Flat 2-Tori in $\mathbb{E}^3$

Vincent Borrelli

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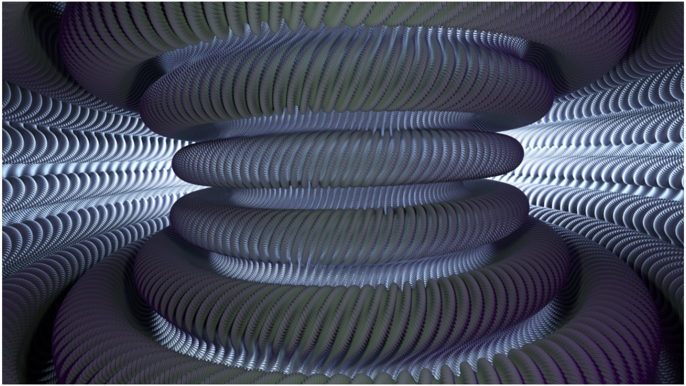


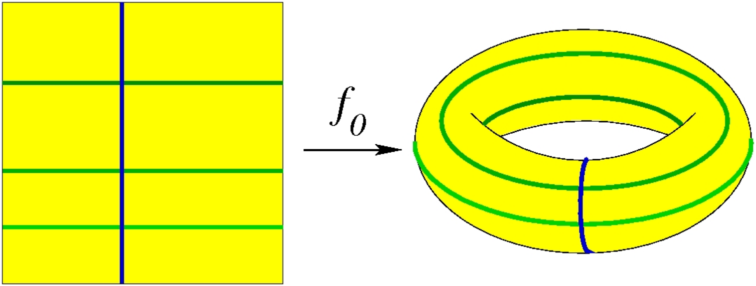
The Hevea Project

Francis Lazarus
Gipsa-Lab, Grenoble

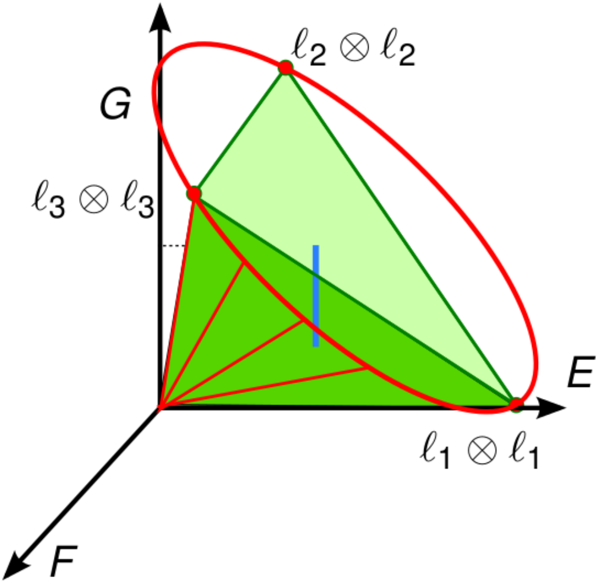
Boris Thibert
LJK, Grenoble

Saïd Jabrane
ICJ, Lyon

The initial map



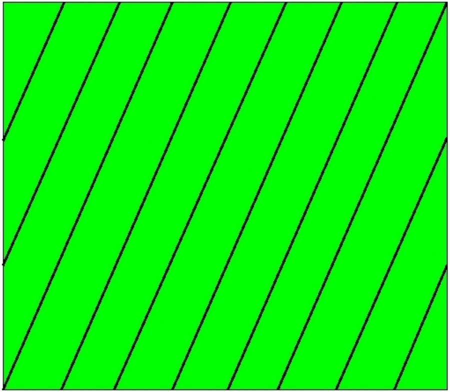
Step 1 : decomposition of Δ



The choice of the ℓ_j

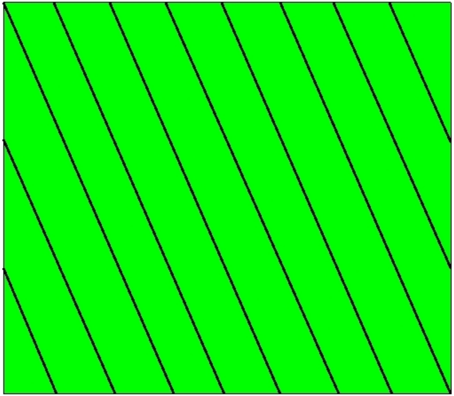
$$\ell_1(.) = \langle e_1, . \rangle_{\mathbb{E}^2}$$

The choice of the ℓ_j



$$\ell_2(.) = \langle \frac{1}{5}(e_1 + 2e_2), . \rangle_{\mathbb{E}^2}$$

The choice of the ℓ_j



$$\ell_3(.) = \langle \frac{1}{5}(e_1 - 2e_2), . \rangle_{\mathbb{E}^2}$$

Step 2 : Three convex integrations

• From f_0 and by applying three convex integrations along the directions given by ℓ_1, ℓ_2 and ℓ_2, we build iteratively three maps (a *salvo*) :

$$f_{1,1}, f_{1,2}, f_{1,3}$$

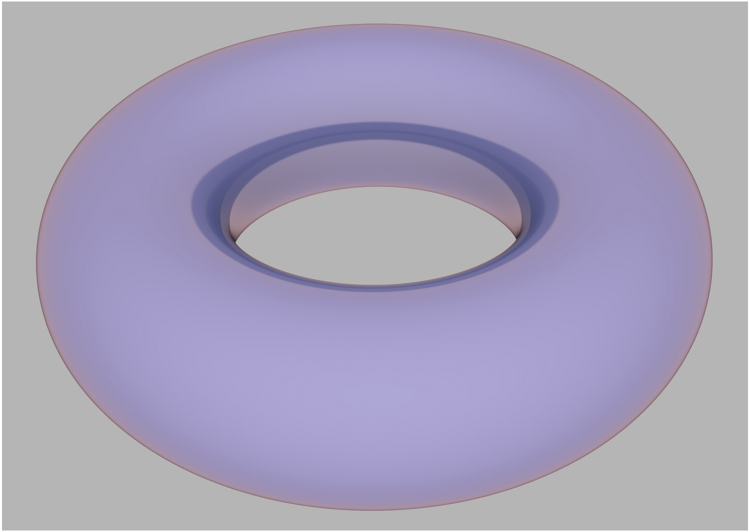
such that

$$\begin{aligned}
g - f_{1,1}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^q} &= \rho_2\ell_2^2 + \rho_3\ell_3^2 + O(\tfrac{1}{N_{1,1}})\\
g - f_{1,2}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^q} &= \rho_3\ell_3^2 + O(\tfrac{1}{N_{1,1}}) + O(\tfrac{1}{N_{1,2}})\\
g - f_{1,3}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^q} &= 0 + O(\tfrac{1}{N_{1,1}}) + O(\tfrac{1}{N_{1,2}}) + O(\tfrac{1}{N_{1,3}})
\end{aligned}$$

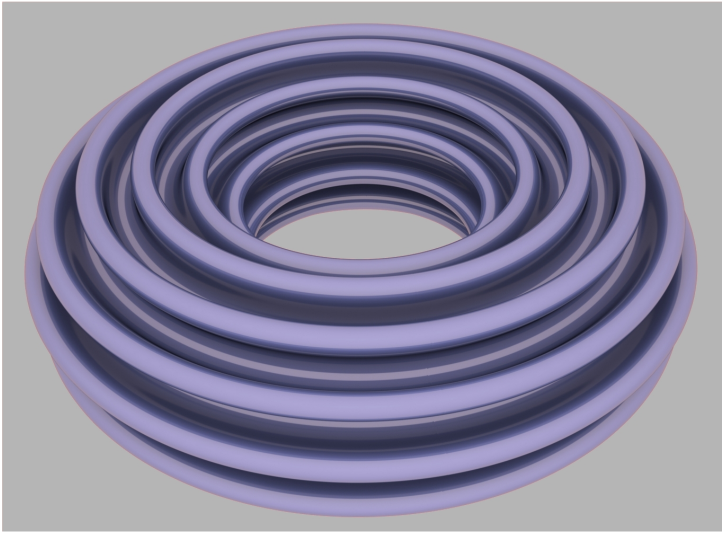
• We choose $N_{1,1}$, $N_{1,2}$ and $N_{1,3}$ such that :

$$\|\langle\cdot,\cdot\rangle_{\mathbb{E}^2} - f_{1,3}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^3}\|_{C^0} \leq \frac{1}{2}\|\langle\cdot,\cdot\rangle_{\mathbb{E}^2} - f_0^*\langle\cdot,\cdot\rangle_{\mathbb{E}^3}\|_{C^0}.$$

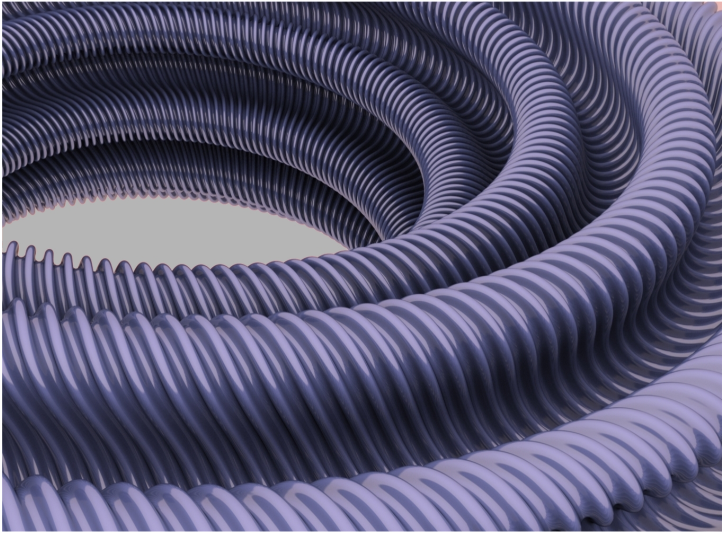
Torus of revolution



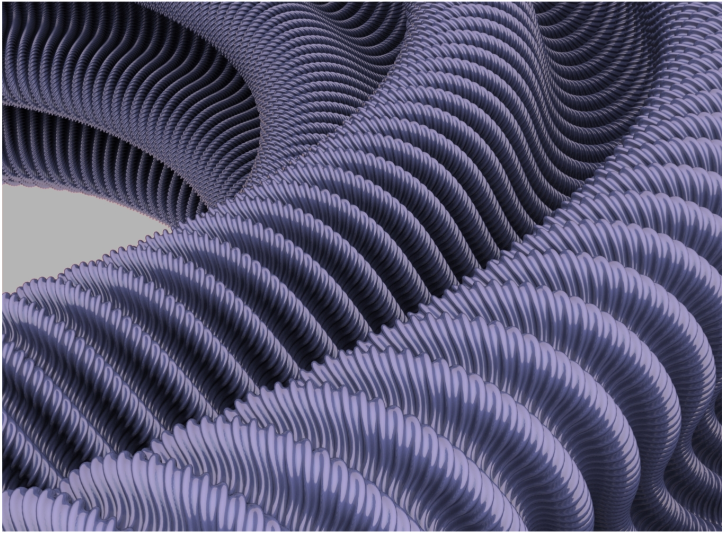
Convex Integration in the First Direction



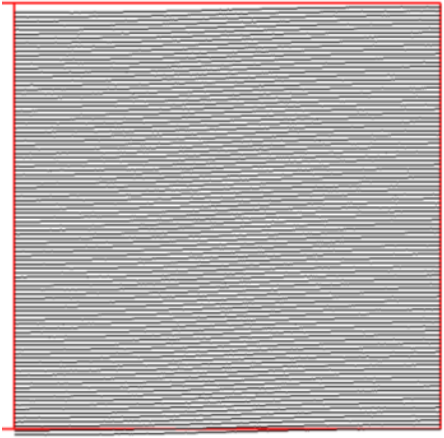
Convex Integration in the Second Direction



Convex Integration in the Third Direction

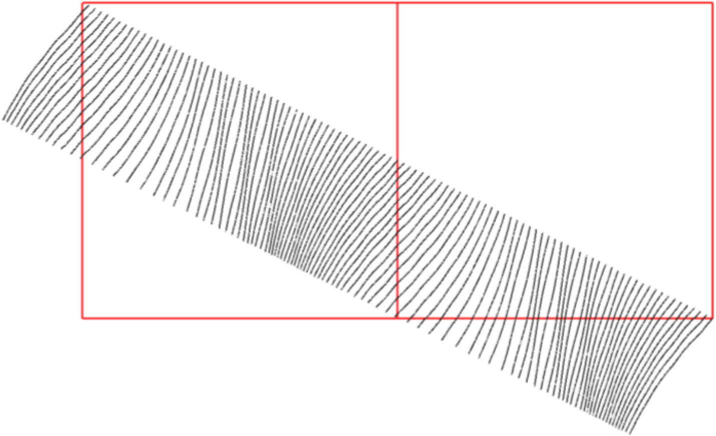


Three directions



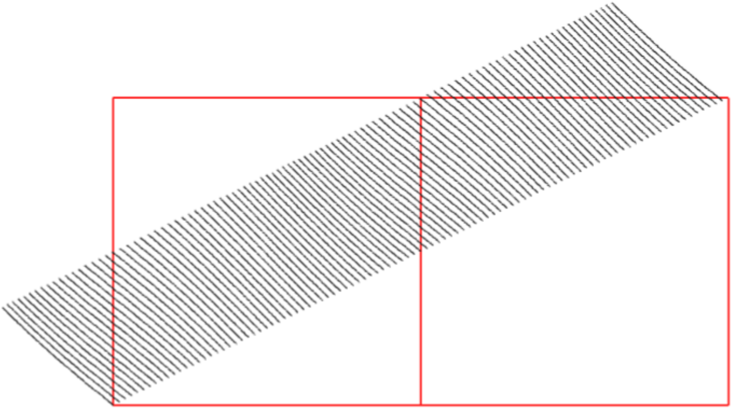
The flow in direction 1

Three directions



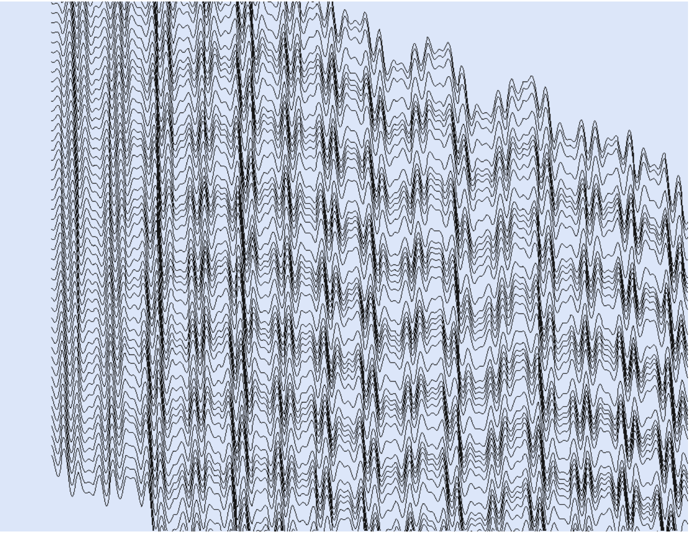
The flow in direction 2

Three directions



The flow in direction 3

A zoom



The vertical scale is exaggerated for emphasis.

Step 3 : Iterating the process

• We continue the process to create infinite sequence of maps

$$f_0, \quad f_{1,1}, f_{1,2}, f_{1,3}, \quad \ldots \quad f_{k,1}, f_{k,2}, f_{k,3}, \quad \ldots$$

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- After the salvo n° k, the isometric default is divided by 2^k

$$\|\langle .,.\rangle_{\mathbb{E}^2} - f_{k,3}^*\langle .,.\rangle_{\mathbb{E}^3}\|_{C^0} \leq \frac{1}{2^k}\|\langle .,.\rangle_{\mathbb{E}^2} - f_0^*\langle .,.\rangle_{\mathbb{E}^3}\|_{C^0}.$$

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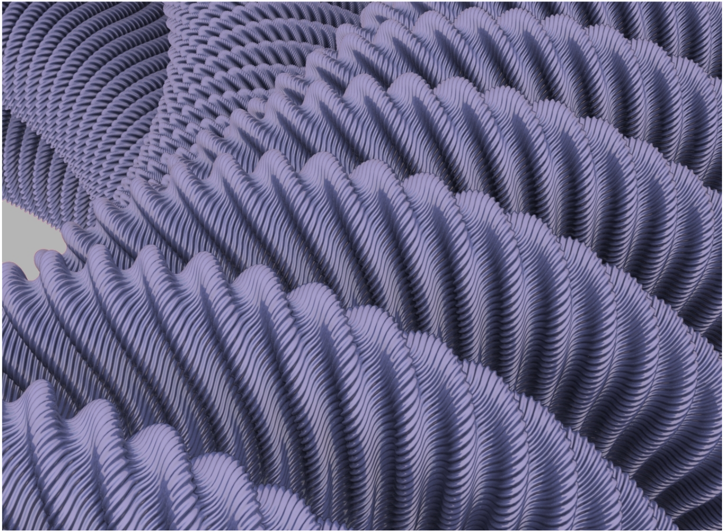
- The limit

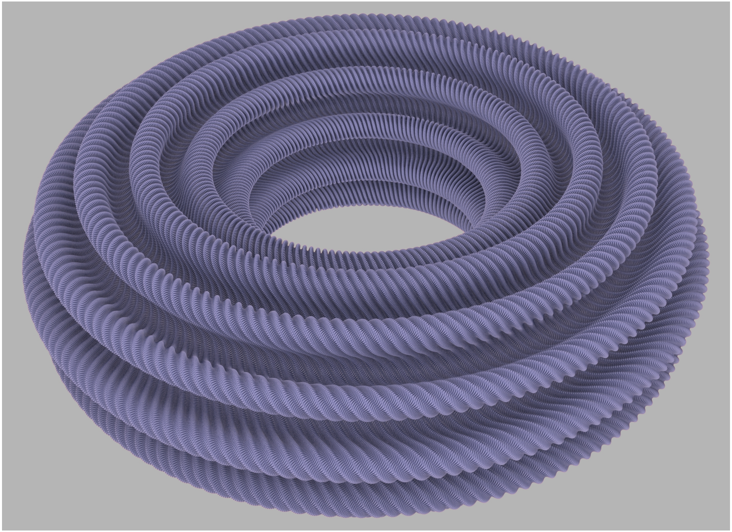
$$f_\infty := \lim_{k \longrightarrow +\infty} f_{k,3}$$

is a C^1 isometric immersion of the flat torus.

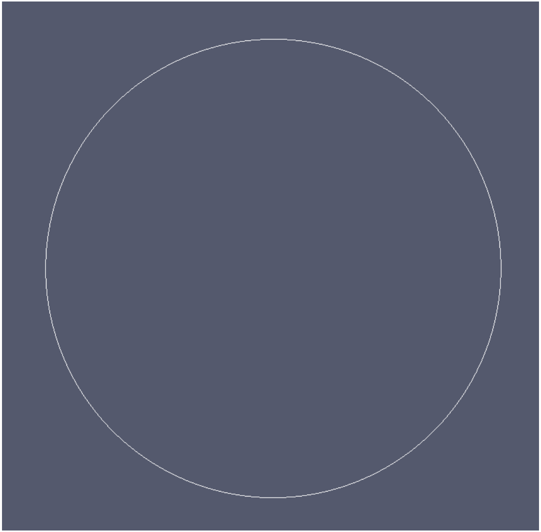
Another CI in the first direction



And so on... up to the Flat Torus



Nash-Kuiper in a 1D setting !



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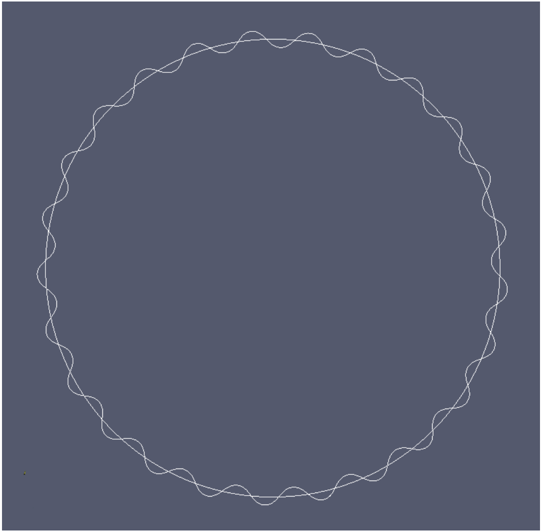

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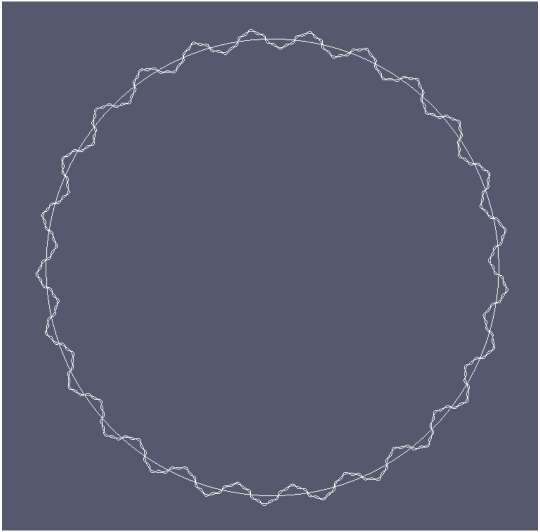

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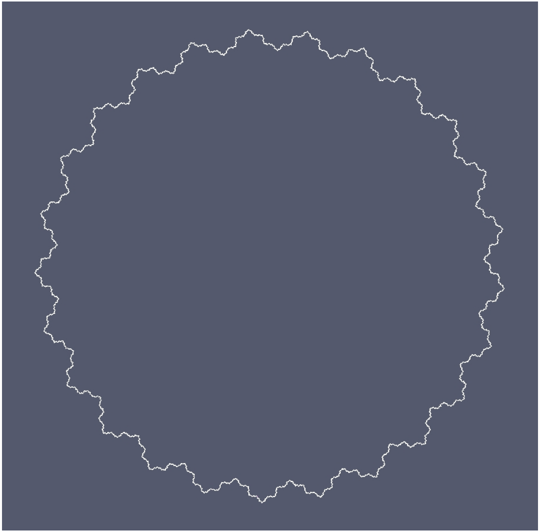
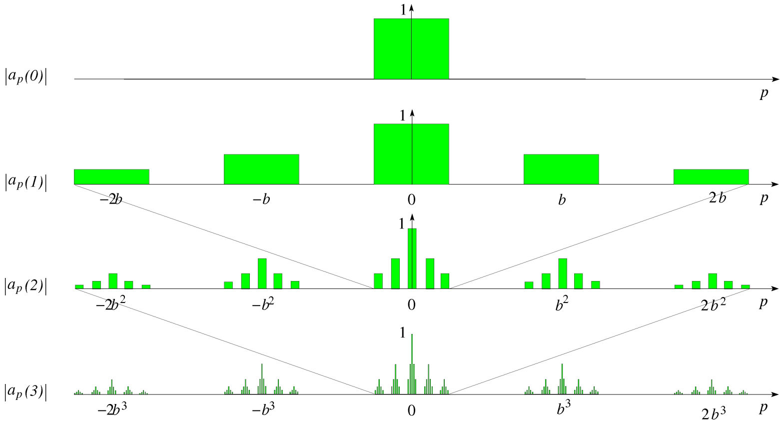


Spectrum of the Gauss map



Moscow University

Frigyes Riesz

Riesz Products

- These spectra arise from the study of *Riesz products* :

$$u \longmapsto \prod_{j=0}^{\infty}(1+\alpha_j \cos(2\pi b^j u)).$$

In that product, $b \geq 3$ is an integer and $(\alpha_j)_{j\in\mathbb{N}}$ is a sequence of real numbers such that, for every $j \in \mathbb{N}$, $|\alpha_j| \leq 1$.

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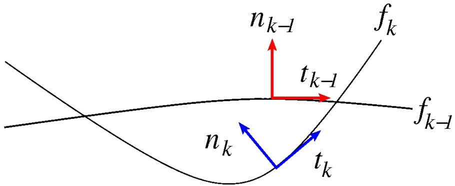
• Here, products of *corrugation matrices* take the place of Riesz products.

Corrugation matrices

- Let $C_k : \mathbb{S}^1 \longrightarrow O(2)$ be the matrix valued map such that

$$\forall u \in \mathbb{S}^1, \quad \begin{pmatrix} t_k(u) \\ n_k(u) \end{pmatrix} = \mathcal{C}_k(u) \cdot \begin{pmatrix} t_{k-1}(u) \\ n_{k-1}(u) \end{pmatrix}$$



We call C_k a **corrugation matrix.**

Structure of the Gauss Map

• Corrugation matrices encode the data of the successive convex integrations :

$$\mathcal{C}_k(u) := \begin{pmatrix} \cos\theta_k(u) & \sin\theta_k(u) \\ -\sin\theta_k(u) & \cos\theta_k(u) \end{pmatrix}$$

with

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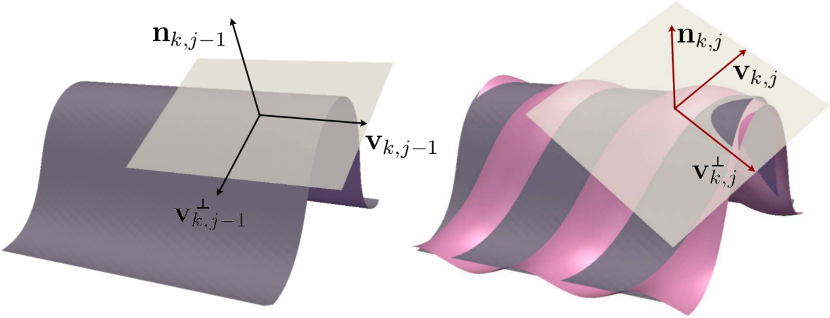
• The Gauss map n_∞ of the limit embedding is given by a Riesz-like product :

$$\begin{pmatrix} t_\infty \\ n_\infty \end{pmatrix} = \left(\prod_{k=1}^{\infty} \mathcal{C}_k\right) \cdot \begin{pmatrix} t_0 \\ n_0 \end{pmatrix}$$

From the circle to the torus



We denote by $\mathcal{C}_{k,j}$ the $SO(3)$ matrix such that :

$$\begin{pmatrix} v^{\perp}_{k,j} \\ v_{k,j} \\ n_{k,j} \end{pmatrix} = \mathcal{C}_{k,j} \cdot \begin{pmatrix} v^{\perp}_{k,j-1} \\ v_{k,j-1} \\ n_{k,j-1} \end{pmatrix}.$$

From the circle to the torus

Let $f_\infty : \mathbb{T}^2 \longrightarrow \mathbb{E}^3$ be the limit of the maps :

$$f_0, \quad f_{1,1}, f_{1,2}, f_{1,3}, \quad f_{2,1}, f_{2,2}, f_{2,3}, \quad \ldots$$

We have

$$\begin{pmatrix} v_\infty^\perp \\ v_\infty \\ n_\infty \end{pmatrix} = \prod_{k=1}^{\infty} \left(\prod_{j=1}^{3} \mathcal{C}_{k,j} \right) \cdot \begin{pmatrix} v_0^\perp \\ v_0 \\ n_0 \end{pmatrix}$$

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Beware ! – Unlike the 1-dimensional case, the analytic expressions of the $\mathcal{C}_{k,j}$'s are simply ugly...

From the circle to the torus

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Beware ! – Unlike the 1-dimensional case, the analytic expressions of the $\mathcal{C}_{k,j}$'s are simply ugly... but fortunately, their asymptotic expressions are nice.

Loss of derivatives

- In the one dimensional setting we have :

$$f(t) := f_0(0) + \int_0^t r(u)\mathbf{e}^{i\alpha(u)\cos 2\pi N u}\,\mathrm{d}u.$$

where $\mathbf{e}^{i\theta} := \cos\theta\,\mathbf{t} + \sin\theta\,\mathbf{n}$ and $\mathbf{t} := \frac{f_0'}{\|f_0'\|}$.

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• In particular

$$\frac{\partial f}{\partial t}(t) = r(t)\mathbf{e}^{i\alpha(t)\cos 2\pi N t},$$

therefore, if f_0 is C^k then f is C^k also.

Loss of derivatives

- In the two dimensional setting we have :

$$f(t,s) := f_0(0,s) + \int_0^t r(u,s)\mathbf{e}^{i\alpha(u,s)\cos 2\pi N u}\,\mathrm{d}u + \text{gluing term}$$

where $\mathbf{e}^{i\theta} := \cos\theta\,\mathbf{t} + \sin\theta\,\mathbf{n}$ with

$$\mathbf{t} := \frac{\partial_t f_0}{\|\partial_t f_0\|} \quad \text{and} \quad \mathbf{n} := \frac{\partial_t f_0 \wedge \partial_s f_0}{\|\partial_t f_0 \wedge \partial_s f_0\|}$$

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• The integral over the variable t can not recover the loss of derivative due to the presence of the partial derivative $\partial_s f$ in the definition of **n**. Therefore if f_0 is C^k then, generically, f is C^{k-1} only.

Corrugation Theorem

Corrugation Theorem ($\sim$, Jabrane, Lazarus, Thibert).– *For every $p \in \mathbb{T}^2$, we have*

$$\mathcal{C}_{k,j+1}(p) = \mathcal{L}_{k,j+1}(p) \cdot \mathcal{R}_{k,j}(p)$$

where

$$\mathcal{L}_{k,j+1} := \begin{pmatrix} \cos\theta_{k,j+1} & 0 & \sin\theta_{k,j+1} \\ 0 & 1 & 0 \\ -\sin\theta_{k,j+1} & 0 & \cos\theta_{k,j+1} \end{pmatrix} + O\left(\frac{1}{N_{k,j+1}}\right)$$

with $\theta_{k,j+1}(p) := \alpha_k(p)\cos(2\pi N_k X_{j+1})$ *and*

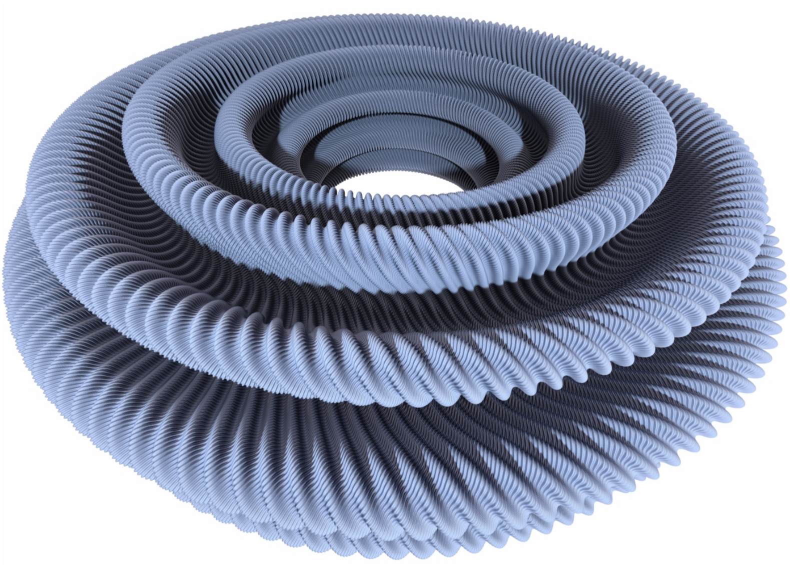
$$\mathcal{R}_{k,j} := \begin{pmatrix} -\sin\beta_j & -\cos\beta_j & 0 \\ \cos\beta_j & -\sin\beta_j & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(\Omega_{k,j})$$

where $\Omega_{k,j} = \|\langle .,.\rangle_{\mathbb{E}^2} - f_{k,j}^*\langle .,.\rangle_{\mathbb{E}^3}\|$ *is the isometric default.*

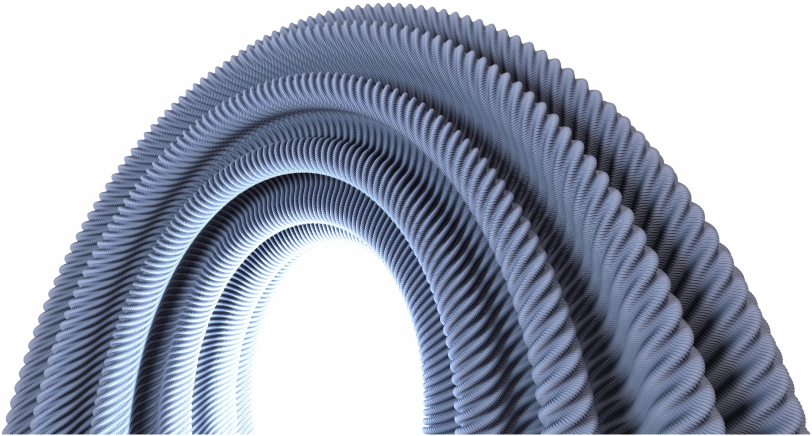
It is time for pictures !

The new CSCS (Swiss National Supercomputing Centre) building in Lugano-Cornaredo.

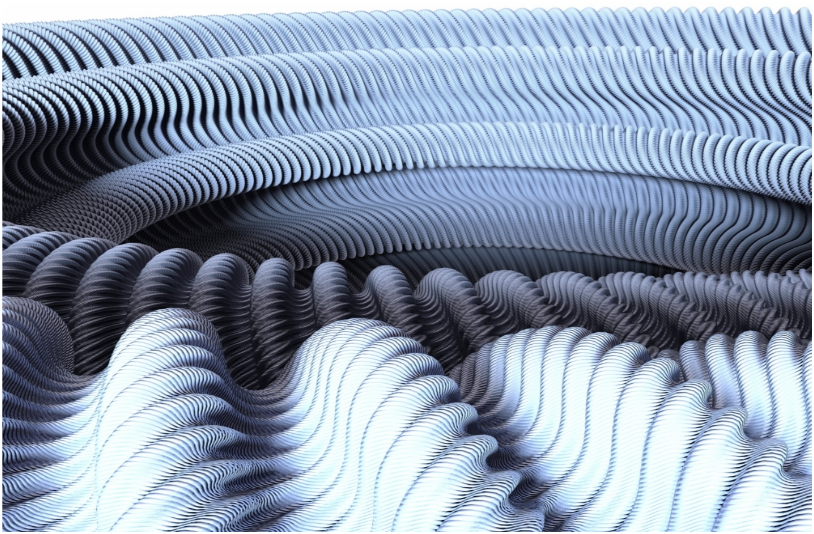
A wide-angle view



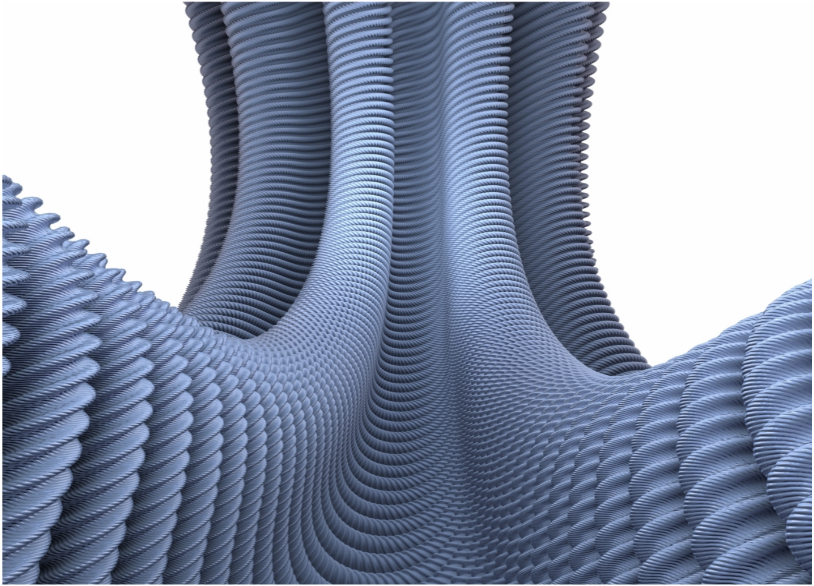
As an arch



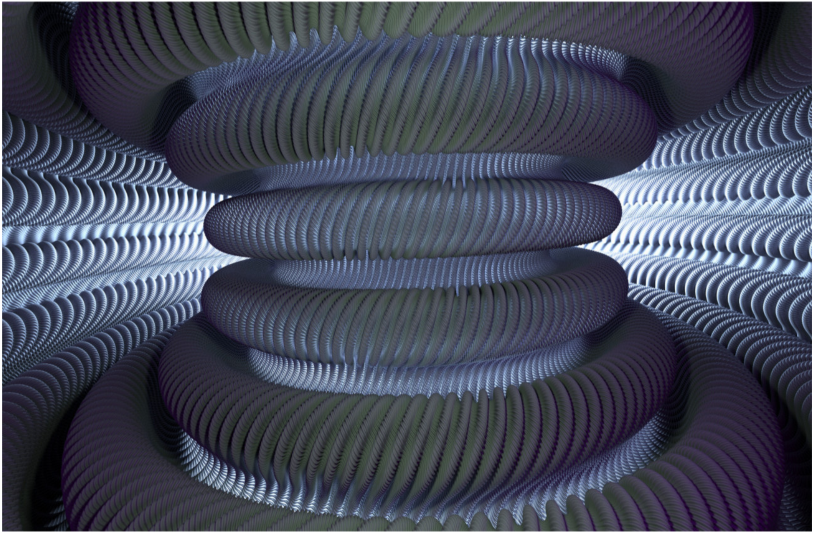
Tangential view



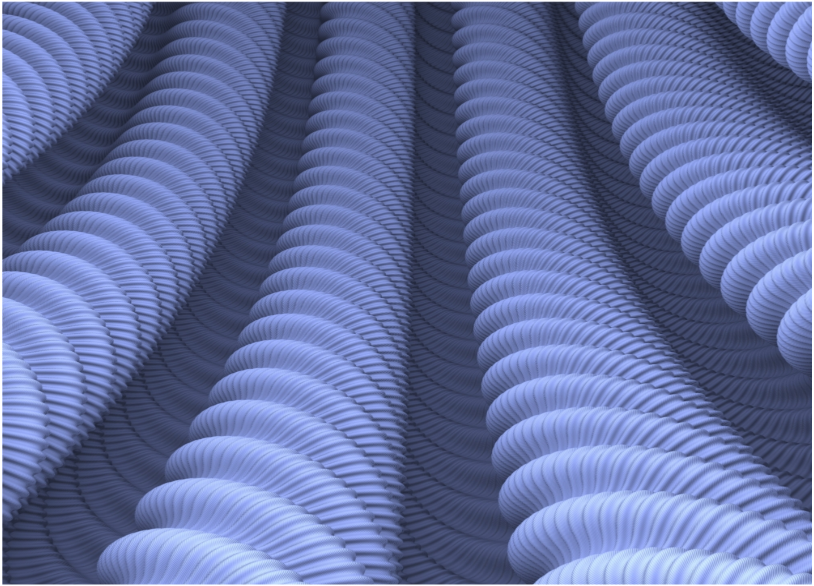
The bobsleigh track



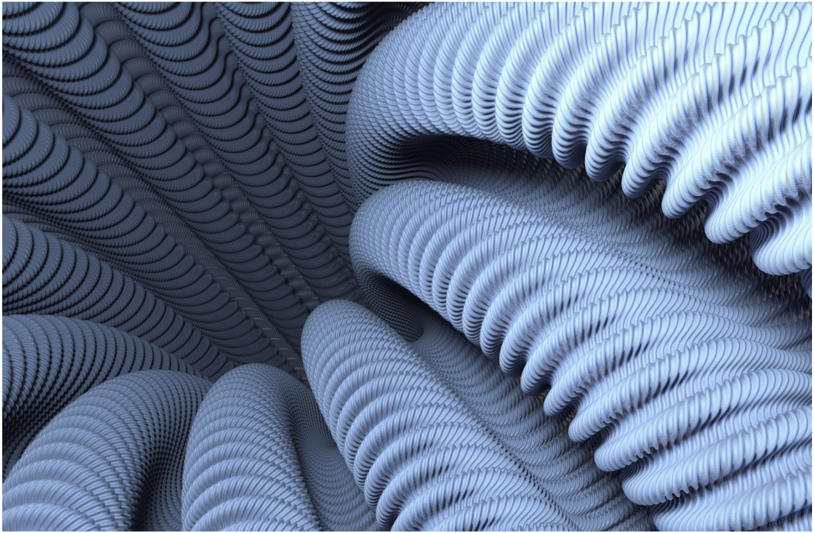
The tokamak



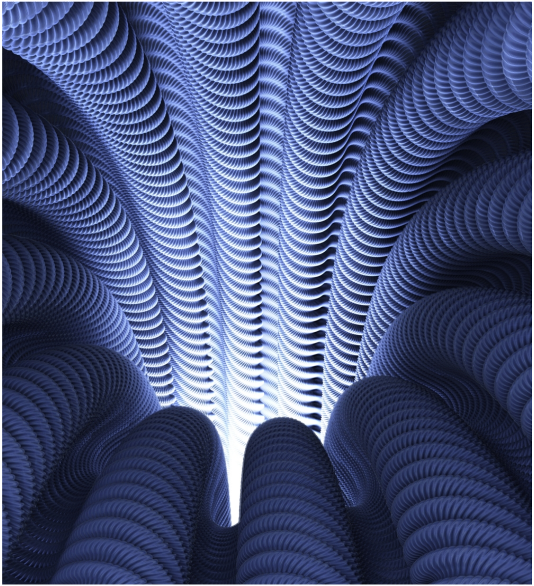
"Cinq colonnes à la une"



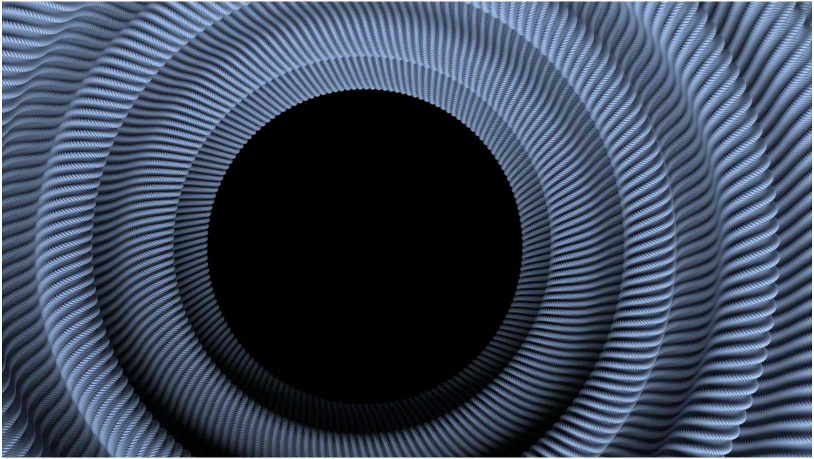
Guts and tripe



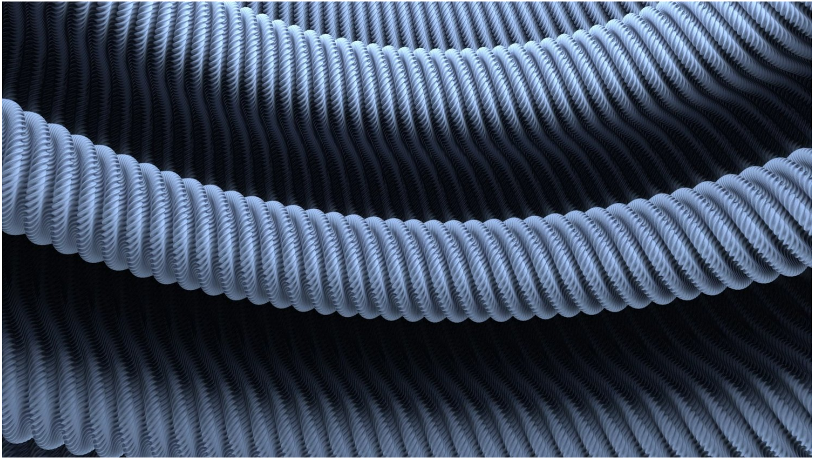
In the belly of the beast



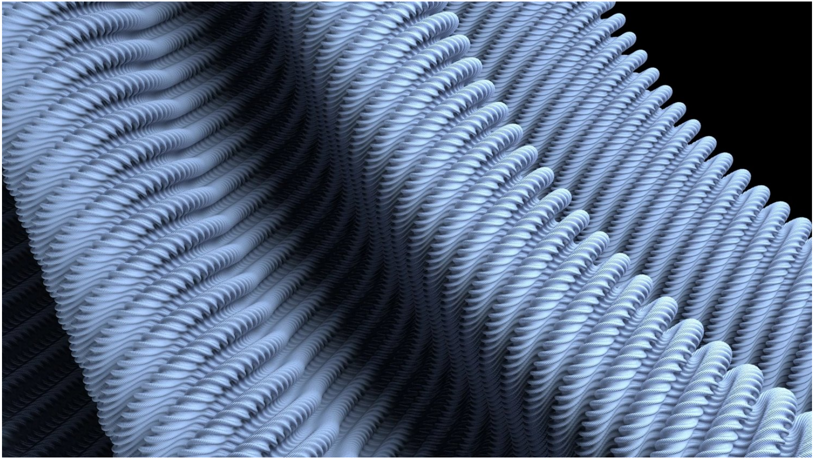
The eye



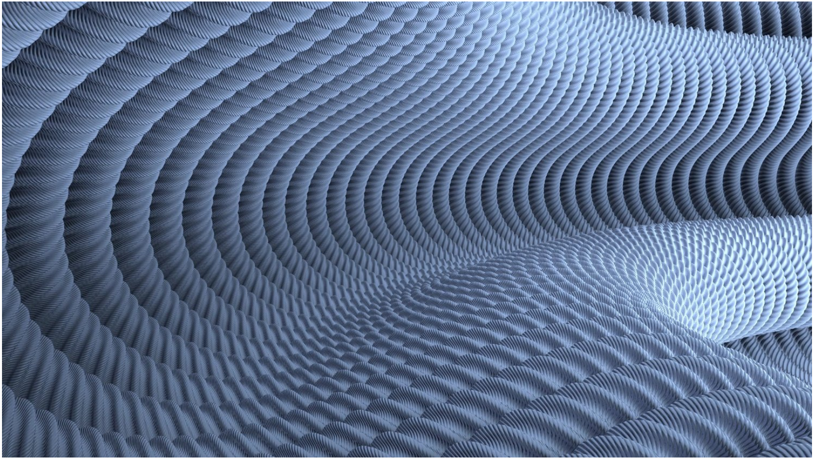
Flexible electrical conduits



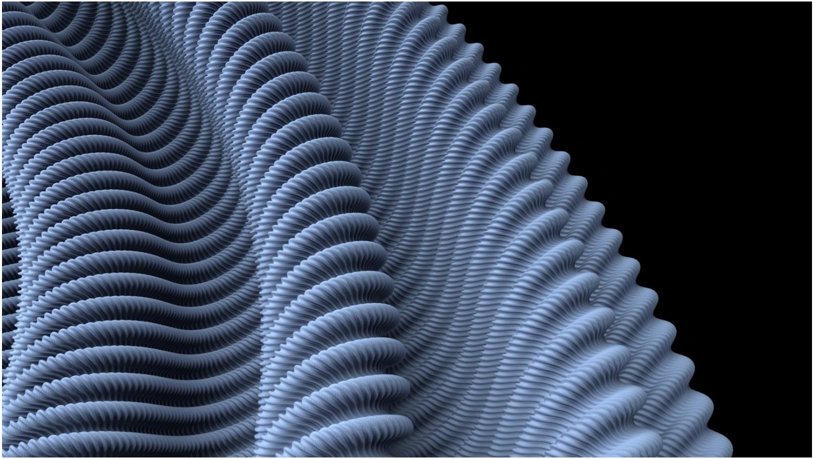
The truck suspension



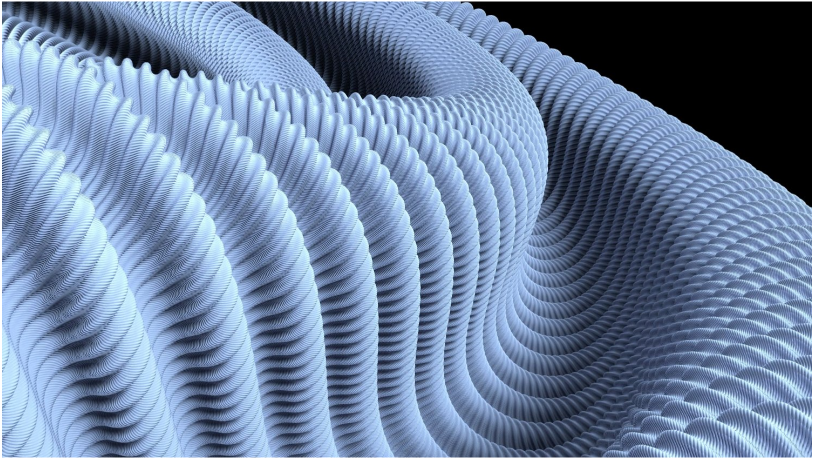
Velodrome



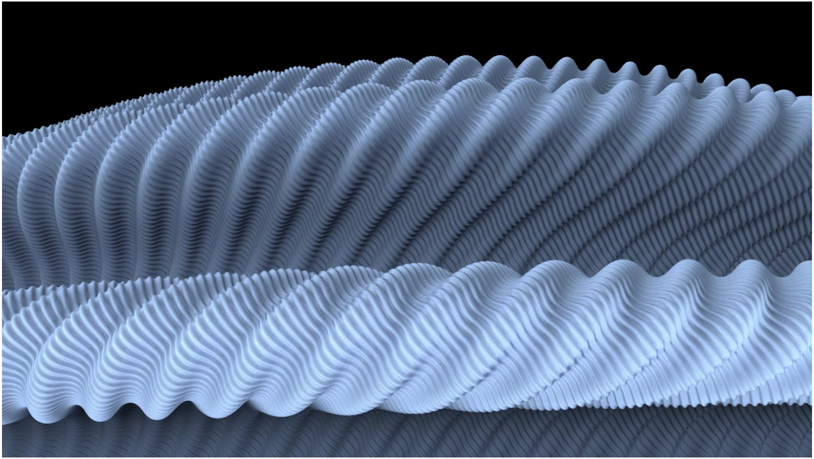
The tire



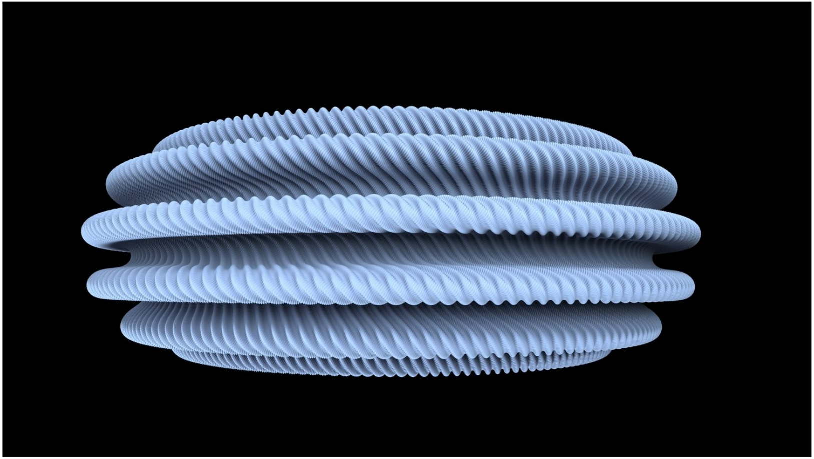
The landing



The rope

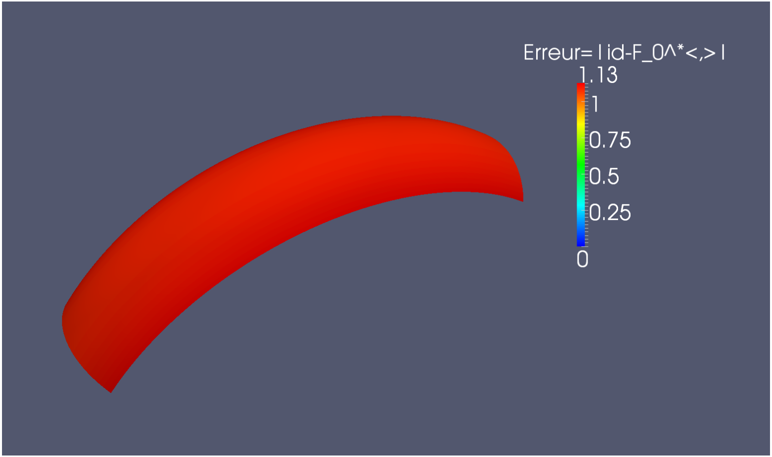


UFO



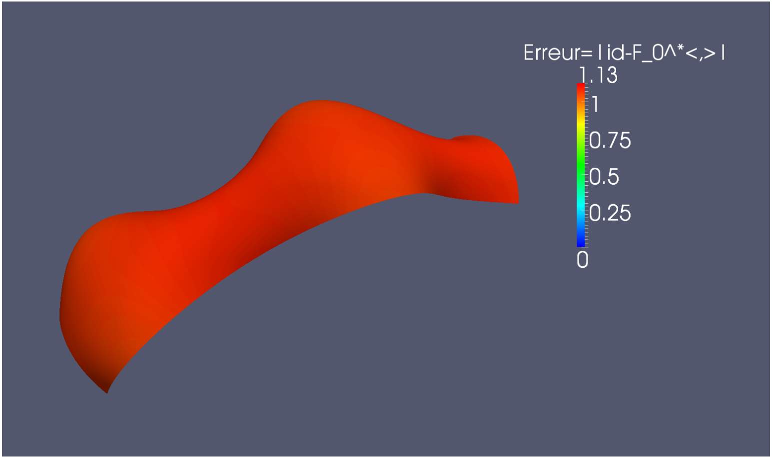
YouTube : Flat Torus

Zoom !



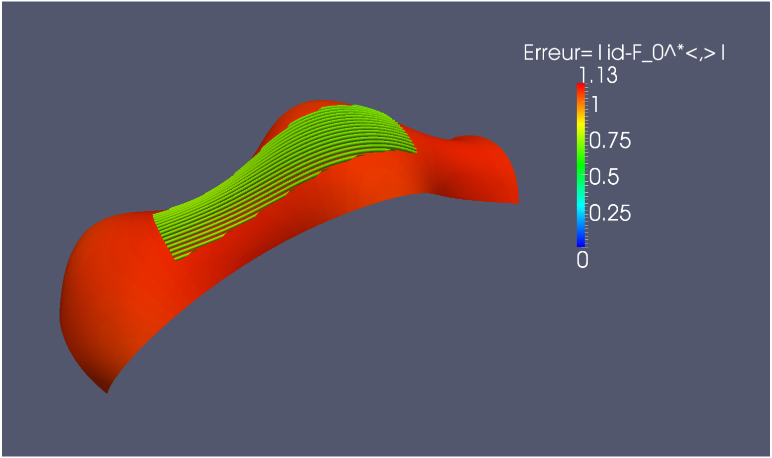
A part of the initial torus of revolution

Zoom !



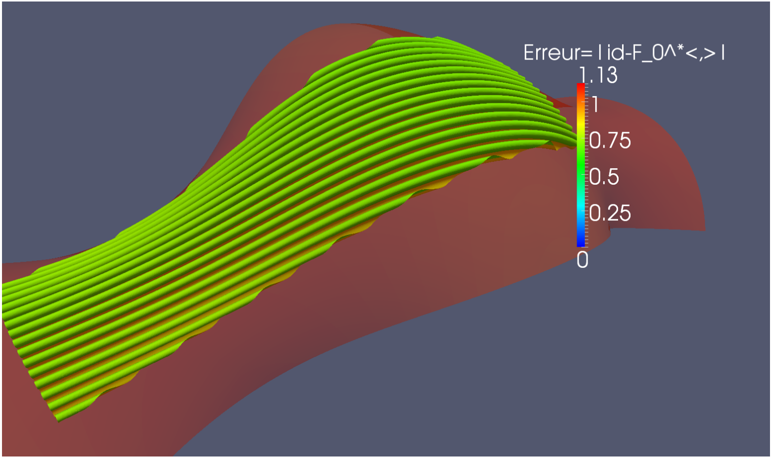
First integration : 8 oscillations

Zoom !



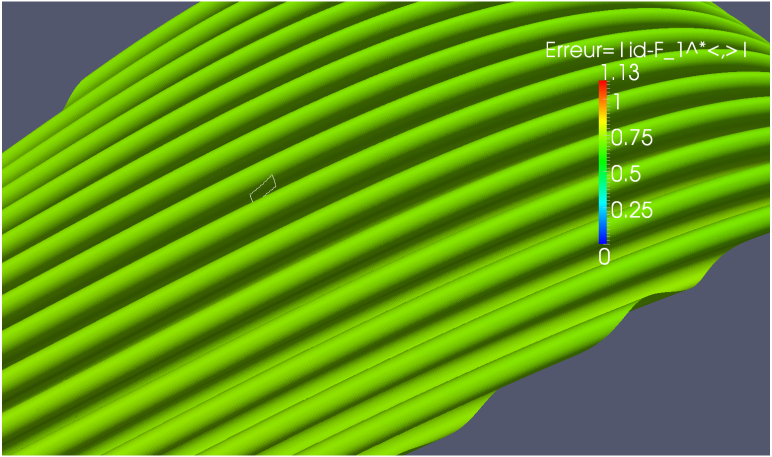
Second integration : 64 oscillations

Zoom !



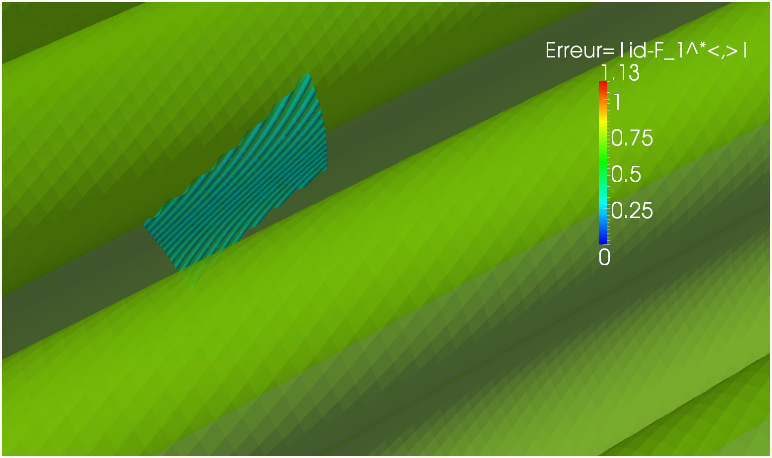
Zoom in on the second integration

Zoom !



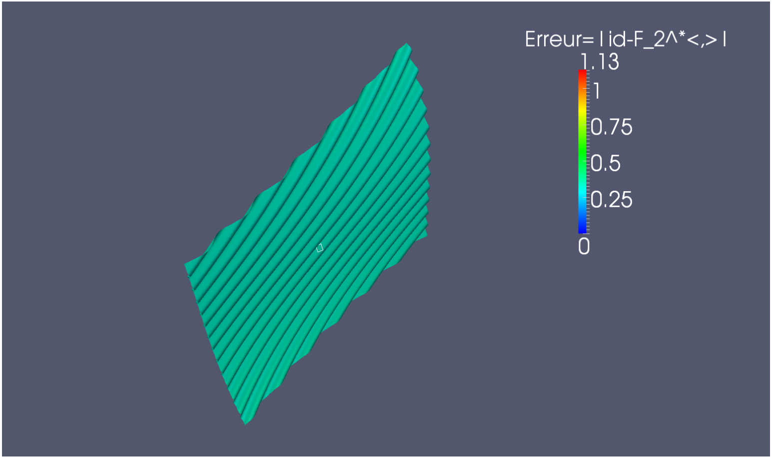
More closely

Zoom !



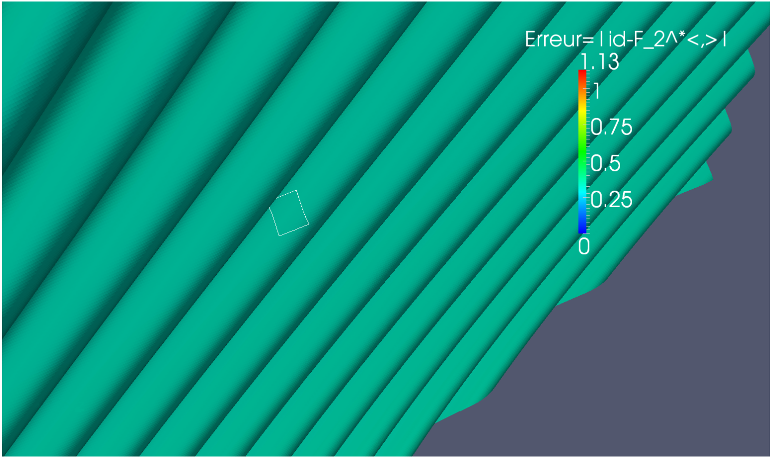
Third integration : 4096 oscillations

Zoom !



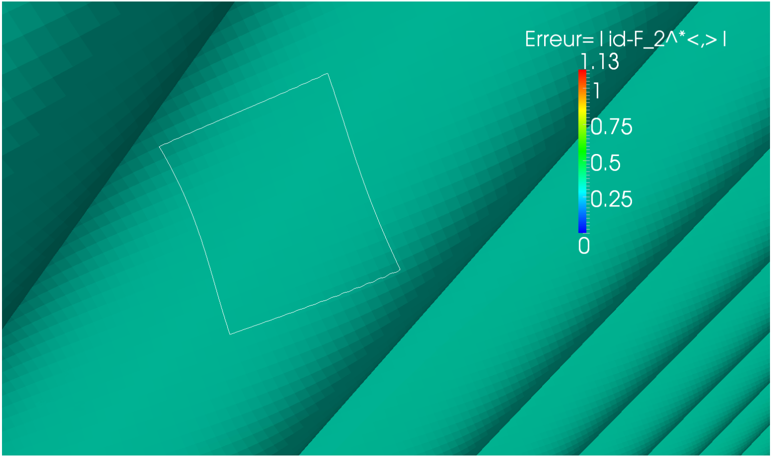
Third integration : 4096 oscillations

Zoom !



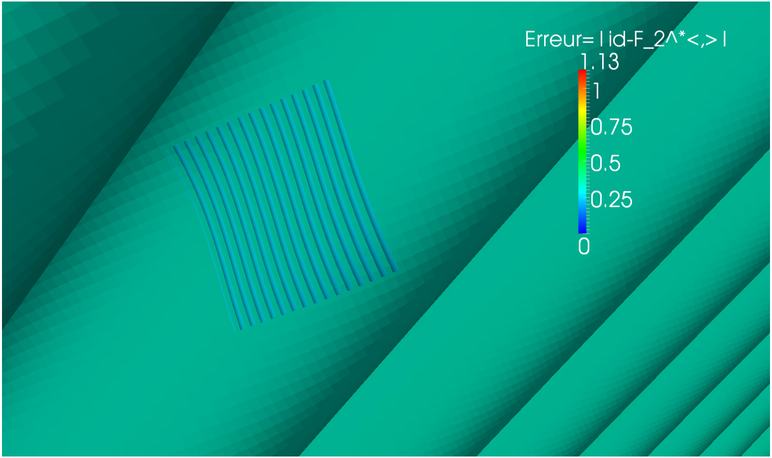
Zoom in on the third integration

Zoom !



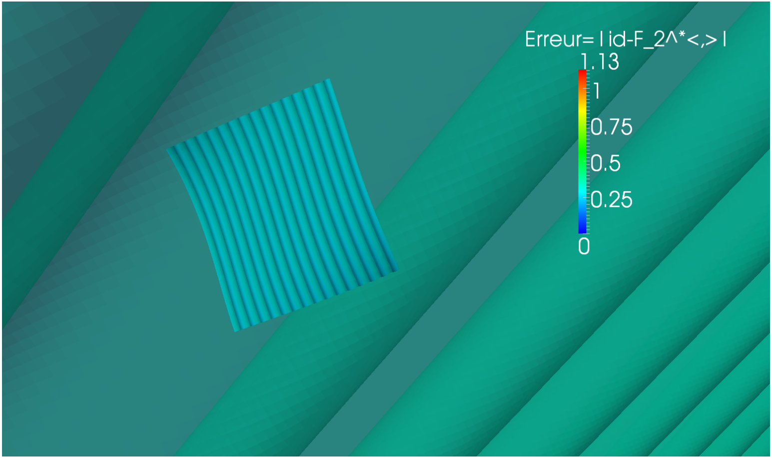
More closely

Zoom !



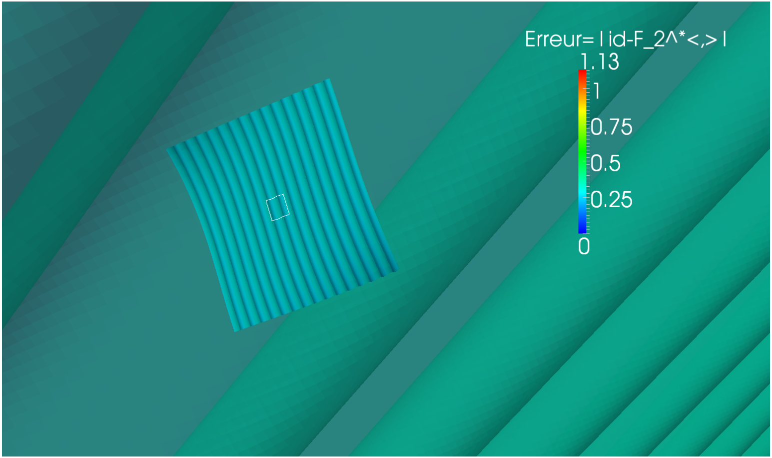
The fourth integration : 524 288 oscillations

Zoom !



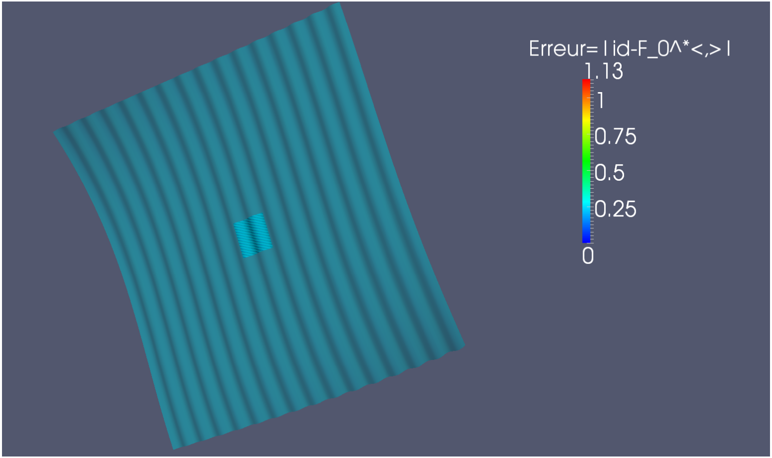
The fourth integration : 524 288 oscillations

Zoom !



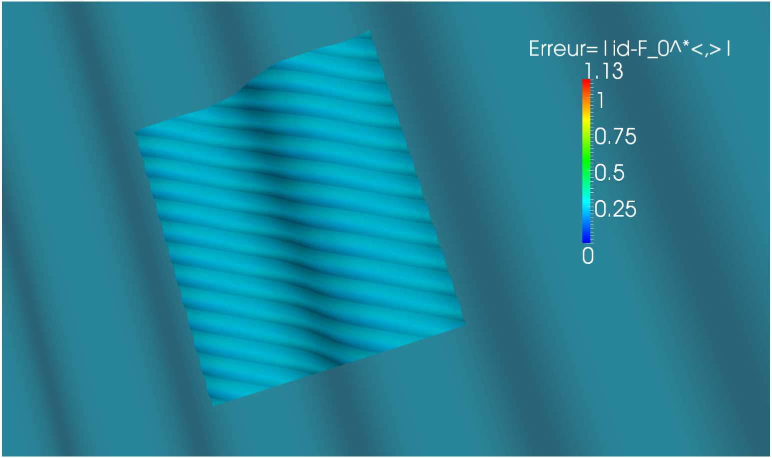
The fourth integration : 524 288 oscillations

Zoom !



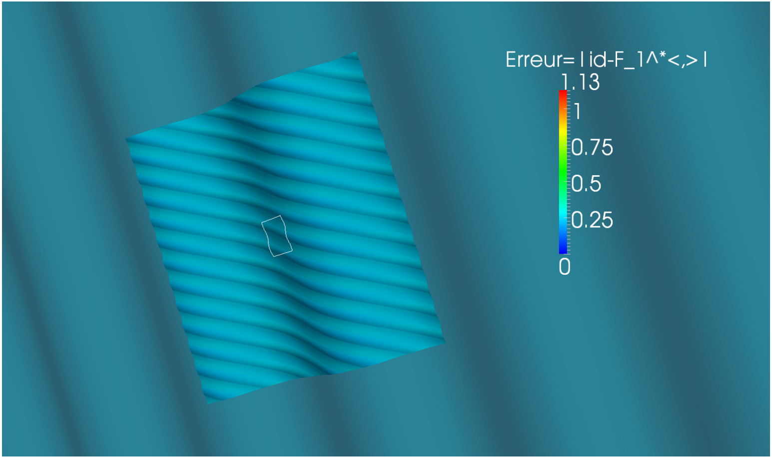
The fifth integration : 2 097 152 oscillations

Zoom !



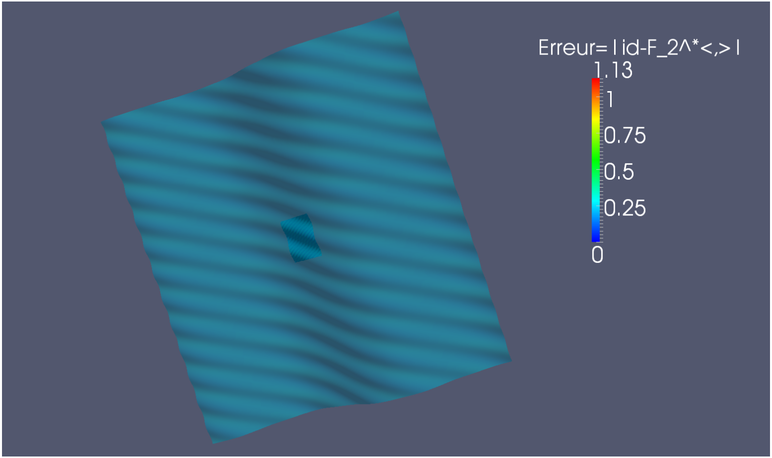
The fifth integration : 2 097 152 oscillations

Zoom !



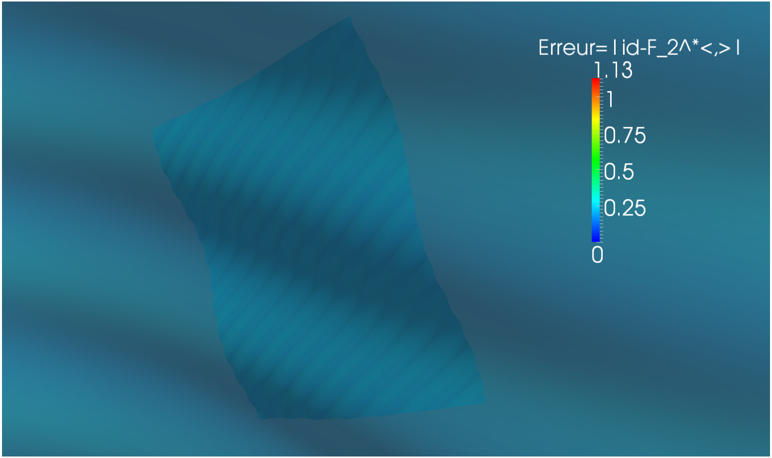
The fifth integration : 2 097 152 oscillations

Zoom !



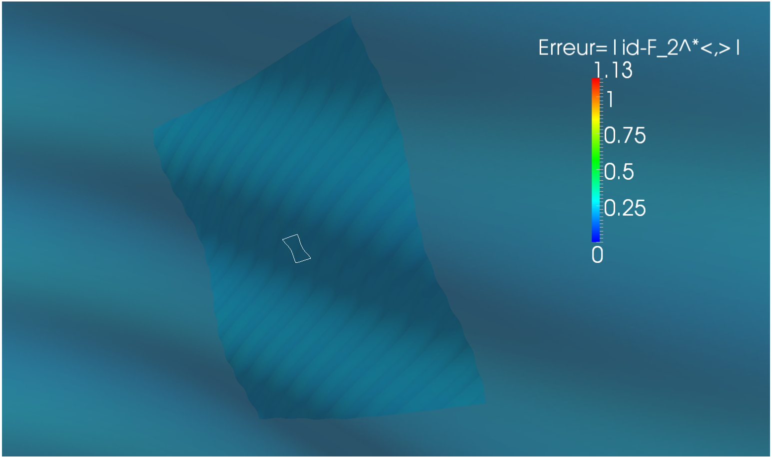
The sixth integration : 16 777 216 oscillations

Zoom !



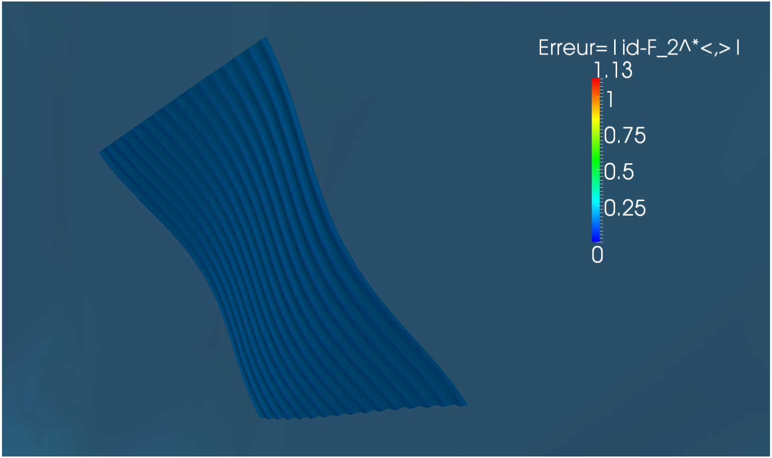
Zoom in on the sixth integration

Zoom !



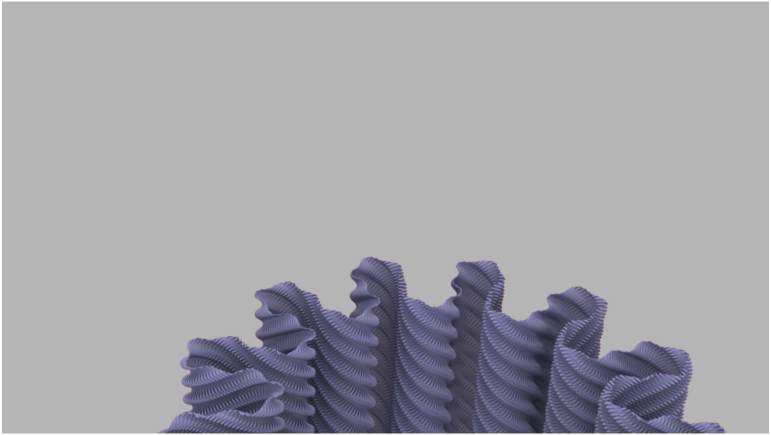
Zoom in on the sixth integration

Zoom !

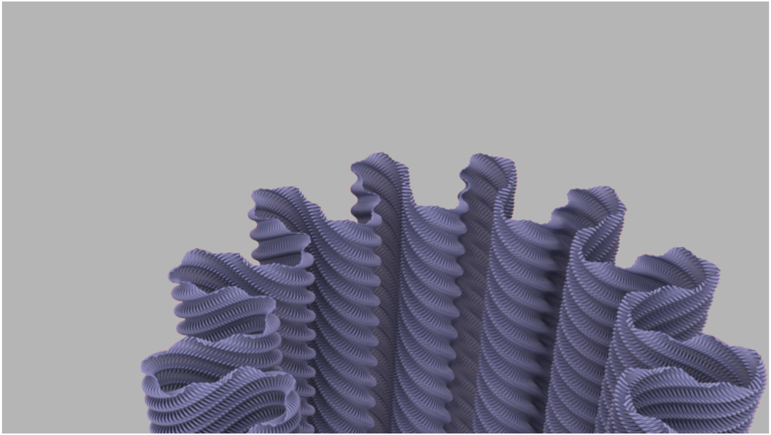


The seventh integration : 536 870 912 oscillations

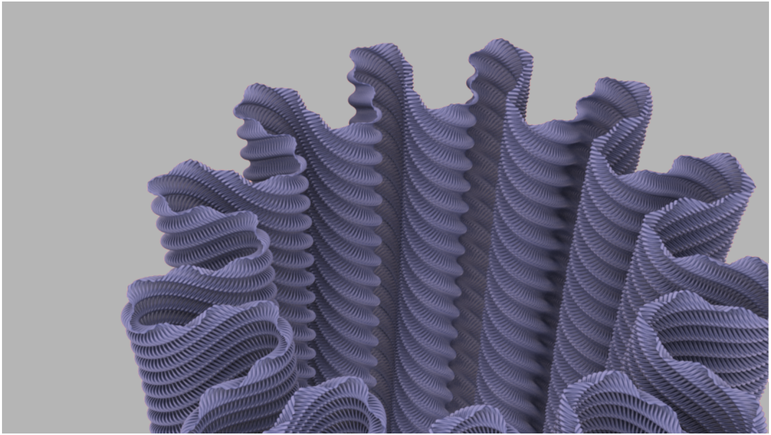
"Dites 33 !"



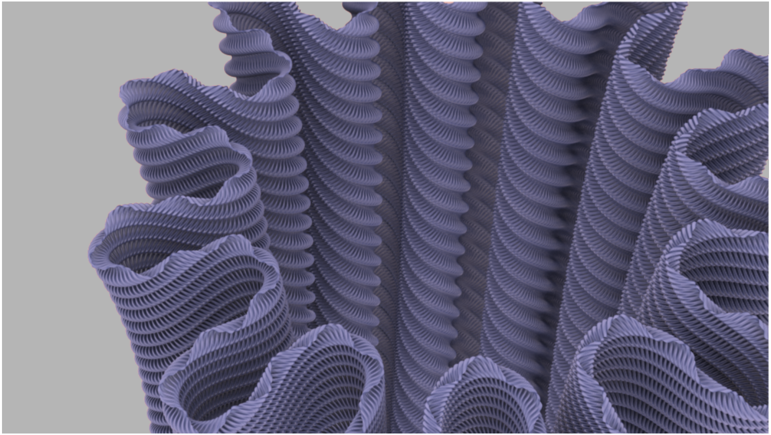
"Dites 33 !"



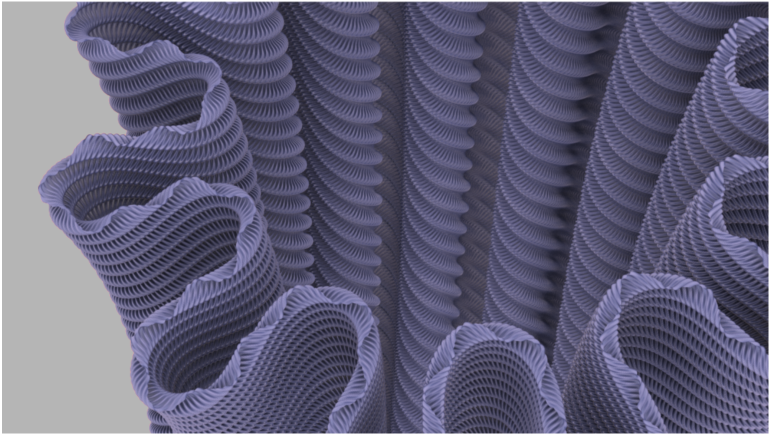
"Dites 33 !"



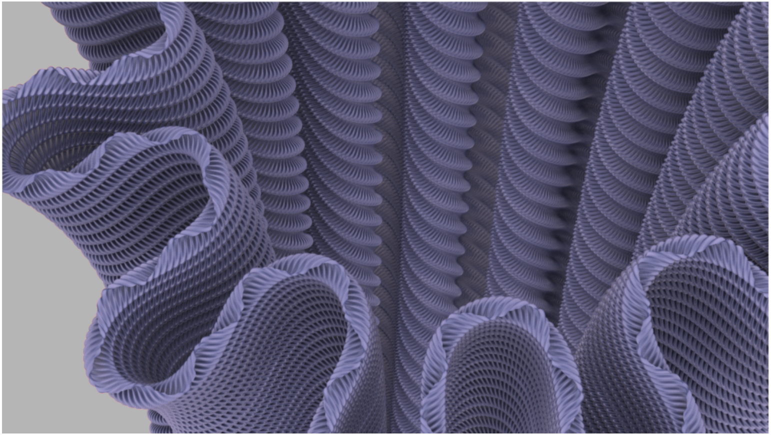
"Dites 33 !"



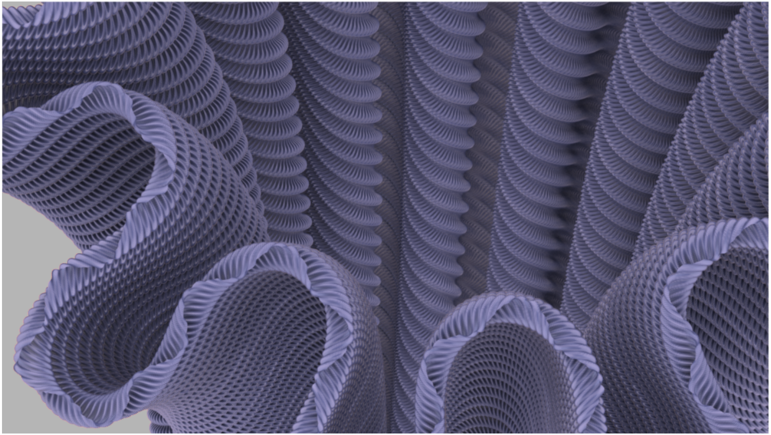
"Dites 33 !"



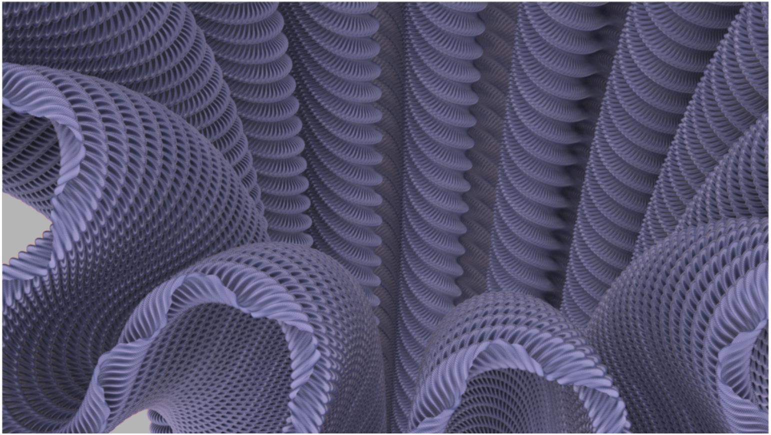
"Dites 33 !"



"Dites 33 !"



"Dites 33 !"



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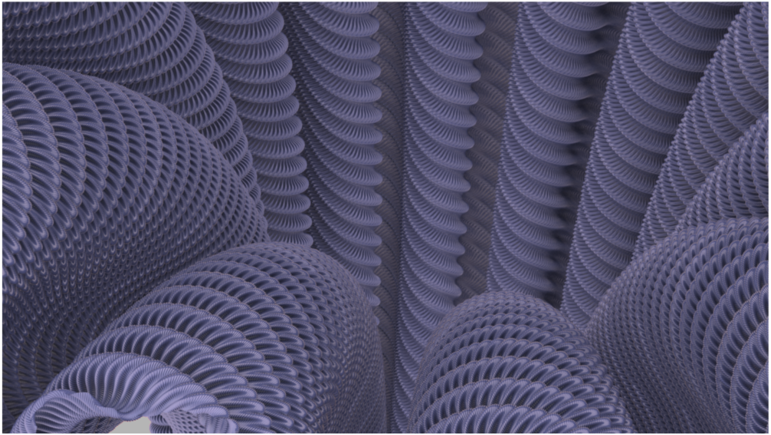
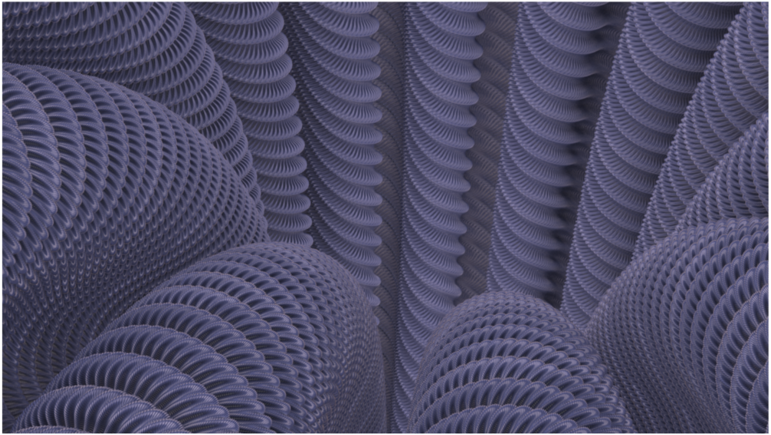
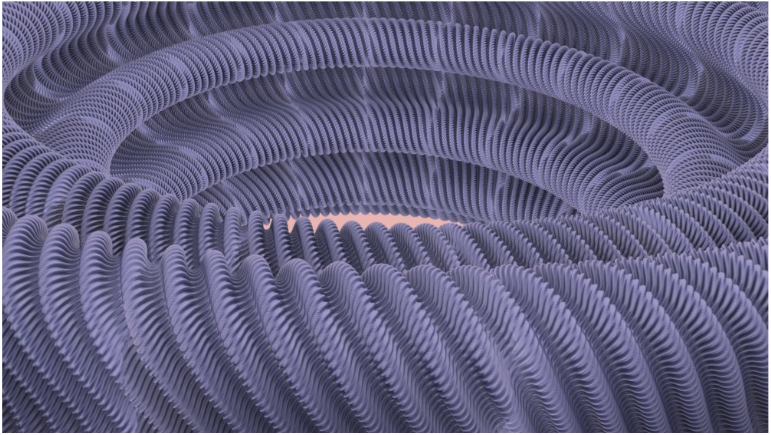


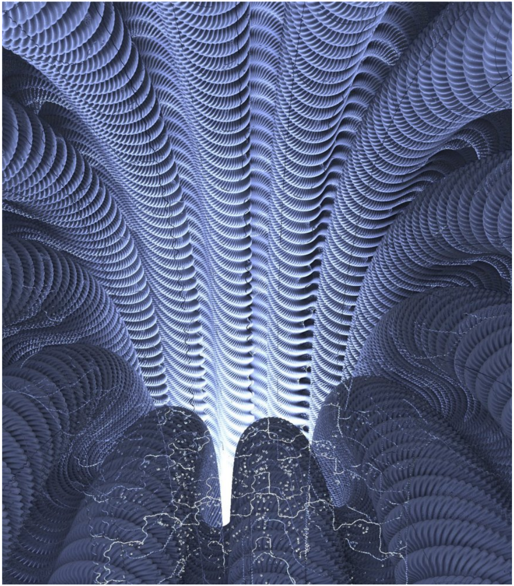
"Dites 33 !"



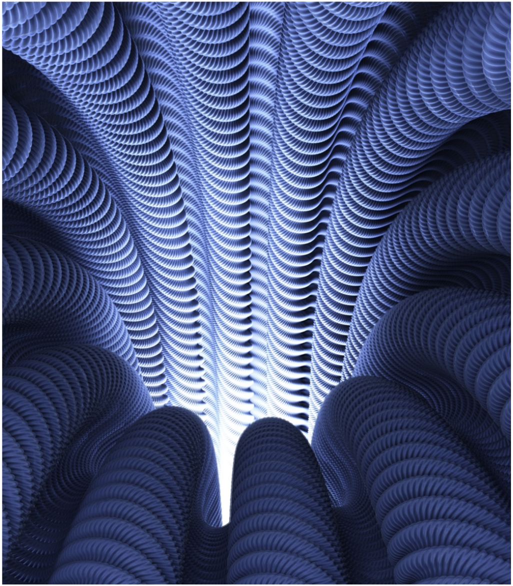
Artefacts



Artefacts



Artefacts



Corrugations in real life ?

An ammonite : Douvilleiceras Mammillatum

Corrugations in real life ?

A whelk

Corrugations in real life ?

Close up picture of a whelk

Corrugations in real life ?

The Hevea Team