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A HOMOGENIZED MODEL OF AN UNDERGROUND WASTE REPOSITORY INCLUDING A DISTURBED ZONE*

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Abstract. The mathematical model describing the leaking of an underground waste repository should include the multiscale geometry and the large variation of the geological coefficients. Numerical simulations for performance assessments using such a local and detailed model are unrealistic, and there is a need to replace this local model (mesoscopic model) by a global one (macroscopic model). After introducing a small parameter ε, linking the relative size of the waste packages to the repository module size and to geological parameters, a first-order accurate macroscopic model of a repository module is obtained by studying the asymptotic behavior of the mesoscopic model when ε tends to 0. The mathematical homogenization method that we use herein leads to an accurate macroscopic model which could be used as a global repository model for far field numerical simulations in performance assessment.

Key words. homogenization, singular measures, underground waste repository, convection-diffusion-reaction equation

AMS subject classifications. 35B27, 35B25, 35B40, 35K57

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1. Introduction.

In our previous papers [4] and [5] we gave a mathematical model describing the global behavior of an entire underground waste repository, assuming it was made of a high number of repository modules. In the present paper we give a mathematical model describing the global behavior of only one repository module in the underground waste repository. For this, we assume that a repository module is made of a high number of disposal holes, filled with waste packages, and located inside a low permeable rock, lying on a hypersurface Σ and linked by parallel loading and handling drifts (see Figure 1.1). Moreover, in a repository module, all these parallel loading and handling drifts were backfilled and are connected, at their head, to a main connecting gallery, which was also backfilled. The entire repository is embedded in a low permeability layer, called host layer, like, for example, clay. As in the previous paper (see the introduction in [5]), we study the worst possible scenario where all the waste packages start leaking at the same time. We mathematically represent the leaking of a waste package by a time-dependent flux given on the boundary of the disposal holes. This leaking lasts over a period of time $]0, t_m[$, which is small compared to the millions of years over which the radioelements are transported. The case where the components, and the starting of the leaking, of each waste package are assumed to be random will be presented in a forthcoming paper.

The pressure drop, or hydraulic head, in the region produces a water flow crossing the repository array. The solute is then transported both by the convection produced by the water flowing slowly (creeping flow) through the rocks and by the diffusion/dispersion coming from the dilution in the water. Following the test case [10]

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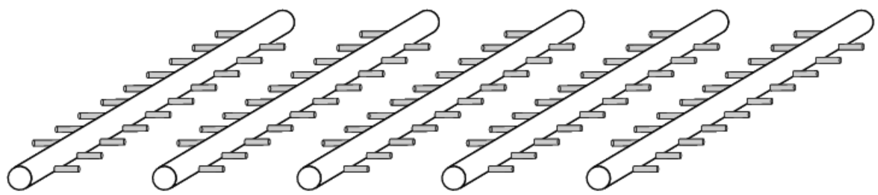

FIG. 1.1. *A part of a waste repository module, with five rows of handling drifts and disposal holes.*

the general solute transport model will include possible chemical effects and radioactive decay, and the fluid viscosity will not depend on the concentration. Hence the hydrology will be decoupled from the transport, and we will assume, without loss of generality, the convection velocity field to be given.

We are making no attempt here to exactly represent a real repository, and we have designed a simplified mesoscopic model, but with interesting mathematical difficulties, for obtaining a global model by homogenization. In our model, following the Couplex benchmark [10] we assume that the typical dimensions of a drift are 10 meters for the diameter, 500 meters for the length, and 50 meters for the distance between the drift axes. The order of a typical waste package diameter is assumed to be a few meters, and for the distance between two waste packages it is assumed to be 10 meters. Since there is a large number of waste packages, each of them with a small size compared to the host geological layer size and to the drift diameter (see Figures 1.1 and 2.1), a direct numerical simulation of a single repository module, based on a *mesoscopic* model taking into account all the details, is unrealistic. The ratio between the width of a module (500 meters) and the distance between the drift axes (50 meters) is considered as a small parameter, ε, in the detailed *mesoscopic* model. According to this rescaling, the waste package now has a diameter, of order ε^γ, with $\gamma \geq 1$, and finally there are now in the renormalized model three scales: 1 for the scale of a repository module, ε for both the scale of a row of waste packages and the periodicity of drifts, and ε^γ for the waste package thickness.[1] Contrary to the test case [10], in the present model we take into account the existence of a possible disturbed zone, the excavation disturbed zone (or EDZ in short) created by the excavation of the handling drifts, connecting galleries, and shafts. Since the excavation backfill has similar geological properties to the EDZ, which is located around the excavations, we will for mathematical simplicity not distinguish the cylindrical backfilled drift itself from the surrounding disturbed zone (annulus), and we will call, in short, the disturbed drift the cylinder including the EDZ. The three different scenarii (considered in the European exercise BENIPA) corresponding to the connected shafts galleries and drifts being either perfectly sealed, poorly sealed, or not sealed will be mathematically represented by different concentration rates on the drifts heads. We will also assume the existence of a barrier or a perfect seal situated on the bottom of each loading and handling drift and between the waste packages and the EDZ or the undisturbed host rock. The purpose of these last two types of modeling assumptions is to simplify the mathematical discussion by

[1]According to the Couplex test case [10] the actual γ is close to three. We have chosen to treat the more general situation here.

producing simpler interface and boundary conditions for the drifts or the disturbed zone; other types of interface and boundary conditions would only add more complications in the mathematical discussion without significant change in the homogenized global model.

In section 2, starting from the geometrical situation and the above described rescaling, we define in subsection 2.1 the detailed geometry of a repository module. In subsection 2.2, we give the equations describing the solute transport in such detailed geometry.

In section 3, we present the results according to the $\varepsilon^{-\beta}$ range, the mathematical parameter used for describing the water inflow and the concentration on the head drift. We remark that there are three typical different behaviors for β equal to, strictly greater than, or lower than one.

In section 4, we derive the a priori estimates for all the situations. In order to obtain convergence all over the domain, we introduce the extension operator adapted to the three scales $O(1)$, ε, and ε^γ, as defined in [5].

In sections 5, 6, and 7, starting from the previous a priori estimates and taking the weak limits in the original detailed model, we give in Theorems 5.1, 6.1, 7.3, and 7.4 the global models corresponding to the three situations: β equal to, strictly greater than, or lower than one.

The a priori estimates are obtained thanks to a sharp estimate of the integral over all the waste package boundaries Γ_ε, which is presented in Appendix B.

The global models obtained at the limit are defined on the hypersurface Σ, and the general two-scale convergence has to be adapted to this situation. Following the "two-scale convergence with respect to a singular measure" for parabolic problems, as developed in [3], we precisely present the compactness properties we use herein in Appendix A.

2. Setting the problem.

2.1. Detailed description of the geometry. The repository module is located inside a domain $\Omega =]0,L[^2 \times]-L/2,L/2[\subset \mathbf{R}^3$, the hypersurface on which the waste packages lie is denoted $\Sigma =]0,L[^2 \times \{0\}$, and for simplicity we assume that $L/\varepsilon = m \in \mathbf{N}$.

In the following we will use the notation

$$x = (x_1, x_2, x_3), \quad x' = (x_1, x_2), \quad y_i = x_i/\varepsilon, \quad i = 2, 3.$$

The cell Y on Figure 2.1 consists of three parts: the disturbed-drift cylinder $\mathcal{S} = \mathcal{C} \times]-1/2, 1/2[$ with cross-section $\mathcal{C}$, the waste packages P_ε, and $\mathcal{Y}$ the rest of the cell called "host rock." The rescaled waste packages P_ε are of cylindric shape with length of order 1 and thickness of order $\varepsilon^{\gamma-1}$, with $\gamma \geq 1$. By the change of variable $x = \varepsilon y$ we shrink the unit cell Y to the actual size ε, and we denote by $Y_\varepsilon(i,j) = \varepsilon\ ((i,j,0) + Y)$ the (i,j)th ε-cell. Inside this ε-cell we have $\mathcal{P}_\varepsilon(i,j) = \varepsilon\ ((i,j,0) + P_\varepsilon)$, the waste package (each waste package having length $O(\varepsilon)$ and cross-section diameter $O(\varepsilon^\gamma)$), and $\mathcal{S}_\varepsilon(i,j) = \varepsilon\ ((i,j,0) + \mathcal{S})$, the disturbed-drift part. Repeating Y_ε periodically m^2 times over the surface Σ, we obtain the whole repository module, and we denote the different parts of this repository module:

$$(2.1) \quad \mathcal{P}_\varepsilon = \bigcup_{i,j=1}^{m} \mathcal{P}_\varepsilon(i,j) \text{ the union of all waste packages,}$$

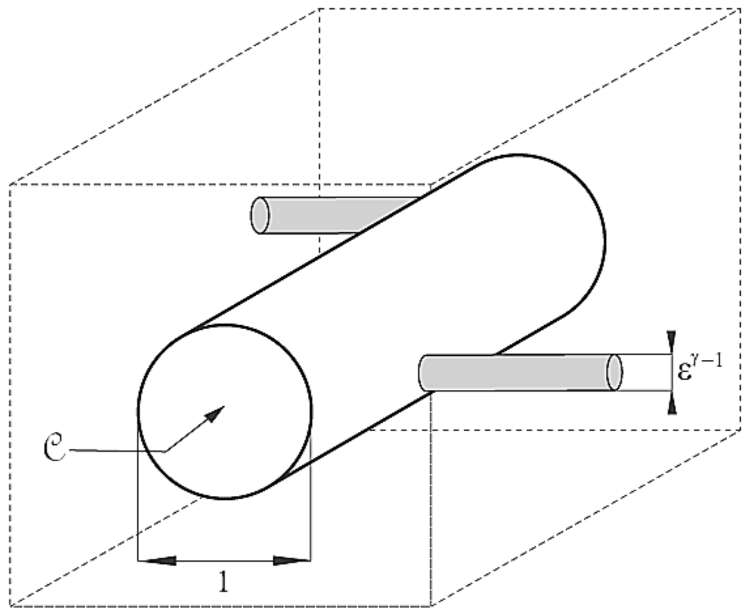

FIG. 2.1. *Cell of periodicity Y containing a disturbed-drift cylinder $\mathcal{S} = \mathcal{C} \times]-1/2, 1/2[$ and a waste package P_ε (gray part).*

$$\Gamma_\varepsilon = \partial \mathcal{P}_\varepsilon = \bigcup_{i,j=1}^{m} \mathcal{P}_\varepsilon(i,j) \text{ the boundary of waste packages,} \tag{2.2}$$

$$\mathcal{S}_\varepsilon = \bigcup_{i,j=1}^{m} \mathcal{S}_\varepsilon(i,j) \text{ the union of all disturbed-drift cylinders,} \tag{2.3}$$

$$\mathcal{C}_\varepsilon = \bigcup_{i,j=1}^{m} \varepsilon\,(\,(i,j) + \mathcal{C}) \text{ the union of all disturbed-drift cylinder sections.} \tag{2.4}$$

Assuming that the disturbed zone is not intersecting any of the waste packages we assume that $\mathcal{P}_\varepsilon \cap \mathcal{S}_\varepsilon = \emptyset$. Finally, we denote by $\Omega_\varepsilon = \Omega \backslash \mathcal{P}_\varepsilon$ the part of the domain Ω, left after removing all the deposition holes $\mathcal{P}_\varepsilon$. In what follows, all functions defined on $\mathcal{C}$ are assumed to be extended by 1-periodicity in the y_2 direction to a periodic set $\cup_{i=0}^{m}(\,(i,0) + \mathcal{C}\,)$.

2.2. Mesoscopic model and equations describing the solute transport. Let $T > 0$, and let us denote

$$\Omega_\varepsilon^T = \Omega_\varepsilon \times]0,T[\,, \quad \Gamma_\varepsilon^T = \Gamma_\varepsilon \times]0,T[\,. \tag{2.5}$$

The time behavior of the flux produced from a waste package is given by the function $\Phi \in L^\infty([0,T])$ which has, as mentioned before, a compact support $[0, t_m] \subset [0, T[$. The radioactive decay constant is $\lambda = \frac{\log 2}{\tau} > 0$, where τ is the half-life of the radio-element, and the initial concentration of the radio material in the soil is $f_0 \in L^\infty(\Omega)$.

In the very worst possible scenario, a higher permeability in the disturbed zone with a possible water inflow if the disturbed zone is connected to the handling galleries and shafts will lead to a Darcy velocity $\mathbf{v}^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon)$ much larger in the disturbed zone than in the nondisturbed host rock. In order to describe the range of situations, from the best to the worst, we introduce a parameter $\beta \geq 0$ and write the velocity in the disturbed and nondisturbed zone:

$$\mathbf{v}^\varepsilon(x) = \begin{cases} \mathbf{v}^h(x) \text{ in the nondisturbed host rock } \Omega_\varepsilon \backslash \mathcal{S}_\varepsilon, \\ \varepsilon^{-\beta}\, \mathbf{v}^d(x', x_2/\varepsilon, x_3/\varepsilon) \text{ in the disturbed-drift zone } \mathcal{S}_\varepsilon. \end{cases}$$

In this way, we relate the intensity of the water inflow to the drift diameter, in order to see this water inflow in the macroscopic model.

Here we suppose, for simplicity, that the Darcy velocity in the drifts is unidirectional, i.e., that $\mathbf{v}^d = v_1^d\, \mathbf{e}_1$. This is a natural assumption for our model since only the fast convection in the direction of the shaft should count in a macroscopic model. General situations can be treated using our methods but with certain technical difficulties (see Remark 1 in section 6); if $\mathbf{v}^d$ was not unidirectional the notations would be more complicated, but the ideas would be the same. Moreover, due to the incompressibility of the underground water flow, we also assume that the velocity $\mathbf{v}^\varepsilon$ is divergence free; and consequently v_1^d will not depend on x_1. We assume that $\mathbf{v}^h \in C^{0,1}(\overline{\Omega})$, $v_1^d \in C^{0,1}([0, L] \times \overline{\mathcal{C}})$ and that $v_1^d \neq 0$.

As in [10], the effective diffusion/dispersion tensor is given by

$$\mathbf{A}^\varepsilon = d(x)\, \mathbf{I} + |\mathbf{v}^\varepsilon|\, \{\alpha_L^\varepsilon\, \mathbf{E}(\mathbf{v}^\varepsilon) + \alpha_T^\varepsilon\, (\mathbf{I} - \mathbf{E}(\mathbf{v}^\varepsilon)\,)\},$$

with α_L^ε the longitudinal and α_T^ε the transversal dispersion coefficients, and

$$[\mathbf{E}(\mathbf{v}^\varepsilon)]_{ij} = \frac{v_i^\varepsilon\, v_j^\varepsilon}{|\mathbf{v}^\varepsilon|^2}.$$

For $k = L, T$

$$\alpha_k^\varepsilon = \begin{cases} \alpha_k^h \text{ in the nondisturbed host rock } \Omega_\varepsilon \backslash \mathcal{S}_\varepsilon, \\ \alpha_k^d \text{ in the disturbed-drift zone } \mathcal{S}_\varepsilon, \end{cases}$$

while the molecular diffusion

$$d(x) \geq d_0 > 0, \quad d \in L^\infty(\Omega). \tag{2.6}$$

Thus, we will assume herein that the effective diffusion/dispersion tensor has the form

$$\mathbf{A}^\varepsilon(x) = \begin{cases} \mathbf{A}^h(x) \text{ in the nondisturbed host rock } \Omega_\varepsilon \backslash \mathcal{S}_\varepsilon, \\ d(x/\varepsilon)\, \mathbf{I} + \varepsilon^{-\beta}\, \mathbf{A}^d(x_2, x_2/\varepsilon, x_3/\varepsilon) \text{ in the disturbed-drift zone } \mathcal{S}_\varepsilon, \end{cases}$$

where, assuming that the convection in the drifts goes only in the direction of the drift and that the transversal component of the dispersion, α_T^ε, is negligible, the matrix $\mathbf{A}^d$ now has the form

$$\mathbf{A}^d(x_2, y_2, y_3) = |v_1^d|\, (x_2, y_2, y_3)\ \ (\, \mathbf{e}_1 \otimes \mathbf{e}_1\,).$$

Although the porosity ω^ε is also higher in the disturbed zone than in the nondisturbed host rock, the difference is not as large as for the permeability, and we will assume that in both zones the porosities are of the same order, i.e.,

$$\omega^\varepsilon = \begin{cases} \omega^h \text{ in the nondisturbed host rock } \Omega_\varepsilon \backslash \mathcal{S}_\varepsilon, \\ \omega^d \text{ in the disturbed-drift zone } \mathcal{S}_\varepsilon, \end{cases}$$

where ω^h, ω^d are strictly positive, continuous, and uniformly bounded functions.

The leaking of the waste packages is described by giving a boundary flux on $\Gamma_\varepsilon = \partial\mathcal{P}_\varepsilon = \cup_{i,j=1}^m \partial\mathcal{P}_\varepsilon(i,j)$, the union of all interfaces between the waste packages and the domain Ω_ε. In order to keep a nontrivial limit, we will assume that the flux Φ_ε on Γ_ε depends on ε in such a way that there exists a continuous $\Phi(t)$:

$$\lim_{\varepsilon\to 0} \varepsilon^{\gamma-1}\, \Phi_\varepsilon = \Phi \quad \text{uniformly in } \; t. \tag{2.7}$$

In the following, for brevity, for any set X, we will use the superscript T to denote $X^T = X \times]0,T[$.

According to all the previous assumptions and notations, the transport of the solute concentration φ in a repository module is now described by

$$\omega^\varepsilon \, \frac{\partial \varphi_\varepsilon}{\partial t} - \operatorname{div}(\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon) + (\mathbf{v}^\varepsilon \cdot \nabla)\varphi_\varepsilon + \lambda\, \omega^\varepsilon\, \varphi_\varepsilon = 0 \quad \text{in } \; \Omega_\varepsilon^T, \tag{2.8}$$

$$\varphi_\varepsilon(0,x) = f_0(x), \quad x \in \Omega_\varepsilon, \tag{2.9}$$

$$\mathbf{n} \cdot (\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon - \mathbf{v}^\varepsilon\, \varphi_\varepsilon) = \; \Phi_\varepsilon(t) \quad \text{on } \; \Gamma_\varepsilon^T. \tag{2.10}$$

We also need to impose some boundary condition on the exterior boundary of the domain $\partial\Omega$. For this, we split the exterior boundary of the repository module $\partial\Omega$ in three parts:

$\mathcal{F}_\varepsilon = \mathcal{S}_\varepsilon \cap \{x_1 = L\}$ the head of the disturbed-drift cylinders, intersecting with the connecting gallery,

$\mathcal{B}_\varepsilon = \mathcal{S}_\varepsilon \cap \{x_1 = 0\}$ the back of the disturbed-drift cylinders, sealed frontier,

$\mathcal{R}_\varepsilon = \partial\Omega \backslash (\mathcal{F}_\varepsilon \cup \mathcal{B}_\varepsilon)$ the rest of the exterior boundary of Ω.

On the back sides of the drifts, $\mathcal{B}_\varepsilon$, we have seals, and thus we assume that the concentration is equal to zero. But on $\mathcal{R}_\varepsilon \cup \mathcal{F}_\varepsilon$ including the drifts heads, we assume that the concentration obeys the Fourier law, i.e., that the rate of flow is proportional to the difference between concentrations inside and outside the repository module. Denoting by $g_\varepsilon \in L^2(0,T;L^2(\partial\Omega))$ the trace of the exterior concentration and by $\kappa(x) \in L^\infty(\partial\Omega)$, $0 < \kappa_0 \le \kappa(x) \le \kappa_1$ the function describing the rate of proportionality, we write the boundary conditions:

$$\mathbf{n} \cdot (\mathbf{A}^\varepsilon \nabla \varphi_\varepsilon - \mathbf{v}^\varepsilon\, \varphi_\varepsilon) = \kappa \;\, (\varphi_\varepsilon - g_\varepsilon) \quad \text{on } \; \mathcal{R}_\varepsilon^T \cup \mathcal{F}_\varepsilon^T, \tag{2.11}$$

$$\varphi_\varepsilon = 0 \quad \text{on } \; \mathcal{B}_\varepsilon^T; \tag{2.12}$$

we will make precise later the value of g_ε on $\mathcal{F}_\varepsilon$, the drift head/connecting gallery intersection, according to the intensity of the water flow in the repository.

3. The three global models. The three different scenarii (considered in the European exercise BENIPA) corresponding to the connected shafts galleries and drifts being either perfectly sealed, poorly sealed, or not sealed at all are now mathematically represented by means of different values of $\varepsilon^{-\beta}$, the mathematical parameter used for describing the water inflow regime, and the concentration rates on the head drifts. We remark that there are three typical different behaviors for β equal to, strictly greater than, or lower than one. The solutions φ_ε of (2.8)–(2.10) are defined on a family of domains Ω_ε^T depending on the parameter ε, and in order to use the weak convergence methods we should extend these solutions to the whole domain Ω^T. After this extension the a priori estimates are no longer in $H^1(\Omega)$ but only in $W^{1,\gamma^*}(\Omega)$, where $\gamma^* = \frac{1}{1-\frac{1}{2\gamma}}$. The detailed proofs for these extension properties follow the ideas from [6] and were given in a previous paper [5]. In what follows we will denote φ_ε the solution of (2.8)–(2.10) and its extension by the same symbol.

In what follows we use the notation $\stackrel{2-d\mu}{\longrightarrow}$ for the two-scale convergence with respect to the singular measure, defined in Appendix A.

Depending on the power in $\varepsilon^{-\beta}$ we distinguish three different cases:

$\beta < 1$ This is the simplest case when the EDZ (disturbed drifts) do not make any contribution; i.e., the repository behaves as if this zone was not there. Then $\varphi_\varepsilon \to \varphi$ weakly in $L^2(0,T;W^{1,\gamma^*}(\Omega))$, where φ is the unique solution of the problem (5.2)–(5.4).

$\beta = 1$ This is the most interesting case when the intensity of the processes inside and outside the EDZ (disturbed drifts) are of the same order and there is strong interaction between them. In this case $\varphi_\varepsilon \to \varphi$ weakly in $L^2(0,T;W^{1,\gamma^*}(\Omega))$ and $\varphi_\varepsilon \stackrel{2-d\mu}{\longrightarrow} \varphi(x_1,x_2,0)$, where φ is the unique solution of the coupled problem (6.4).

$\beta > 1$ In this case the process in the drifts is dominant, and we do not see the rest of the domain in the limit. Indeed $\varepsilon^{(1-\beta)/2}\ \varphi_\varepsilon \stackrel{2-d\mu}{\longrightarrow} \varphi^0$, where φ^0 is the unique solution of a one-dimensional problem posed on $]0,L[$. The cases $1<\beta<2$, $\beta=2$, and $\beta>2$ do not have the same dependency of the two-scale limit φ^0 on the fast variable y. Nevertheless, the mean value of the limit remains the same.

4. A priori estimates. As always, the asymptotic analysis starts with sharp a priori estimates based on the variational formulation

$$
\begin{aligned}
&-\int_{\Omega_\varepsilon^T} \omega_\varepsilon\ \varphi_\varepsilon\ \frac{\partial z_\varepsilon}{\partial t} + \int_0^T \int_{\Omega_\varepsilon \setminus \mathcal{S}_\varepsilon} \mathbf{A}^h\ \nabla\varphi_\varepsilon\ \nabla z_\varepsilon \\
&+ \varepsilon^{-\beta} \int_0^T \int_{\mathcal{S}_\varepsilon} (\mathbf{A}^d + \varepsilon^\beta d\,\mathbf{I})\ \nabla\varphi_\varepsilon\ \nabla z_\varepsilon + \varepsilon^{-\beta} \int_0^T \int_{\mathcal{C}_\varepsilon} \mathbf{v}^d \cdot \nabla\varphi_\varepsilon\ z_\varepsilon \\
&+ \lambda \int_{\Omega_\varepsilon^T} \omega_\varepsilon \varphi_\varepsilon\ z_\varepsilon + \int_0^T \int_{\Omega_\varepsilon \setminus \mathcal{S}_\varepsilon} \mathbf{v}^h \cdot \nabla\varphi_\varepsilon\ z_\varepsilon \\
&= \int_{\Omega_\varepsilon} \omega^h f_0\ z_\varepsilon(0,x) + \int_{\mathcal{F}_\varepsilon^T \cup \mathcal{R}_\varepsilon^T} \kappa\ (g_\varepsilon - \varphi_\varepsilon)\ z_\varepsilon + \int_0^T \Phi_\varepsilon \int_{\Gamma_\varepsilon} z_\varepsilon.
\end{aligned} \tag{4.1}
$$

PROPOSITION 4.1. *Let $\{\varphi_\varepsilon\}$ be the sequence of solutions to the problem* (2.8)–(2.12)*. Then there exists a constant $C>0$, independent from ε, such that*

$$|\nabla\varphi_\varepsilon|_{L^2(0,T;L^2(\Omega_\varepsilon))} \le C, \tag{4.2}$$

$$|\varphi_\varepsilon|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \le C, \tag{4.3}$$

$$|\varphi_\varepsilon|_{L^2(0,T;L^2(\mathcal{S}_\varepsilon))} \le C\,\varepsilon^{\frac{\beta}{2}}, \tag{4.4}$$

$$\left|\frac{\partial\varphi_\varepsilon}{\partial x_1}\right|_{L^2(0,T;L^2(\mathcal{S}_\varepsilon))} \le C\,\varepsilon^{\frac{\beta}{2}}. \tag{4.5}$$

Proof. The main difficulty is to estimate the surface integral over the waste package boundaries Γ_ε. To do so we use the study of such a surface integral given in Appendix B. In particular we use the strong convergence (B.1) and the trace estimate on Σ, leading to

$$\left|\varepsilon^{1-\gamma}\int_{\Gamma_\varepsilon} z\right| \le \left|\varepsilon^{1-\gamma}\int_{\Gamma_\varepsilon} z - \mathcal{M}\int_{\Sigma} z(x',0)dx'\right| + \left|\mathcal{M}\int_{\Sigma} z(x',0)dx'\right|$$
$$\le C|z|_{H^1(\Omega)}, \quad z \in H^1(\Omega). \tag{4.6}$$

We use φ_ε as the test function in (2.8)–(2.12) and apply (4.6). We obtain

$$\frac{1}{2}\left|\sqrt{\omega^\varepsilon}\,\varphi_\varepsilon(\,\cdot\,,T)\right|^2_{L^2(\Omega_\varepsilon)} + (\mathbf{A}^\varepsilon\nabla\varphi_\varepsilon|\nabla\varphi_\varepsilon)_{L^2(\Omega_\varepsilon^T)} + \lambda\,|\sqrt{\omega^\varepsilon}\,\varphi_\varepsilon|^2_{L^2(\Omega_\varepsilon^T)} + |\sqrt{\kappa}\,\varphi_\varepsilon|^2_{L^2(\partial\Omega^T)}$$
$$= \int_0^T \Phi_\varepsilon \int_{\Gamma_\varepsilon}\varphi_\varepsilon + \int_{\partial\Omega^T}\kappa\varphi_\varepsilon\, g_\varepsilon + \frac{1}{2}\left|\sqrt{\omega^\varepsilon}\, f_0\right|^2_{L^2(\Omega_\varepsilon)}.$$

The first integral on the right-hand side is estimated using (4.6). Thus it remains only to estimate the integral over $\partial\Omega$. Depending on β, we have three different cases:

$$\left|\int_{\partial\Omega^T}\kappa\varphi_\varepsilon\, g_\varepsilon\right| \le C|\varphi_\varepsilon|_{L^2(0,T;H^1(\Omega_\varepsilon))} \text{ in case } \beta<1,$$
$$\left|\int_{\partial\Omega^T}\kappa\varphi_\varepsilon\, g_\varepsilon\right| = \left|\varepsilon^{-p}\int_{\mathcal{F}_\varepsilon^T}\kappa\, g^d\,\varphi_\varepsilon + \int_{\mathcal{R}_\varepsilon^T}\kappa\, g^h\,\varphi_\varepsilon\right|$$
$$\le C + \frac{1}{2}\varepsilon^{1-2p}\,|v_1^d|\,\left|\frac{\partial\varphi_\varepsilon}{\partial x_1}\right|^2_{L^2(\mathcal{S}_\varepsilon^T)} + C\,|\varphi_\varepsilon|_{L^2(0,T;H^1(\Omega_\varepsilon))} \text{ in case } \beta\ge 1,$$

with

$$p = \begin{cases} \beta, & \beta = 1, \\ \frac{\beta+1}{2}, & \beta > 1. \end{cases}$$

To prove (4.4) we need the Poincaré inequality

$$|\varphi|_{L^2(\mathcal{S}_\varepsilon)} \le 2\,L\,\left|\frac{\partial\varphi}{\partial x_1}\right|_{L^2(\mathcal{S}_\varepsilon)}, \tag{4.7}$$

which can be proved by direct integration, using the Dirichlet condition on the bottom. ∎

5. The simplest case, $0 \le \beta < 1$, where the EDZ effects do not appear at the global scale. This case is an attempt to mathematically describe a situation corresponding to a scenario where the connected shaft galleries and drifts are perfectly sealed. The concentration rate on the drifts heads (inside the connecting gallery) is of same order as everywhere else on the domain Ω_ϵ boundaries, and consequently we assume that $g_\varepsilon = g$ does not depend on ε. In this case the process in the disturbed-drift zone is not important enough to appear in the corresponding global model. According to the volume of the waste packages, $\varepsilon^{\gamma-1}$, we assume that the flux Φ_ε on the boundaries of the waste packages is big enough, i.e., that (2.7) holds.

THEOREM 5.1. *Let $\beta < 1$, and let $\{\varphi_\varepsilon\}$ be the sequence of solutions to the problem* (2.8)–(2.12). *Then*

$$\varphi_\varepsilon \rightharpoonup \varphi \text{ weak* in } L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;W^{1,\gamma^*}(\Omega)), \tag{5.1}$$

where φ is the unique solution of the problem

$$\omega^h \frac{\partial\varphi}{\partial t} - \operatorname{div}(\mathbf{A}^h\nabla\varphi) + (\mathbf{v}^h \cdot \nabla)\varphi + \lambda\,\omega^h\,\varphi = 0 \ \text{in} \ \tilde{\Omega}^T = (\Omega\backslash\Sigma)\times\,]0,T[\,, \tag{5.2}$$

$$\varphi(x,0) = f_0(x), \ \ x\in\tilde{\Omega} = \Omega\backslash\Sigma, \ \ \mathbf{n}\cdot(\mathbf{A}^h\nabla\varphi - \mathbf{v}^h\,\varphi) = \kappa\,(\varphi - g) \ \text{on} \ (\partial\Omega)^T, \tag{5.3}$$

$$[\varphi] = 0, \ \ \big[\mathbf{e}_3\cdot\mathbf{A}^h\nabla\varphi - (\mathbf{v}^h\cdot\mathbf{e}_3)\,\varphi\big] = -\Phi\,\mathcal{M} \ \text{on} \ \Sigma, \tag{5.4}$$

where $[w](x') = w(x',0+) - w(x',0-)$, $x' = (x_1,x_2)$, *denotes the jump over* Σ *and where* $\mathcal{M}$ *denotes the limit of the rescaled area of a waste package, i.e.,*

$$\mathcal{M} = \lim_{\varepsilon\to 0}\varepsilon^{1-\gamma}\,|\partial P_\varepsilon|. \tag{5.5}$$

Proof. The proof is essentially the same as the proof of Theorem 1 in [5], and it relies on (B.1) with additional properties of the a priori estimates (4.2)–(4.5). The only difference comparing to [5] is that in the variational form of (2.8)–(2.12) written in (4.1) we now have two additional integrals

$$J_1^\varepsilon = \varepsilon^{-\beta}\int_0^T\int_{\mathcal{C}_\varepsilon}\mathbf{A}^d\,\nabla\varphi_\varepsilon\,\nabla\psi,$$
$$J_2^\varepsilon = \varepsilon^{-\beta}\int_0^T\int_{\mathcal{C}_\varepsilon}(\mathbf{v}^d\cdot\nabla)\,\varphi_\varepsilon\,\psi$$

coming from the drifts. Here $\psi\in C^1(\Omega^T)$ is a test function. For given ψ, we estimate those two integrals as

$$|J_1^\varepsilon| \le C\varepsilon^{-\beta}|\nabla\varphi_\varepsilon|_{L^2(0,T;L^2(\mathcal{C}_\varepsilon))}\,|\mathcal{C}_\varepsilon|^{\frac12} \le C\,\varepsilon^{-\beta+\frac{\beta}{2}+\frac12} = C\varepsilon^{\frac12(1-\beta)}\to 0$$

and similarly for J_2^ε. Additional technical difficulty comes from the Dirichlet condition on $\mathcal{B}_\varepsilon$. It can be overcome using the construction from Appendix D and the existence for any $z\in H^1(\Omega)$ of a sequence of functions $\{z_m\}_{m\in\mathbf{N}}$, $z_m\in C^1(\overline{\Omega})$, such that $z_m(0,x_2,0)=0$ and $z_m\to z$ in $H^1(\Omega)$ since on a one-dimensional line $c = \{x\in\mathbf{R}^3;\ x_1=0,\ x_3=0\}$ the trace of a function from $H^1(\Omega)$ cannot be specified. □

6. The critical case, $\beta = 1$, when contributions from the EDZ and from the undisturbed host rock are of the same order. This case is an attempt to mathematically describe a situation corresponding to a scenario where the connected shafts galleries and drifts are poorly sealed. In order to keep the influence of the disturbed drifts at the global level, we assume the concentration rate g_ε to be stronger on the drifts heads (inside the connecting gallery) and to have the form

$$g_\varepsilon = \begin{cases} \varepsilon^{-1}\,g^d(t,x_2,\frac{x_2}{\varepsilon},\frac{x_3}{\varepsilon}) & \text{on the disturbed-drift head } \mathcal{F}_\varepsilon,\\ g^h(t,x) & \text{on } \mathcal{R}_\varepsilon \text{ except on the interface with the connecting gallery,}\end{cases} \tag{6.1}$$

with $g^h\in L^2(0,T;L^2(\partial\Omega))$, $g^d\in C([0,T]\times[0,L]\times\overline{\mathcal{C}})$. We assume again that the leaking is described by (2.7). In this case both contributions from the disturbed drifts and from the undisturbed host rock will appear in the corresponding global model.

In what follows we will use the notation $\langle f\rangle = \int_{\mathcal{C}} f(x,y_2,y_3)\,dy_2\,dy_3$ for the mean value of any function f over the rescaled drift cross-section $\mathcal{C}$. Before we continue we need to define an appropriate functional space

$$V = \left\{\phi\in H^1(\Omega);\ \ \frac{\partial\phi}{\partial x_1}(x_1,x_2,0)\in L^2(\Sigma),\ \ \phi(0,x_2,0) = 0\right\}$$

equipped by the norm

$$|\phi|_V = |\phi|_{H^1(\Omega)} + \left|\frac{\partial \phi}{\partial x_1}\right|_{L^2(\Sigma)}.$$

Obviously, V is a Hilbert space and the term in the norm $\left|\frac{\partial\phi}{\partial x_1}\right|_{L^2(\Sigma)}$ gives sense to a trace $\phi(0, x_2, 0)$ in $L^2(0, L)$. Indeed, it is easy to prove that

$$|\phi(0, x_2, 0)|_{L^2(0,L)} \leq C|\phi|_V.$$

The space $V \cap C^1(\overline{\Omega})$ is dense in V. In this section we use the notion of the two-scale convergence with respect to the singular measure denoted $\stackrel{2-d\mu}{\longrightarrow}$, introduced in Appendix A.

THEOREM 6.1. *Let* $\beta = 1$, *and let* $\{\varphi_\varepsilon\}$ *be the sequence of solutions to the problem* (2.8)–(2.12). *Then*

$$\varphi_\varepsilon \rightharpoonup \varphi \text{ weak* in } L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;W^{1,\gamma^*}(\Omega)), \tag{6.2}$$

$$\varphi_\varepsilon \stackrel{2-d\mu}{\longrightarrow} \varphi^0(t, x_1, x_2) = \varphi(t, x_1, x_2, 0), \tag{6.3}$$

where $\varphi \in L^2(0, T; V)$ *is the unique solution of the following variational problem:*

$$\begin{aligned} &\int_0^T \int_\Omega \left(-\omega^h \varphi \frac{\partial \psi}{\partial t} + \mathbf{A}^h \nabla\varphi\nabla\psi + (\mathbf{v}^h \cdot \nabla)\varphi\,\psi + \lambda\omega^h\varphi\,\psi \right) + \int_{\partial\Omega^T} \kappa\varphi\psi \\ &+ \int_0^T \int_\Sigma \left(\langle \mathbf{A}^d_{11}\rangle \frac{\partial\varphi}{\partial x_1}(t, x', 0) \frac{\partial\psi}{\partial x_1}(t, x', 0) + \langle v^d_1 \rangle \frac{\partial\varphi}{\partial x_1}(t, x', 0)\,\psi(t, x', 0) \right) \\ &= \int_0^T \int_\Sigma \Phi\,\psi(t, x', 0)\,\mathcal{M} + \int_\Omega \omega^h f_0\,\psi(0, x) + \int_{\partial\Omega^T} \kappa\psi\, g^h + \int_0^T \int_0^L \kappa\,\psi(t, L, x_2, 0)\, g^d \\ &\qquad \text{for any } \psi \in H^1(0,T;V) \text{ such that } \psi(T, x) = 0. \end{aligned} \tag{6.4}$$

REMARK 1. *If* $\mathbf{A}^d$ *was not diagonal, as, for instance, for nonunidirectional fluid flow in disturbed drifts, it would be necessary to introduce* R, *the solution of the auxiliary problem on the cross-section of the drift:*

$$\begin{aligned} &\mathrm{div}_y\,(\mathbf{A}^d\,\nabla_y R) = -\mathrm{div}_y\,(\mathbf{A}^d\,\mathbf{e}_1) \quad \text{in } \mathcal{C}, \\ &\mathbf{n}\cdot\mathbf{A}^d\,\nabla_y\,R = -\mathbf{n}\cdot\mathbf{A}^d\,\mathbf{e}_1 \quad \text{on } \partial\mathcal{C}. \end{aligned} \tag{6.5}$$

The solution R *would then enter the homogenized diffusion tensor on* Σ, *and it would also produce an oscillatory part in the gradient of the two-scale limit* φ_1 *which will have the form*

$$\varphi_1 = R(y)\,\frac{\partial\varphi^0}{\partial x_1}.$$

Moreover, if the concentration is nonunidirectional in the drifts, the convergence proof from Theorem 6.1 *requires additional assumptions in order to get a priori estimates on the gradient and an additional compactness theorem for the* $2 - d\mu$ *convergence of the gradient.*

Proof of Theorem 6.1. Due to the compactness theorem, Theorem A.2 in Appendix A, there exists some $\varphi \in L^2(0,T;W^{1,\gamma^*}(\Omega))$, $\varphi^0 \in L^2(0,T;L^2(\Sigma\times\mathcal{C})\ )$, $\frac{\partial\varphi^0}{\partial x_1} \in L^2(0,T;L^2(\Sigma\times\mathcal{C})\ )$ such that

$$\begin{aligned}
&\varphi_\varepsilon \to \varphi \quad \text{weakly in } \ L^2(0,T;W^{1,\gamma^*}(\Omega)),\\
&\varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0,\\
&\frac{\partial\varphi_\varepsilon}{\partial x_1} \overset{2-d\mu}{\longrightarrow} \frac{\partial\varphi^0}{\partial x_1}.
\end{aligned}$$

Furthermore, due to Proposition A.3 in Appendix A, $\varphi|_\Sigma = \varphi^0(t,x_1,x_2)$. Taking now a simple test function $z = z(t,x) \in H^1(0,T;V)\cap C^1(\overline{\Omega^T})$ such that $z(T,x)=0$, we modify it and get z_ε, as described in Appendix D, which is then used as a test function in the variational form:

$$\begin{aligned}
&-\int_{\Omega_\varepsilon^T} \omega_\varepsilon\,\varphi_\varepsilon\,\frac{\partial z_\varepsilon}{\partial t} + \int_0^T\int_{\Omega_\varepsilon\setminus\mathcal{S}_\varepsilon} \mathbf{A}^h\,\nabla\varphi_\varepsilon\,\nabla z_\varepsilon + \varepsilon^{-1}\int_0^T\int_{\mathcal{S}_\varepsilon}(\mathbf{A}^d+\varepsilon d\,\mathbf{I})\,\nabla\varphi_\varepsilon\,\nabla z_\varepsilon\\
&+\varepsilon^{-1}\int_0^T\int_{\mathcal{S}_\varepsilon}\mathbf{v}^d\cdot\nabla\varphi_\varepsilon\,z_\varepsilon + \lambda\int_{\Omega_\varepsilon^T}\omega_\varepsilon\varphi_\varepsilon\,z_\varepsilon + \int_0^T\int_{\Omega_\varepsilon\setminus\mathcal{S}_\varepsilon}\mathbf{v}^h\cdot\nabla\varphi_\varepsilon\,z_\varepsilon\\
&=\int_{\Omega_\varepsilon}\omega^h f_0\,z_\varepsilon(0,x) + \int_{\mathcal{F}_\varepsilon^T}\kappa\,(\varepsilon^{-1}g^d-\varphi_\varepsilon)\,z_\varepsilon + \int_{\mathcal{R}_\varepsilon^T}\kappa\,(g^h-\varphi_\varepsilon)\,z_\varepsilon + \int_0^T \Phi_\varepsilon\int_{\Gamma_\varepsilon} z_\varepsilon.
\end{aligned}$$

Using the above two-scale convergence and the weak convergence, we pass to the limit and obtain

$$\begin{aligned}
&\int_0^T\int_\Omega\left(-\omega^h\,\varphi\,\frac{\partial z}{\partial t} + \mathbf{A}^h\nabla\varphi\nabla z + (\mathbf{v}^h\cdot\nabla)\varphi\,z + \lambda\omega^h\varphi\,z\right)\\
&+\int_0^T\int_\Sigma\left(\langle\mathbf{A}^d_{11}\rangle\,\frac{\partial\varphi}{\partial x_1}(t,x',0)\,\frac{\partial z}{\partial x_1}(t,x',0) + \langle\mathbf{v}^d_1\rangle\,\frac{\partial\varphi}{\partial x_1}(t,x',0)\,z(t,x',0)\right)\\
&=\int_0^T\int_\Sigma \Phi\,z(t,x',0)\,\mathcal{M} + \int_\Omega\omega^h\,f_0\,z(0,x) + \int_0^T\int_0^L\kappa\,\langle g^d\rangle\,z(L,x_2,0)\\
&+\int_{\partial\Omega^T}\kappa\,(g^h-\varphi)\,z.
\end{aligned}$$

The strong convergence of z_ε towards z (see Appendix D) enables us to drop the subscript ε in the test function. Indeed, the only critical term is the integral

$$\varepsilon^{-\beta}\int_{\mathcal{S}_\varepsilon}\mathbf{A}^d_{11}\frac{\partial\varphi_\varepsilon}{\partial x_1}\,\frac{\partial z_\varepsilon}{\partial x_1} = \varepsilon^{-\beta}\int_{\mathcal{S}_\varepsilon}\mathbf{A}^d_{11}\frac{\partial\varphi_\varepsilon}{\partial x_1}\,\frac{\partial(z-z_\varepsilon)}{\partial x_1} + \varepsilon^{-\beta}\int_{\mathcal{S}_\varepsilon}\mathbf{A}^d_{11}\frac{\partial\varphi_\varepsilon}{\partial x_1}\,\frac{\partial z}{\partial x_1}.$$

But, due to the construction of z_ε (see Appendix D) we estimate the last integral as

$$\left|\varepsilon^{-\beta}\int_{\mathcal{S}_\varepsilon}\mathbf{A}^d_{11}\frac{\partial\varphi_\varepsilon}{\partial x_1}\,\frac{\partial(z-z_\varepsilon)}{\partial x_1}\right| \le \varepsilon^{-\beta}\left|\frac{\partial(z-z_\varepsilon)}{\partial x_1}\right|_{L^2(\Omega)}\ \left|\frac{\partial\varphi_\varepsilon}{\partial x_1}\right|_{L^2(\mathcal{S}_\varepsilon)} \le C\,\varepsilon^{1-\frac{\beta}{2}}.$$

Thus we get (6.4).

7. The case $\beta > 1$, where the dominant process is located in the EDZ. This case is an attempt to mathematically describe a situation corresponding to a scenario where the connected shafts galleries and drifts have no seals or barriers of any sort. In order to keep the influence of the disturbed drifts at the global level, we assume the concentration rate g_ε to be much stronger on the drifts heads (inside the connecting gallery) than on all the others drift or disposal hole boundaries. We assume then that the concentration, g_ε, on the drifts heads (inside the connecting gallery) and everywhere else on the boundaries is given by

$$g_\varepsilon = \begin{cases} \varepsilon^{-\frac{\beta+1}{2}}\, g^d & \text{on the disturbed-drift head } \mathcal{F}_\varepsilon, \\ g^h & \text{on } \mathcal{R}_\varepsilon, \text{ everywhere except on the connection with the gallery,} \end{cases} \tag{7.1}$$

with the same assumptions on g^h, g^d as in the previous section. In this last case the conductivity in the EDZ (disturbed-drift zone) is so high that the process inside this zone will dominate the global scale model. Again, for the flux Φ_ε on the boundaries of the waste packages, we assume that (2.7) holds.

Due to the a priori estimates (4.2)–(4.5), with assumption (7.1) we get the two-scale limits.

LEMMA 7.1. *If $\beta > 1$, there exists $\varphi^0 \in L^2(\Sigma \times \mathcal{C})$ such that $\frac{\partial \varphi^0}{\partial x_1} \in L^2(\Sigma \times \mathcal{C})$ and*

$$\varepsilon^{\frac{1-\beta}{2}}\, \varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0, \tag{7.2}$$

$$\varepsilon^{\frac{1-\beta}{2}}\, \frac{\partial \varphi_\varepsilon}{\partial x_1} \overset{2-d\mu}{\longrightarrow} \frac{\partial \varphi^0}{\partial x_1}. \tag{7.3}$$

Furthermore, in case $\beta \le 2$ we have in addition the following lemma.

LEMMA 7.2. *If $\beta < 2$, then φ^0 does not depend on y_2, y_3. If $\beta = 2$, then φ^0 may depend on y_2, y_3, and*

$$\varepsilon^{\frac{1-\beta}{2}}\, \frac{\partial}{\partial x_\alpha}\, \varepsilon\, \varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \frac{\partial \varphi^0}{\partial y_\alpha}, \quad \alpha = 2, 3. \tag{7.4}$$

Proof. Using the a priori estimate (4.2) we get

$$\varepsilon^{-\frac{1}{2}} \left| \varepsilon^{\frac{1-\beta}{2}}\, \frac{\partial}{\partial x_\alpha}\, \varepsilon\, \varphi_\varepsilon \right|_{L^2(\mathcal{S}_\varepsilon)} \le C\, \varepsilon^{1-\frac{\beta}{2}}.$$

Thus there exists some $f_\alpha \in L^2(\Sigma \times \mathcal{C})$ such that

$$\varepsilon^{\frac{1-\beta}{2}}\, \frac{\partial}{\partial x_\alpha}\, \varepsilon\, \varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} f_\alpha$$

with $f_\alpha = 0$ for $\beta < 2$. Since for $\eta \in C_0^\infty(\mathcal{C})$ and $\psi \in L^2(0,T; H_0^1(\Omega))$ we have

$$\begin{aligned} \int_0^T \int_{\mathcal{S}_\varepsilon} \frac{\partial \varphi_\varepsilon}{\partial x_\alpha}\, \eta\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\, \psi &= -\int_0^T \int_{\mathcal{S}_\varepsilon} \frac{\partial \psi}{\partial x_\alpha}\, \eta\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\, \varphi_\varepsilon \\ &\quad - \varepsilon^{-1} \int_0^T \int_{\mathcal{S}_\varepsilon} \frac{\partial \eta}{\partial y_\alpha}\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\, \varphi_\varepsilon\, \psi, \end{aligned} \tag{7.5}$$

multiplying (7.5) by $\varepsilon^{\frac{1-\beta}{2}}$ and passing to the two-scale limit, we obtain that $f_\alpha = \frac{\partial \varphi^0}{\partial x_\alpha}$, proving the claim. □

Now we characterize the two-scale limits obtained in Lemmas 7.1 and 7.2. We start with the case $1 < \beta < 2$.

THEOREM 7.3. *Let $1 < \beta < 2$, and let $\{\varphi_\varepsilon\}$ be the sequence of solutions to the problem* (2.8)–(2.12). *Then*

$$\varepsilon^{\frac{1-\beta}{2}}\,\varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0, \tag{7.6}$$

$$\varepsilon^{\frac{1-\beta}{2}}\,\frac{\partial \varphi_\varepsilon}{\partial x_1} \overset{2-d\mu}{\longrightarrow} \frac{\partial \varphi^0}{\partial x_1}, \tag{7.7}$$

where $\varphi^0(t, x_1, x_2)$ is the unique solution of the following problem:

$$-\frac{\partial}{\partial x_1}\left(\langle|v_1^d|\rangle\,\frac{\partial \varphi^0}{\partial x_1}\right) + \langle v_1^d\rangle\,\frac{\partial \varphi^0}{\partial x_1} = 0 \quad in \ \ ]0, L[\,, \tag{7.8}$$

$$\varphi^0(0) = 0, \quad -\langle|v_1^d|\rangle\,\frac{\partial \varphi^0}{\partial x_1}(L) + (\langle v_1^d\rangle + \kappa)\,\varphi^0(L) = \kappa\,\langle g^d\rangle.$$

Furthermore, this problem can be solved explicitly, and

$$\varphi^0(t, x_1, x_2) = \frac{\langle g^d\rangle}{1 + \frac{\langle v_1^d\rangle}{\kappa} - e^{sgn(v_1^d)\,L}}\,(1 - e^{sgn(v_1^d)\,x_1}). \tag{7.9}$$

Proof. We already know that the two-scale limit φ^0 does not depend on y_2, y_3. We choose $z \in C^1(\overline{\Omega^T})$ and modify it to get z_ε as in the proof of Theorem 6.1 (the construction described in Appendix D). Taking such z_ε for a test function in (2.8)–(2.12) and multiplying the variational form (4.1) by $\varepsilon^{\frac{\beta-1}{2}}$ we get

$$\begin{aligned}
&-\varepsilon^{\frac{\beta-1}{2}}\int_{\Omega_\varepsilon^T}\omega_\varepsilon\,\varphi_\varepsilon\,\frac{\partial z_\varepsilon}{\partial t} + \varepsilon^{\frac{\beta-1}{2}}\int_0^T\int_{\Omega_\varepsilon\setminus\mathcal{S}_\varepsilon}\mathbf{A}^h\,\nabla\varphi_\varepsilon\,\nabla z_\varepsilon\\
&+\varepsilon^{-\frac{\beta+1}{2}}\int_0^T\int_{\mathcal{S}_\varepsilon}(\mathbf{A}^d + d\varepsilon^\beta\mathbf{I})\,\nabla\varphi_\varepsilon\,\nabla z_\varepsilon\\
&+\varepsilon^{-\frac{\beta+1}{2}}\int_0^T\int_{\mathcal{S}_\varepsilon}\mathbf{v}^d\cdot\nabla\varphi_\varepsilon\,z_\varepsilon + \varepsilon^{\frac{\beta-1}{2}}\lambda\int_{\Omega_\varepsilon^T}\omega_\varepsilon\varphi_\varepsilon\,z_\varepsilon + \varepsilon^{\frac{\beta-1}{2}}\int_0^T\int_{\Omega_\varepsilon\setminus\mathcal{S}_\varepsilon}\mathbf{v}^h\cdot\nabla\varphi_\varepsilon\,z_\varepsilon\\
&=\varepsilon^{\frac{\beta-1}{2}}\int_{\Omega_\varepsilon}\omega^h f_0\,z_\varepsilon(0,x) + \varepsilon^{\frac{\beta-1}{2}}\int_{\mathcal{F}_\varepsilon^T}\kappa\,(\varepsilon^{-\frac{\beta+1}{2}}g^d - \varphi_\varepsilon)\,z_\varepsilon\\
&+\varepsilon^{\frac{\beta-1}{2}}\int_{\mathcal{R}_\varepsilon^T}\kappa\,(g^h - \varphi_\varepsilon)\,z_\varepsilon + \varepsilon^{\frac{\beta-1}{2}}\int_0^T\Phi_\varepsilon\int_{\Gamma_\varepsilon}z_\varepsilon.
\end{aligned}$$

Due to the a priori estimates (4.2)–(4.5) all the terms on the left-hand side disappear on the limit except

$$\varepsilon^{-\frac{\beta+1}{2}}\int_0^T\int_{\mathcal{S}_\varepsilon}\mathbf{A}^d\,\nabla\varphi_\varepsilon\,\nabla z_\varepsilon, \quad \varepsilon^{-\frac{\beta+1}{2}}\int_0^T\int_{\mathcal{S}_\varepsilon}\mathbf{v}^d\cdot\nabla\varphi_\varepsilon\,z_\varepsilon.$$

Applying the two-scale convergence, as defined in Appendix A, leads to

$$\varepsilon^{-1}\int_0^T\int_{\mathcal{S}_\varepsilon}|v_1^d|\,\frac{\partial(\varepsilon^{\frac{1-\beta}{2}}\,\varphi_\varepsilon)}{\partial x_1}\,\frac{\partial z}{\partial x_1} \to \int_0^T\int_\Sigma\int_{\mathcal{C}}\left(|v_1^d|\,\frac{\partial\varphi^0}{\partial x_1}\right)\,\frac{\partial z}{\partial x_1}$$

and the same for the second integral. On the right-hand side one term remains:

$$\varepsilon^{-1}\int_{\mathcal{F}_\varepsilon^T}\kappa\,g^d\,z.$$

Due to the assumption (7.1) it converges to

$$\int_0^T \int_0^L \kappa \int_{\mathcal{C}} (g^d) \ z(L, x_2, 0) \ dx_2.$$

Thus we obtain for the two-scale limit the following variational problem:

Find $\varphi \in L^2(0,T;H^1(0,L))$, such that

$$\int_0^T \int_\Sigma \left(\langle |v_1^d| \rangle \frac{\partial \varphi^0}{\partial x_1} \frac{\partial z}{\partial x_1} + \langle v_1^d \rangle \frac{\partial \varphi^0}{\partial x_1} z \right) = \int_0^T \int_0^L \langle g^d \rangle \ z(t, L, x_2, 0) \ dx_2 \, dt,$$

$$\text{for any} \quad z \in H^1(0,T;H^1(0,L)) \ \text{ such that } \ z(T,x) = 0. \tag{7.10}$$

As v_1^d has a constant sign, by assumption, then $\langle |v_1^d| \rangle = \operatorname{sgn}(v_1^d) \ \langle v_1^d \rangle$. Obviously, y_2, y_3, t, x_2 are just parameters in that problem, and it can be explicitly solved, giving (7.9). ◻

Now we characterize the two-scale limit obtained in Lemmas 7.1 and 7.2 in situation $\beta > 2$.

THEOREM 7.4. *Let $\beta > 2$, and let $\{\varphi_\varepsilon\}$ be the sequence of solutions to the problem* (2.8)–(2.12). *Then*

$$\varepsilon^{\frac{1-\beta}{2}} \ \varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0, \tag{7.11}$$

$$\varepsilon^{\frac{1-\beta}{2}} \ \frac{\partial \varphi_\varepsilon}{\partial x_1} \overset{2-d\mu}{\longrightarrow} \frac{\partial \varphi^0}{\partial x_1}, \tag{7.12}$$

where $\varphi^0(t, x_1, x_2, y_2, y_3)$ is the unique solution of the following problem:

$$-\frac{\partial}{\partial x_1} \left(|v_1^d| \ \frac{\partial \varphi^0}{\partial x_1} \right) + v_1^d \ \frac{\partial \varphi^0}{\partial x_1} = 0 \quad \text{in} \ \ ]0, L[\, , \tag{7.13}$$

$$\varphi^0(0) = 0, \quad -|v_1^d| \ \frac{\partial \varphi^0}{\partial x_1}(L) + (v_1^d + \kappa) \, \varphi^0(L) = \kappa \ g^d.$$

Furthermore, this problem can be solved explicitly, and

$$\varphi^0(t, x_1, x_2, y_2, y_3) = \frac{g^d}{1 + \frac{v_1^d}{\kappa} - e^{sgn(v_1^d) \ L}} \ (1 - e^{sgn(v_1^d) \ x_1}). \tag{7.14}$$

Proof. We first choose z_ε as in the proof of Theorem 6.1; then we take $\psi \in C_0^\infty(\mathcal{C})$, extend it by zero outside $\mathcal{C}$, and define $\psi_\varepsilon(x) = \psi(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon})$. The basic difference from Theorem 7.3 is that here we use $z_\varepsilon \ \psi_\varepsilon$ as a test function for (2.8)–(2.12). We introduce another test function ψ_ε depending on the fast variable, since the limit is expected to depend on y. Again, we multiply the variational form of (2.8)–(2.12) by $\varepsilon^{\frac{\beta-1}{2}}$ to get

$$\begin{aligned}
&-\varepsilon^{\frac{\beta-1}{2}} \int_{\mathcal{S}_\varepsilon^T} \omega^d \ \varphi_\varepsilon \ \frac{\partial z_\varepsilon}{\partial t} \psi_\varepsilon + \varepsilon^{-\frac{\beta+1}{2}} \int_0^T \int_{\mathcal{S}_\varepsilon} (\mathbf{A}^d + d\varepsilon^\beta \mathbf{I}) \, \nabla \varphi_\varepsilon \, \nabla z_\varepsilon \psi_\varepsilon \\
&+ \varepsilon^{-\frac{\beta+1}{2}} \int_0^T \int_{\mathcal{S}_\varepsilon} \mathbf{v}^d \ \cdot \ \nabla \varphi_\varepsilon \ z_\varepsilon \psi_\varepsilon \\
&+ \varepsilon^{\frac{\beta-1}{2}} \lambda \int_{\mathcal{S}_\varepsilon^T} \omega_\varepsilon \varphi_\varepsilon \ z_\varepsilon \psi_\varepsilon + \varepsilon^{\frac{\beta-3}{2}} \int_0^T \int_{\mathcal{S}_\varepsilon} d \, \nabla \varphi_\varepsilon \ \nabla_{y_2 y_3} \psi\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) \ z_\varepsilon \\
&= \varepsilon^{\frac{\beta-1}{2}} \int_{\mathcal{S}_\varepsilon} \omega^d f_0 \ z_\varepsilon(0,x) \psi_\varepsilon + \varepsilon^{\frac{\beta-1}{2}} \int_{\mathcal{F}_\varepsilon^T} \kappa \ (\varepsilon^{-\frac{\beta+1}{2}} g^d - \varphi_\varepsilon) \, z_\varepsilon \psi_\varepsilon.
\end{aligned}$$

Similarly to Theorem 7.3, all the terms on the left-hand side disappear on the limit except

$$\varepsilon^{-\frac{\beta+1}{2}} \int_0^T \int_{\mathcal{S}_\varepsilon} \mathbf{A}^d \, \nabla\varphi_\varepsilon \, \nabla z_\varepsilon \; \psi_\varepsilon, \quad \varepsilon^{-\frac{\beta+1}{2}} \int_0^T \int_{\mathcal{S}_\varepsilon} \mathbf{v}^d \cdot \nabla\varphi_\varepsilon \; z_\varepsilon \; \psi_\varepsilon.$$

Here, the test function depends on the fast variable and thus, passing to the two-scale limit, leads to

$$\varepsilon^{-1} \int_0^T \int_{\mathcal{S}_\varepsilon} |v_1^d| \, \frac{\partial(\varepsilon^{\frac{1-\beta}{2}} \; \varphi_\varepsilon)}{\partial x_1} \; \frac{\partial z}{\partial x_1} \psi_\varepsilon \to \int_0^T \int_\Sigma \int_{\mathcal{C}} \left(|v_1^d| \, \frac{\partial \varphi^0}{\partial x_1} \; \psi \right) \; \frac{\partial z}{\partial x_1}$$

and the same for the second integral. On the right-hand side, one term remains:

$$\varepsilon^{-1} \int_{\mathcal{F}_\varepsilon^T} \kappa \; g^d \; z \; \psi_\varepsilon.$$

Due to the assumption (7.1) it converges to

$$\int_0^T \int_0^L \kappa \; \int_{\mathcal{C}} (\; g^d \; \psi \;) \; z(L, x_2, 0) \; dx_2.$$

Thus we obtain for the two-scale limit the following variational problem:

Find $\varphi \in L^2(0,T;H^1(0,L))$, such that

$$\int_{\mathcal{C}} \psi \; \int_0^T \int_\Sigma \left(|v_1^d| \, \frac{\partial \varphi^0}{\partial x_1} \; \frac{\partial z}{\partial x_1} + v_1^d \, \frac{\partial \varphi^0}{\partial x_1} \; z \; \right) = \int_{\mathcal{C}} \psi \; \int_0^T \int_0^L g^d \; z(t, L, x_2, 0) \; dx_2 \, dt,$$

$$\text{(7.15)} \quad \text{for any} \quad z \in H^1(0,T;H^1(0,L)) \;\; \text{such that} \;\; z(T,x) = 0, \quad \psi \in H_0^1(\mathcal{C}).$$

This problem is a boundary value problem for an ODE where y_2, y_3, t, x_2 are just parameters. Thus it can be written in the differential form (7.13) and then explicitly solved. ◻

Only the case $\beta = 2$ remains. In this case, using the same techniques, it is not possible to get the uniqueness either for the two-scale limit or for the weak limit for $\langle \varphi_\varepsilon \rangle = \varepsilon^{-1} \int_{\mathcal{S}_\varepsilon} \varepsilon^{\frac{1-\beta}{2}} \; \varphi_\varepsilon$ (see Remark 2). Only with an additional assumption $v_1^d = v_1^d(x_2)$ are we able to prove the uniqueness of the weak limit $\langle \varphi^0 \rangle$ of $\langle \varphi_\varepsilon \rangle$. Repeating the proof of Theorem 7.3 we get, in the case where $\beta = 2$ and v_1^d is independent of y_2, y_3, the convergence of the mean values towards a mean value $\langle \varphi^0 \rangle$ that can be explicitly computed as in (7.9).

PROPOSITION 7.5. *For $\beta = 2$ and v_1^d independent of y_2, y_3 we have*

$$\text{(7.16)} \qquad \frac{1}{\varepsilon} \int_{\mathcal{S}_\varepsilon} \varepsilon^{\frac{1-\beta}{2}} \; \varphi_\varepsilon \rightharpoonup \langle \varphi^0 \rangle \quad \textit{weakly in} \;\; H^1(0,L),$$

where

$$\text{(7.17)} \qquad \langle \varphi^0 \rangle = \frac{\langle g^d \rangle}{1 + \frac{v_1^d}{\kappa} - e^{sgn(v_1^d) \; L}} \; (1 - e^{sgn(v_1^d) \; x_1}).$$

REMARK 2. *Actually, we can prove, in the case where $\beta = 2$ and v_1^d depends on y_2, y_3, the two-scale convergence, up to a subsequence:*

$$\varepsilon^{\frac{1-\beta}{2}} \; \varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0,$$

where φ^0 satisfies the partial differential equation

$$-d\Delta_{y_2y_3}\,\varphi^0 - |v_1^d|\,\frac{\partial^2\varphi^0}{\partial x_1^2} + v_1^d\,\frac{\partial\varphi^0}{\partial x_1} = 0$$

in $]0,L[\,\times\,\mathcal{C}$. *But we cannot identify the boundary conditions satisfied on* $]0,L[\,\times\,\partial\mathcal{C}$. *Thus we are unable to prove the uniqueness of such* φ^0*, and the convergence remains up to a subsequence. However, under the assumption that* v_1^d *is independent of* y_2, y_3*, all possible two-scale accumulation points have the same mean value* (7.17). *To see how the two-scale limit interacts with the rest of the domain the higher-order asymptotic effects need to be studied.*

Remark 3. *The conditions* (7.1)*, which we imposed on* g_ε *in order to see the influence of the EDZ at the global scale, correspond to the same type of conditions (see, for instance,* (6.1)*) imposed in all previous cases for* $0 \le \beta \le 1$.

Appendix A. Two-scale convergence. At the global scale, all the important effects of the process will be concentrated on Σ, a two-dimensional domain. To see them, it is necessary to consider the limit with respect to the rescaled measure $d\mu^\varepsilon(x) = \varepsilon^{-1}\,\mathbf{1}_{\mathcal{S}_\varepsilon}\,d\mathcal{L}_3$, where $d\mathcal{L}_3$ is the Lebesgue measure in $\mathbf{R}^3$ and $\mathbf{1}_{\mathcal{S}_\varepsilon}$ is the characteristic function of the set $\mathcal{S}_\varepsilon$. Following the definition of two-scale convergence with respect to the concentrated measure $d\mu^\varepsilon(x)$ defined in [2] and adapted to thin domains (as in [7] or [1]) and to parabolic problems (as in [3]), we recall the following definition and properties of the two-scale convergence associated to the singular measure $d\mu(x)$.

Definition A.1. *A sequence* $\{\varphi_\varepsilon\}_{\varepsilon>0}$, $\varphi_\varepsilon \in L^p(\Omega_\varepsilon)$*, is said to converge two-scale, with respect to the singular measure* $d\mu(x)$*, to* $\varphi^0 \in L^p(\Sigma\times\mathcal{C})$ *if for any* $\psi \in C(\Omega; L^{p'}(\mathcal{C})\,)$

(A.1)

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon(x)\psi\left(x,\frac{x_2}{\varepsilon},\frac{x_3}{\varepsilon}\right)d\mu^\varepsilon(x) = \int_\Omega d\delta_\Sigma(x)\ \int_{\mathcal{C}}\varphi^0(x_1,x_2,y_2,y_3)\,\psi(x,y_2,y_3)\,dy_2\,dy_3.$$

In other words (A.1) *can be written simply as*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathcal{S}_\varepsilon}\varphi_\varepsilon(x)\psi\left(x,\frac{x'}{\varepsilon}\right)dx = \int_\Sigma dx_1\,dx_2\ \int_{\mathcal{C}}\varphi^0(x_1,x_2,y_2,y_3)\,\psi(x_1,x_2,0,y_2,y_3)\,dy_2\,dy_3.$$

In the following, we will denote shortly the two-scale convergence, with respect to the singular measure $d\mu(x)$,

$$\varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0.$$

The crucial result related to the above convergence is the following compactness theorem.

Theorem A.2. *Let*

$$\varepsilon^{-\frac{1}{p}}\,|\varphi_\varepsilon|_{L^p(\mathcal{S}_\varepsilon)} \le C.$$

Then there exists $\varphi^0 \in L^p(\Sigma\,\times\,\mathcal{C})$ *such that*

$$\varphi_\varepsilon \overset{2-d\mu}{\longrightarrow} \varphi^0.$$

Proof. We need only to prove that for any $\psi \in C(\Omega; L^{p'}(\mathcal{C}))$

$$\frac{1}{\varepsilon}\int_{\mathcal{S}_\varepsilon}\Big|\psi\Big(x, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\Big|^{p'} \to \int_\Omega d\delta_\Sigma(x)\ \int_{\mathcal{C}} |\psi(x, y_2, y_3)|^{p'}\, dy_2\, dy_3.$$

But we have

$$\frac{1}{\varepsilon}\int_{\mathcal{S}_\varepsilon}\Big|\psi\Big(x, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\Big|^{p'} = \int_0^\ell \left(\sum_{i=1}^{M(\varepsilon)} \varepsilon \int_{\mathcal{C}} |\psi(x_1, \varepsilon(i+y_2), y_2, y_3)|^{p'}\, dy_2\, dy_3\right) dx_1$$

$$= \int_0^\ell \sum_{i=1}^{M(\varepsilon)} \varepsilon\ \Big[\int_{\mathcal{C}} |\psi(x_1, \varepsilon\, i, 0, y_2, y_3)|^{p'}\, dy_2\, dy_3 + o(\varepsilon)\Big]$$

$$\to \int_\Sigma \int_{\mathcal{C}} |\psi(x', 0, y_2, y_3)|^{p'}\, dy_2\, dy_3\, dx_1\, dx_2. \qquad \square$$

Finally, under the assumptions of Theorem 6.1 we find the link between the weak limit and the two-scale limit with respect to the singular measure $d\mu(x)$.

PROPOSITION A.3. *Let φ be the weak limit of $\{\varphi_\varepsilon\}_{\varepsilon>0}$, the sequence of solutions of* (2.8)–(2.10) *for $\beta = 1$, and let φ^0 be its two-scale limit, with respect to the singular measure $d\mu(x)$, as in Theorem* 6.1. *Then φ^0 does not depend on y_2, y_3 and*

$$\varphi(t, x_1, x_2, 0) = \varphi^0(t, x_1, x_2), \quad (x_1, x_2) \in \Sigma.$$

Furthermore, $\varphi \in L^2(0, T; V)$.

Proof. Let $\eta(y_2, y_3) \in C_0^\infty(\mathcal{C})$ (extended by 0 to Ω), and let $\Omega_\varepsilon^+ = \Omega_\varepsilon \cap \{x_3 > 0\}$, $\Omega^+ = \Omega \cap \{x_3 > 0\}$.

Denoting $\mathcal{C}^+ = \mathcal{C} \cap \{y_3 > 0\}$ and $\mathcal{S}_\varepsilon^+ = \mathcal{S}_\varepsilon \cap \{x_3 > 0\}$, for any function ψ from $H_0^1(\Omega)$ we obtain

$$\int_{\Omega_\varepsilon^+} \frac{\partial \varphi_\varepsilon}{\partial x_2}\, \psi\ \Big(1 + \eta\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\Big) \to \int_{\Omega^+} \psi\, \frac{\partial \varphi}{\partial x_2}.$$

The integral on the left-hand side can be transformed by partial integration to

$$-\int_{\Omega_\varepsilon^+} \varphi_\varepsilon\, \frac{\partial \psi}{\partial x_2}\,(1+\eta) - \varepsilon^{-1}\int_{\mathcal{S}_\varepsilon^+} \varphi_\varepsilon\, \psi\, \frac{\partial \eta}{\partial y_2}\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) \to -\int_{\Omega^+} \varphi\, \frac{\partial \psi}{\partial x_2}$$
$$-\int_\Sigma \psi \int_{\mathcal{C}} \varphi^0\, \frac{\partial \eta}{\partial y_2}.$$

Thus φ^0 does not depend on y_2. We proceed in the same manner with y_3:

$$\int_{\Omega_\varepsilon^+} \frac{\partial \varphi_\varepsilon}{\partial x_3}\, \psi\ \Big(1 + \eta\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big)\Big) \to \int_{\Omega^+} \psi\, \frac{\partial \varphi}{\partial x_3}.$$

As before, the integral on the left-hand side can be transformed to

$$-\int_{\Omega_\varepsilon^+} \varphi_\varepsilon\, \frac{\partial \psi}{\partial x_3}\,(1+\eta) - \varepsilon^{-1}\int_{\mathcal{S}_\varepsilon^+} \varphi_\varepsilon\, \psi\, \frac{\partial \eta}{\partial y_3}\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) - \int_{\Omega_\varepsilon \cap \{x_3=0\}} \varphi_\varepsilon\, \psi\, \Big(1 + \eta\Big(\frac{x_2}{\varepsilon}, 0\Big)\Big).$$

Using the weak convergence for the first integral, the two-scale convergence for the second, and the strong convergence of the trace for the third, we pass to the limit and get

$$\int_{\Omega^+} \psi\, \frac{\partial \varphi}{\partial x_3} = -\int_{\Omega^+} \varphi\, \frac{\partial \psi}{\partial x_3} - \int_\Sigma \psi \int_{\mathcal{C}^+} \varphi^0\, \frac{\partial \eta}{\partial y_3} - \int_\Sigma \varphi\, \psi\, \left(1 + \int_{\mathcal{C} \cap \{y_3=0\}} \eta(y_2, 0)\, dy_2\right).$$

Choosing η equal to zero on $y_3 = 0$, we first conclude that φ^0 does not depend on y_3 either. Second, we have

$$\int_\Sigma \varphi^0 \, \psi \, \int_{\mathcal{C}^+} \frac{\partial \eta}{\partial y_3} = - \int_\Sigma \psi(x_1, x_2, 0) \, \varphi(t, x_1, x_2, 0) \, \int_{\mathcal{C} \cup \{y_3 = 0\}} \eta(y_2, 0) \, dy_2 \, dx_1 \, dx_2.$$

Since

$$\int_{\mathcal{C}^+} \frac{\partial \eta}{\partial y_3} = - \int_{\mathcal{C} \cap \{y_3 = 0\}} \eta(y_2, 0) \, dy_2,$$

choosing η such that $\int_{\mathcal{C} \cap \{y_3 = 0\}} \eta(y_2, 0) \, dy_2 \neq 0$, due to the fact that ψ is arbitrary, we prove that $\varphi|_\Sigma = \varphi^0$. It remains to prove that $\varphi \in L^2(0, T; V)$. It is obvious that $\frac{\partial \varphi}{\partial x_1}(t, x_1, x_2, 0) \in L^2(0, T; L^2(\Sigma))$. To prove that $\varphi(t, 0, x_2, 0) = 0$ we take a test function $\psi \in H^1(\Omega)$ such that $\psi(L, x_2, x_3) = 0$ and η is the same as before. Then we pass to the two-scale limit in

$$\frac{1}{\varepsilon} \int_{\mathcal{S}_\varepsilon} \frac{\partial \varphi_\varepsilon}{\partial x_1} \, \psi \, \eta\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) = - \frac{1}{\varepsilon} \int_{\mathcal{S}_\varepsilon} \frac{\partial \psi}{\partial x_1} \, \varphi_\varepsilon \, \eta\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big),$$

leading to

$$\int_0^L \varphi(t, 0, x_2, 0) \, \psi(0, x_2, 0) \, dx_2 = 0,$$

implying the claim. $\square$

Appendix B. Estimating the flux on Γ_ε, the interface host rock/waste packages. For any function ψ from $H^1(\Omega)$ we can define the integral $\mathcal{L}^\varepsilon$ by

$$\mathcal{L}^\varepsilon \psi = \varepsilon^{1-\gamma} \int_{\Gamma_\varepsilon} \psi;$$

obviously, $\mathcal{L}^\varepsilon \in [H^1(\Omega)]'$.

For continuous ψ we have

$$\mathcal{L}^\varepsilon \psi = \varepsilon^{1-\gamma} \sum_{i,j=1}^m \int_{\partial \mathcal{P}_\varepsilon(i,j)} \psi \approx \varepsilon^{1-\gamma} \sum_{i,j=1}^m |\partial \mathcal{P}_\varepsilon(i,j)| \psi(x_\varepsilon^{ij}),$$

with x_ε^{ij} an arbitrary point from $\mathcal{P}_\varepsilon(i,j)$. As (see Figure 2.1)

$$|\partial \mathcal{P}_\varepsilon(i,j)| = \varepsilon^{\gamma+1} \, (\mathcal{M} + o(1)\,), \quad \text{where } \mathcal{M} \text{ is defined from (5.5)},$$

we have

$$\mathcal{L}^\varepsilon \psi \approx \varepsilon^2 \sum_{i,j=1}^m \mathcal{M} \, \psi(x_\varepsilon^{ij}).$$

The expression on the right-hand side is the Riemann integral sum for $\mathcal{M} \, \psi(x', 0)$, and thus it converges to $\int_\Sigma \mathcal{M} \, \psi$, i.e.,

$$\lim_{\varepsilon \to 0} \mathcal{L}^\varepsilon \psi = \int_\Sigma \mathcal{M} \, \psi.$$

Using the density argument we could prove the same for $\psi \in H^1(\Omega)$. We want to do better. We want to prove that

$$\mathcal{L}^\varepsilon \to \mathcal{M}\,\delta_\Sigma \text{ strongly in } [H^1(\Omega)]'. \tag{B.1}$$

To do so we estimate the difference $|\mathcal{L}^\varepsilon - \mathcal{M}\,\delta_\Sigma|_{[H^1(\Omega)]'}$ by using the solution w^ε of the auxiliary problem (C.1).

We start with

$$\begin{aligned}\mathcal{L}^\varepsilon\psi &= \varepsilon^{1-\gamma}\int_{\Gamma_\varepsilon}\psi = \varepsilon^{2-\gamma}\int_{\Gamma_\varepsilon}\psi\,\mathbf{n}\cdot\mathbf{A}\,\nabla_x\,w^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\\ &= \varepsilon^{2-\gamma}\int_{G_\varepsilon}\nabla\psi\cdot\mathbf{A}\,\nabla_x w^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\\ &- \varepsilon^{2-\gamma}\int_0^L\int_0^L\left[\psi(x',\gamma\,\varepsilon\,\log(1/\varepsilon))\,\mathbf{e}_3\cdot\mathbf{A}\,\nabla_x w^\varepsilon\left(\frac{x'}{\varepsilon},\gamma\,|\log\varepsilon|\right)\right.\\ &\left.- \psi(x',-\gamma\,\varepsilon\,\log(1/\varepsilon))\,\mathbf{e}_3\cdot\mathbf{A}\,\nabla_x w^\varepsilon\left(\frac{x'}{\varepsilon},-\gamma\,|\log\varepsilon|\right)\right]dx_1\,dx_2,\end{aligned}\tag{B.2}$$

where $x' = (x_1, x_2)$ and

$$G_\varepsilon = \Omega_\varepsilon \cap \{-\gamma\,\varepsilon\,\log(1/\varepsilon) < x_3 < \gamma\,\varepsilon\,\log(1/\varepsilon)\}.$$

Due to (C.2) we have

$$\varepsilon\,\mathbf{A}\,\nabla_x\,w^\varepsilon\left(\frac{x'}{\varepsilon},\pm\gamma\,|\log\varepsilon|\right) = \mp\frac{1}{2}\,|\partial P_\varepsilon| + O(\varepsilon^\gamma),$$

where the term $O(\varepsilon^\gamma)$ comes from the pointwise exponential decay

$$\mathbf{A}\,\nabla_y\,w^\varepsilon(y) \pm \frac{1}{2}\,|\partial P_\varepsilon| \to 0 \quad \text{as} \quad y_3 \to \pm\infty$$

by taking $y_3 = \gamma\,|\log\varepsilon| = |\log\varepsilon^\gamma|$. Thus, the last integral can be written as

$$\frac{\varepsilon^{1-\gamma}}{2}\int_0^L\int_0^L (|\partial P_\varepsilon| + O(\varepsilon^\gamma)\,)[\psi(x',\gamma\,\varepsilon\,|\log\varepsilon|) + \psi(x',-\gamma\,\varepsilon\,|\log\varepsilon|)\,]\,dx'.$$

It remains to estimate the first integral on the right-hand side of (B.2). We have

$$\left|\varepsilon^{2-\gamma}\int_{G_\varepsilon}\nabla\psi\cdot\mathbf{A}\,\nabla_x w^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right| \le C\,\varepsilon^{2-\gamma}\,|\nabla\psi|_{L^2(G_\varepsilon)}\,\left|\nabla_x w^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right|_{L^2(G_\varepsilon)}.$$

Using (C.3), we get

$$\begin{aligned}\left|\nabla_x w^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right|^2_{L^2(G_\varepsilon)} &\le C\left(\left|\nabla_x R^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right|^2_{L^2(G_\varepsilon)} + \left|\nabla_x v_\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right|^2_{L^2(G_\varepsilon)}\right)\\ &\le C(\varepsilon^{2\gamma-3}\,|\log\varepsilon| + |\partial P_\varepsilon|^2\,|G_\varepsilon|\,(\log\varepsilon)^2\,) \le C\varepsilon^{2\gamma-3}\,|\log\varepsilon|.\end{aligned}$$

Thus

$$\left|\varepsilon^{2-\gamma}\int_{G_\varepsilon}\nabla\psi\cdot\mathbf{A}\,\nabla_x w^\varepsilon\left(\frac{\cdot}{\varepsilon}\right)\right| \le C\,\sqrt{\varepsilon\,|\log\,\varepsilon\,|}\,|\nabla\psi|_{L^2(G_\varepsilon)}. \tag{B.3}$$

We now have

$$
\text{(B.4)}\quad \langle \mathcal{L}^\varepsilon - \mathcal{M}\,\delta_\Sigma \mid \psi \rangle = \varepsilon^{2-\gamma} \int_{G_\varepsilon} \nabla\psi \,\cdot\, \mathbf{A}\,\nabla_x\, w^\varepsilon\Big(\frac{\cdot}{\varepsilon}\Big) - \int_0^L \int_0^L \Big\{ \mathcal{M}\psi(x',0) - \frac{\varepsilon^{1-\gamma}}{2}(|\partial P_\varepsilon| + O(\varepsilon^\gamma)\,)\,[\,\psi(x',\gamma\,\varepsilon\,\log(1/\varepsilon)) + \psi(x',-\gamma\,\varepsilon\,\log(1/\varepsilon))\,]\Big\}\,dx'.
$$

We have already estimated the first integral on the right-hand side of (B.4) in (B.3), and we proceed with the second. To estimate the easiest term

$$
\left| \varepsilon \int_0^L \int_0^L \{[\,\psi(x',\gamma\,\varepsilon\,\log(1/\varepsilon)) + \psi(x',-\gamma\,\varepsilon\,\log(1/\varepsilon))\,]\}\,dx' \right|
$$

we use the trace theorem

$$
\left| \int_0^L \int_0^L \{[\,\psi(x',\gamma\,\varepsilon\,\log(1/\varepsilon)) + \psi(x',-\gamma\,\varepsilon\,\log(1/\varepsilon))\,]\}\,dx' \right| \le C|\psi|_{H^1(\Omega)}.
$$

Next, since

$$
\mathcal{M} = \lim_{\varepsilon\to 0} \varepsilon^{1-\gamma}\,|\partial P_\varepsilon\,|
$$

and

$$
\begin{aligned}
&\left| \int_0^L \int_0^L \{2\psi(x',0) - [\,\psi(x',\gamma\,\varepsilon\,\log(1/\varepsilon)) + \psi(x',-\gamma\,\varepsilon\,\log(1/\varepsilon))\,]\,dx_1\,dx_2\,\} \right| \\
&= \left| \int_0^L \int_0^L \left(\int_0^{\gamma\,\varepsilon\,\log(\frac{1}{\varepsilon})} \frac{\partial\psi}{\partial x_3} - \int_{-\gamma\,\varepsilon\,\log(1/\varepsilon)}^0 \frac{\partial\psi}{\partial x_3} \right) dx_1\,dx_2 \right| \\
&\le\; C\,\sqrt{\varepsilon\,\log(1/\varepsilon)}\;|\nabla\psi|_{L^2(\Omega)},
\end{aligned}
$$

we have proved (B.1).

Appendix C. The auxiliary problem in the infinite strip $\boldsymbol{P_\varepsilon}$. In Appendix B we made use of the following auxiliary problem, introduced in a previous paper [5]:

$$
\text{(C.1)}\qquad \begin{cases} -\mathrm{div}\,(\mathbf{A}\,\nabla w_\varepsilon) = 0 \quad \text{in } \mathcal{G}_\varepsilon, \\ \mathbf{n}\,\cdot\,\mathbf{A}\,\nabla w_\varepsilon = 1 \quad \text{on } \partial P_\varepsilon, \\ w_\varepsilon \quad \text{is 1-periodic in } y' = (y_1,y_2), \\ \lim_{y_3\to\pm\infty} \mathbf{A}\,\nabla w_\varepsilon(y) = \mp\frac{1}{2}\,|\partial P_\varepsilon|\,\mathbf{e}_3, \end{cases}
$$

where $\mathcal{G}_\varepsilon = (\,]0,1[^2 \times \mathbf{R}\,)\setminus P_\varepsilon$ represents an infinite strip in the direction y_3 where the waste package P_ε has been removed. Here $\mathbf{A}$ is some positive matrix (not necessarily related to our previous diffusion tensor). Solvability of the problem (C.1) is classical (see, e.g., [5]). Indeed the problem (C.1) admits a unique solution, and the convergence in the last line of (C.1) is exponential. Furthermore, for large $|y_3|$, w_ε possesses the following asymptotic behavior for large $|y_3|$:

$$
\text{(C.2)}\qquad w_\varepsilon(y) \approx -(\mathbf{A}_{33})^{-1}|y_3|\,\frac{1}{2}\,|\partial P_\varepsilon| + \text{exponentially decaying part}
$$

(see [5] and [8] for details).

We need the following auxiliary result.

LEMMA C.1. *Let*

$$\eta(y_3) = \begin{cases} 0, & |y_3| < 1, \\ 1, & |y_3| > 2, \\ \textit{smooth otherwise} \end{cases}$$

such that $0 \le \eta \le 1$, *and let*

$$v_\varepsilon(y) = -\mathbf{A}_{33}^{-1} \frac{1}{2} |y_3| \, |\partial P_\varepsilon| \, \eta(y_3).$$

The function $R_\varepsilon = w^\varepsilon - v_\varepsilon$ *satisfies the estimate*

$$|\nabla R_\varepsilon|_{L^2(\tilde{\mathcal{G}}_\varepsilon)} \le C \, \varepsilon^{\gamma-1} \sqrt{|\log \varepsilon\,|}\,, \tag{C.3}$$

with $\tilde{\mathcal{G}}_\varepsilon = \mathcal{G}_\varepsilon \cap \{-\gamma \, |\log \varepsilon| < y_3 < \gamma \, |\log \varepsilon|\}$.

Proof. By direct computation we obtain

$$\begin{aligned} &-\mathrm{div}\,(\mathbf{A}\,\nabla R_\varepsilon) = \mathrm{div}\,(\mathbf{A}\,\nabla v_\varepsilon) \\ &= -\mathrm{div}\left\{ \mathbf{A}\,(|y_3|\eta'(y_3) + \eta(y_3)\,\mathrm{sgn}\,y_3\,)\,\mathbf{A}_{33}^{-1}\,\frac{1}{2}\,|\partial P_\varepsilon|\,\mathbf{e}_3 \right\}. \end{aligned}$$

Thus

$$-\int_{\tilde{\mathcal{G}}_\varepsilon} \mathrm{div}\,(\mathbf{A}\,\nabla R_\varepsilon) R_\varepsilon = \int_{\tilde{\mathcal{G}}_\varepsilon} \mathbf{A}\,\nabla R_\varepsilon \cdot \nabla R_\varepsilon \le C |\partial P_\varepsilon| \, |\nabla R_\varepsilon|_{L^2(\tilde{\mathcal{G}}_\varepsilon)} \sqrt{|\log \varepsilon|}\,. \qquad \square$$

Appendix D. The test function. In this appendix, starting from a $z \in C^1(\overline{\Omega})$ we construct the test functions z_ε satisfying the Dirichlet condition on $\mathcal{B}_\varepsilon$ that we used in the proofs of Theorems 6.1, 7.3, and 7.4.

LEMMA D.1. *For any* $z \in C^1(\overline{\Omega})$, *such that* $z(0, x_2, 0) = 0$, *there exists* $z_\varepsilon \in C^1(\overline{\Omega})$ *such that* $z_\varepsilon \to z$ *in* $H^1(\Omega)$ *as* $\varepsilon \to 0$, $|z_\varepsilon|_{W^{1,\infty}(\Omega)} \le C$,

$$\left| \frac{\partial}{\partial x_1}(z - z_\varepsilon) \right|_{L^2(\Omega)} \le C \, \varepsilon \, \left| \frac{\partial z}{\partial x_1} \right|_{L^\infty(\Omega)},$$

and $z_\varepsilon = 0$ *on* $\mathcal{B}_\varepsilon$.

Proof. Let $z \in C^1(\overline{\Omega})$ be such that $z(0, x_2, 0) = 0$. We start by noticing that for $|x_1| \le \varepsilon$ and $|x_3| \le \varepsilon$ we have the pointwise behavior

$$|\varepsilon^{-1} z(x)| = \left| \frac{x_1}{\varepsilon} \frac{z(x) - z(0, x_2, x_3)}{x_1} + \frac{x_3}{\varepsilon} \frac{z(0, x_2, x_3) - z(0, x_2, 0)}{x_3} \right| \le |\nabla z|_{L^\infty(\Omega)}.$$

We now define the auxiliary function $\eta \in C^\infty(Y)$ on unit cell Y such that $0 \le \eta \le 1$ and

$$\begin{aligned} \eta &= 0 \quad \text{for} \;\; y_1 = 1, \\ \eta &= 0 \quad \text{for} \;\; y_2 = 0, 1, \\ \eta &= 0 \quad \text{for} \;\; y_3 = -\frac{1}{2}, \frac{1}{2}. \end{aligned}$$

The function $(y_2, y_3) \mapsto \eta(0, y_2, y_3)$ defined on square $Q = [0,1] \times [-1/2, 1/2]$ belongs to $C_0^\infty(Q)$, and we impose, in addition, that it equals 1 on $\mathcal{C}$. We now pose $\eta_\varepsilon(x) = \eta(x/\varepsilon)$ for $x_1 \in [0, \varepsilon]$ and $x_3 \in [-\varepsilon/2, \varepsilon/2]$. Then we extend η_ε by zero for $x_1 > \varepsilon$ and $|x_3| > \varepsilon/2$. Finally, we define

$$z_\varepsilon(x) = (1 - \eta_\varepsilon(x)\ )\ z(x).$$

Obviously,

$$|z_\varepsilon|_{L^\infty(\Omega)} \leq |z|_{L^\infty(\Omega)}$$

and

$$|\nabla z_\varepsilon|_{L^\infty(\Omega)} \leq |\nabla z|_{L^\infty(\Omega)} + |\nabla_y \eta|_{L^\infty(Y)}\ |\varepsilon^{-1}\ z|_{L^\infty(\Omega)} \leq C |\nabla z|_{L^\infty(\Omega)}.$$

Now the convergence of $z_\varepsilon \to z$ follows directly from the fact that the support of η_ε shrinks as $\varepsilon \to 0$. Finally,

$$\left| \frac{\partial}{\partial x_1}(z - z_\varepsilon) \right|_{L^2(\Omega)} = \left| \eta_\varepsilon \frac{\partial z}{\partial x_1} \right|_{L^2(\Omega)} \leq C\ |\mathrm{supp}\ \eta_\varepsilon|^{1/2}\ \left| \frac{\partial z}{\partial x_1} \right|_{L^\infty(\Omega)}. \qquad \square$$

Conclusions. According to the range of the mathematical parameter $\varepsilon^{-\beta}$ we introduced for describing the water inflow regime and the concentration rates on the head drifts, we have captured the different behaviors corresponding to the three scenarii considered in the European exercise BENIPA [9] (the connected shafts galleries and drifts being either perfectly sealed, poorly sealed, or not sealed). Corresponding to each of the three scenarii, we now have a global (macroscopic) model which can be used for numerical far field simulations in performance assessment.

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