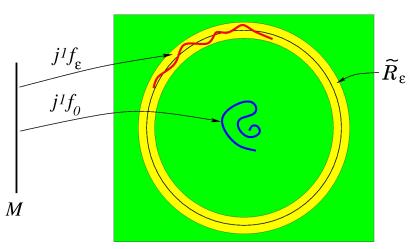
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# Thickening the Differential Relation

### Question.- Is the limit

 $\lim_{\epsilon \longrightarrow 0} f_{\epsilon}$ 

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an isometric map (if it exists)?

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# Thickening the Differential Relation

Question.- Is the limit

 $\lim_{\epsilon \longrightarrow 0} f_{\epsilon}$ 

an isometric map (if it exists)?

Answer.- No!

 $\lim_{\epsilon \longrightarrow 0} f_{\epsilon} = f_0.$ 

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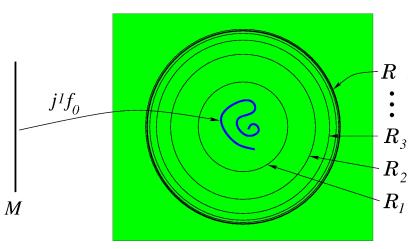
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# Approximating the Differential Relation



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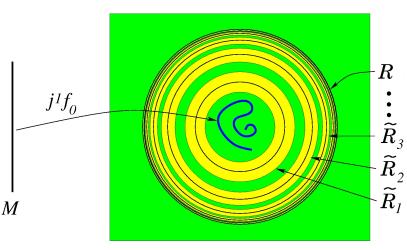
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Approximating the Differential Relation



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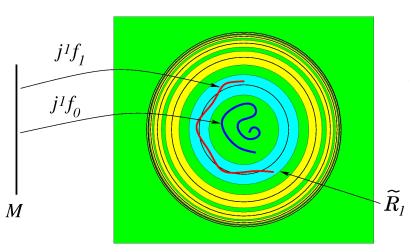
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**Iterated Convex Integrations** 



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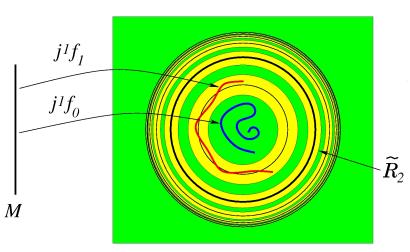
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### **Iterated Convex Integrations**



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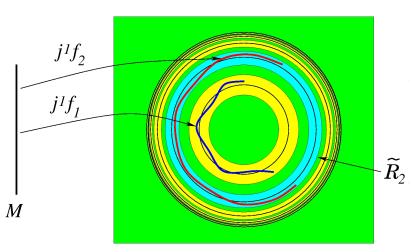
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**Iterated Convex Integrations** 



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# Thickening the Differential Relation

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### Question.- Is the limit

 $\lim_{k \longrightarrow +\infty} f_k$ 

an isometric map (if it exists)?

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# Thickening the Differential Relation

### Question.- Is the limit

 $\lim_{k \longrightarrow +\infty} f_k$ 

an isometric map (if it exists)?

Answer.- Yes!

Let us see why i) it is  $C^0$  converging, ii) it is  $C^1$  converging.

Consequently

$$f_{\infty} := \lim_{k \to +\infty} f_k$$
 is a  $C^1$  isometric map.

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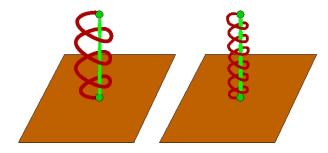
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### $C^0$ Convergence

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### It is enough to control the difference $||f_k - f_{k-1}||_{C^0}$ .



We set

$$f_{\infty} = \lim_{k \to +\infty} f_k.$$

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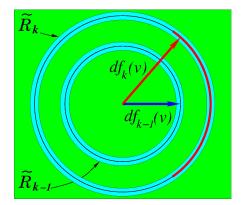
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## $C^1$ convergence

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It is enough to control the difference  $||df_k - df_{k-1}||_{C^0}$ .



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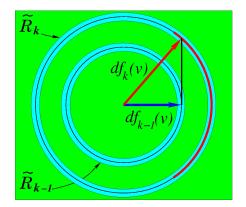
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## $C^1$ convergence

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It is enough to control the difference  $||df_k - df_{k-1}||_{C^0}$ .



 $\|df_k - df_{k-1}\|_{C^0} \leq C^{te} \sqrt{dist(\widetilde{\mathcal{R}}_{k-1}, \widetilde{\widetilde{\mathcal{R}}_k})}$ 

### John Nash

#### Embeddings V.Borrelli

The *H*-principle for

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