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Atomic decomposition for the vorticity of a viscous flow in the whole space

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We show that the vorticity of a viscous flow in $\mathbb{R}^3$ admits an atomic decomposition of the form $\omega(x,t) = \sum_{k=1}^{\infty} \omega_k(x - x_k, t)$, with localized and oscillating building blocks ω_k, if such a property is satisfied at the beginning of the evolution. We also study the long time behavior of an isolated coherent structure and the special behavior of flows with highly oscillating vorticities.

1 Introduction

We study the motion of a viscous incompressible fluid which fills the whole space $\mathbb{R}^3$. The flow is given by the Navier-Stokes equations, which we write in the velocity-vorticity formulation as follows:

$$
\begin{aligned}
&\partial_t \omega - \Delta\omega + \sum_{h=1}^{3} \partial_h \big(u^h \omega - \omega^h u\big) \;=\; 0\,, \\
&\omega(x,0) \;=\; \mu(x), \quad \operatorname{div}(\omega) \;=\; 0\,.
\end{aligned}
\tag{NS}
$$

Here $\partial_t = \partial/\partial t$, $\partial_h = \partial/\partial x_h$ $(h = 1,2,3)$ and Δ is the Laplacian with respect to the space variables.

In (NS) and below, $u = \big(u^1, u^2, u^3\big)$ is the velocity field and

$$
\omega \;=\; \nabla \times u \;=\; \big(\partial_2 u^3 - \partial_3 u^2, -\partial_1 u^3 + \partial_3 u^1, \partial_1 u^2 - \partial_2 u^1\big)
$$

is the corresponding vorticity. The Navier-Stokes system is completed by the Biot-Savart law, which allows us to compute u, if we know the corresponding vorticity:

$$
u(x,t) \;=\; -\frac{1}{4\pi} \int \frac{x-y}{|x-y|^3} \times \omega(y,t)\,dy \quad \big(x \in \mathbb{R}^3,\ t \ge 0\big)\,. \tag{BS}
$$

We refer to [11] and [16] for the study of the local and global well-posedness of (NS) in very general frameworks.

In this paper we deal with the persistence of the spatial localization of the vorticity. In spite of the fact that this property is experimentally easily observed, only few papers are concerned with its mathematical treatment (see e.g. [18], [7], [12], [20]). Our approach is slightly different, since we will solve (NS) in functional spaces which depend on the initial data.

Our main result states that if the initial vorticity μ is localized around a fixed sequence $(x_k)_1^\infty \subset \mathbb{R}^3$, at given rates $\gamma(k) \ge 1$ $(k = 1,2,\ldots)$, then the corresponding solution $\omega(x,t)$ of (NS) conserves this property, at least on a small time interval $[0,T\,]$.

The localization will be measured by means of weighted-L^∞ spaces. This means that the vorticity admits an atomic decomposition

$$
\omega(x,t) \;=\; \sum_{k=1}^{\infty} \omega_k(x - x_k, t)\,, \tag{1.1}
$$

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in which the "building blocks" $\omega_k(x,t)$ are bounded and decay at least as fast as $|x|^{-\gamma(k)}$ as $|x| \to \infty$. Such building blocks are also oscillating $\big($i.e. $\int \omega_k(x,t)\,dx = 0$ for all $k\big)$, if this is true at the beginning of the evolution.

We point out that these properties are specific to the vorticity and do not hold for the velocity field. Indeed, the instantaneous spreading effect of the velocity field (see, [5], [8]) forbids $u(x,t)$ to have a strong decay at infinity. However, as an immediate consequence of our results on the vorticity, we will see that $u(x,t)$ is localized around the points x_k in a weaker sense.

We complement these results studying the localization of the solution with respect to the frequency variable. We will provide some conditions which ensure that the vorticity has a "wavelet like" profile in the Fourier domain (i.e. $|\widehat{\omega}(\xi,t)|$ is mostly carried by a band $\epsilon \le |\xi| \le C(t)$, for some $\epsilon > 0$). The localization of the vorticity at small frequencies is mainly due to the viscosity, which converts short waves into long waves. Such effect will be discussed in Section 3 where we study the pointwise decay of $\widehat{\omega}(\xi,t)$ as $|\xi| \to \infty$. On the other hand, the localization of $\widehat{\omega}$ outside the ball $|\xi| \le \epsilon$ is closely related with the persistence of the oscillating patterns of the vorticity during the evolution. As we shall see in Section 5, this second effect is more evident in the case of flows invariant under rotations.

Our investigation on the space-frequency localization was motivated by a program described in detail in [20] (Section 2.7), about the the long standing problem of decoupling the Navier-Stokes equations into a simpler system. Roughly, the program of [20] is the following: the first stage is to expand the solution into an orthonormal basis of divergence free wavelets (see e.g. [9], [6], [19]). The second stage consists in proving that, for a wide class of flows, the wavelet coefficients form a sparse matrix. This seems to be quite reasonable and have been confirmed in a few numerical experiments. Indeed, one observes that during the evolution the vorticity usually organizes itself into localized and oscillating patterns, or "coherent structures", even if at the beginning the flow appears to be chaotic. Moreover, coherent structures are expected to have a sparse wavelet expansion. Proving that a flow can be described by means of a sparse wavelet series would be an essential point, in order to ensure the efficiency of numerical algorithms based on wavelets (for a description of such algorithms we refer e.g. to [23]). The atomic decomposition of the vorticity is a first step in this direction (due to the localization and the vanishing integral of each term ω_k in (1.1)) and should shed some light on this program.

Another issue of this paper is the study of the long time behavior of the Navier-Stokes equations. In Section 4 we prove the existence of global (small) solutions to (NS) such that

$$|\omega(x,t)| \;\le\; C_\alpha(1+|x|)^{-\alpha}(1+t)^{-(S-\alpha)/2}\,, \quad \text{for all} \quad 0 \;\le\; \alpha \;\le\; \gamma\,. \tag{1.2}$$

Here, γ depends solely on the spatial localization of the initial data and it may be chosen arbitrarily large. Hence, such space-time profiles for the vorticity are slightly more general than the corresponding profiles which have been obtained for the velocity field (see e.g. [24], [21], [13], [3]). In (1.2), the parameter S is closely related to the cancellations of μ and to the possible symmetries of the flow.

2 Spatial localization

In this paper we mostly deal with functional spaces of $\mathbb{R}^3$-valued functions. In our notations, however, we will make no distinction between scalar and vector-valued functions.

For any $\gamma \ge 0$ $(\gamma \ne 1)$ let us denote by L^∞_γ the space of all measurable functions f such that $(1+|x|)^\gamma\,|f(x)| \in L^\infty\big(\mathbb{R}^3\big)$. In the case $\gamma = 1$ we define $f \in L^\infty_1$ if and only if $f \in L^\infty\big(\mathbb{R}^3\big)$ and if there exists a sequence $\epsilon_j \in \ell^1(\mathbb{N})$ such that $|f(x)| \le 2^{-j}\epsilon_j\ \big(2^j \le |x| < 2^{j+1}\big)$. These spaces are normed by (we denote by "sup" the essential supremum):

$$\|f\|_{L^\infty_\gamma} \;=\; \sup_{x\in\mathbb{R}^3}(1+|x|)^\gamma\,|f(x)| \quad (\gamma \ge 0,\ \gamma \ne 1)$$

and, if $\gamma = 1$, by

$$\|f\|_{L^\infty_1} = \sup_{|x|<1}|f(x)| + \sum_{j=0}^{\infty} 2^j \sup_{2^j \le |x| < 2^{j+1}} |f(x)|\,. \tag{2.1}$$

We have the obvious embedding $L^\infty_\gamma \subset L^\infty_{\gamma'}$, for all $\gamma \geq \gamma' \geq 0$.

We now introduce a functional space which is especially suited to describe the localization properties.

Definition 2.1 (a) Let $x_0 \in \mathbb{R}^3$ and $\gamma \geq 0$. We say that f *is localized around* x_0 *at the rate* γ, if $f(\cdot - x_0) \in L^\infty_\gamma$.

(b) Let $(x_k)_1^\infty \subset \mathbb{R}^3$ and let $\gamma(k)$ $(k = 1, 2 \ldots)$ be a sequence of non-negative real numbers. We say that f is *localized around* (x_k) *at rates* $\gamma(k)$, if there exists $f_k \in L^\infty_{\gamma(k)}$ $(k = 1, 2 \ldots)$ such that $f(x) = \sum_{k=1}^\infty f_k(x - x_k)$ and $\sum_{k=1}^\infty \|f_k\|_{L^\infty_{\gamma(k)}}$ is finite. In this case we write $f \in X[x_k, \gamma(k)]$.

This Banach space is normed by

$$\|f\|_{X[x_k,\gamma(k)]} \;=\; \inf\left\{\sum_{k=1}^\infty \|f_k\|_{L^\infty_{\gamma(k)}}\right\}, \tag{2.2}$$

where the infimum is taken over all the possible decompositions of f as above.

We will solve (NS) in the space $\mathcal{C}\big([0, T], X[x_k, \gamma(k)]\big)$ of continuous functions ω which take values in $X[x_k, \gamma(k)]$. When there is no risk of confusion, the norm of $\mathcal{C}([0, T], X[x_k, \gamma(k)])$ will be simply denoted by $\|\cdot\|$. Thus,

$$\|\omega\| \;=\; \sup_{t\in[0,T]} \|\omega(t)\|_{X[x_k,\gamma(k)]}\,.$$

The continuity with respect to the time variable of $\omega(t)$ is defined, for $0 < t \leq T$, by the natural norm of $X[x_k, \gamma(k)]$. The convergence $\omega(t) \to \omega(0)$ should be understood in the distributional sense, as it is usually done in non-separable spaces.

We will obtain a persistence result for the vorticity, in the space $X[x_k, \gamma(k)]$. For this, we will use the fixed point theorem in the integral formulation of (NS), which we write as follows:

$$\begin{cases} \omega(t) \;=\; e^{t\Delta}\mu - A(u,\omega)(t)\,, \\ \operatorname{div}(\mu) \;=\; 0\,, \\ A(u,\omega)(t) \;=\; \displaystyle\int_0^t e^{(t-s)\Delta}\sum_{h=1}^3 \partial_h\big(u^h\omega - \omega^h u\big)(s)\,ds\,, \\ u(t) \;=\; K * \omega(t) \;=\; -\dfrac{1}{4\pi}\displaystyle\int \frac{x-y}{|x-y|^3}\times\omega(y,t)\,dy\,, \end{cases} \tag{IE}$$

where $e^{t\Delta}$ is the heat kernel $\big($the convolution with $g_t(x) = (4\pi t)^{-3/2}e^{-|x|^2/(4t)}\big)$.

Throughout this paper, by "solution" to (IE) we mean a function ω obtained as the limit of an iteration scheme which is formally equivalent to that of Kato [15] (see (2.8) below, and also [11]), in a functional setting which will be precised in each situation. All the solutions that we obtain turn out to be smooth.

Our main result is the following:

Theorem 2.2 (I) *Let* $(x_k)_1^\infty \subset \mathbb{R}^3$ *be a fixed sequence and* μ *a soleinoidal vector field localized around* (x_k)*, uniformly at the rate* 1: *i.e.* $\mu \in X[x_k, \mathbf{1}]$*, where* $\mathbf{1}$ *is the constant sequence* $\gamma(k) = 1$ $(k = 1, 2, \ldots)$*. Then there exists* $T = T(\mu) > 0$ *and a unique solution* $\omega \in \mathcal{C}(]0, T], X[x_k, \mathbf{1}])$ *of* (IE)*, such that* $\omega(t) \to \mu$ *as* $t \to 0$ *in the distributional sense. Moreover, the velocity field* $u(x, t)$ *is uniformly bounded in* $\mathbb{R}^3 \times [0, T]$.

(II) *Let* $\gamma(k)$ *such that* $1 \leq \gamma(k) \leq \Gamma$ *for all positive integer* k *and some* $\Gamma \geq 1$.

a) *We now assume that* μ *is localized around the sequence* (x_k) *at rates* $\gamma(k)$ $(k = 1, 2, \ldots)$*. Then, conclusion* (I) *is improved by* $\omega \in \mathcal{C}(]0, T], X[x_k, \gamma(k)])$.

b) *The following localization property holds for* u*. There exists a sequence of functions* $u_k(x, t)$ *such that* $\sup_{t\in[0,T]}\sum_{k=1}^\infty \|u_k(t)\|_\infty < \infty$ *and*

$$u(x,t) \;=\; \sum_{k=1}^\infty u_k(x - x_k, t)\,.$$

Further, for each k, $u_k(t)$ is localized around x_k in the following sense: uniformly on $[0,T]$, we have

$$\begin{aligned} &u_k(t) \in L^\infty_{\gamma(k)-1} && \textit{if} \quad 1 \le \gamma(k) < 3 \quad \textit{and} \quad \gamma(k) \ne 2\,,\\ &u_k(t) \in \bigcap_{\beta<\gamma(k)-1} L^\infty_\beta\,, && \textit{if} \quad \gamma(k) = 2\,, 3\,,\\ &u_k(t) \in L^\infty_2 && \textit{if} \quad \gamma(k) > 3\,. \end{aligned} \tag{2.3}$$

P r o o f . We start observing that the operator given by the convolution with the Biot-Savart kernel $K(x) = -\frac{1}{4\pi}\, x/|x|^3$ is bounded from L^∞_1 to L^∞_0. Indeed, if $f \in L^\infty_1$, then for any x we may split the convolution integral $K * f(x)$ in three parts, namely $\int_{|y|\le|x|/2}$, $\int_{|x|/2\le|y|\le3|x|/2}$ and $\int_{|y|\ge3|x|/2}$. Using $|K(x-y)| \le C\,|x-y|^{-2}$, we immediately see that the first and the second integral are bounded, up to a constant, by $\sup_x |x|\,|f(x)|$. The third integral is bounded, up to a constant, by

$$\int_{|y|\le1} |y|^{-2}\,|f(y)|\,dy + \sum_{j=1}^\infty \int_{2^j\le|y|\le2^{j+1}} |y|^{-2}\,|f(y)|\,dy\,.$$

Both terms are bounded by $\|f\|_{L^\infty_1}$ and this yields $K * f \in L^\infty_0$ as claimed.

The main step of the proof of the theorem is contained in the following lemma.

Lemma 2.3 *Let $\Gamma \ge 1$, let (x_k) and $\gamma(k)$ be two fixed sequences such that $x_k \in \mathbb{R}^3$ and $1 \le \gamma(k) \le \Gamma$ $(k = 1, 2, \dots)$. Let also $\omega \in \mathcal{C}([0,T], X[x_k,\gamma(k)])$ for some $T > 0$ and $u \in \mathcal{C}\big([0,T], L^\infty(\mathbb{R}^3)\big)$. If A denotes the bilinear operator introduced in* (IE)*, then*

$$A(u,\omega) \in \mathcal{C}([0,T], X[x_k,\gamma(k)])\,.$$

Furthermore, there exists a constant $C(\Gamma,T)$, depending only on Γ and T, such that $C(\Gamma,T) = O\big(T^{1/2}\big)$ as $T \to 0$ and

$$\|A(u,\omega)\| \;\le\; C(\Gamma,T)\Big(\sup_{t\in[0,T]}\|u(t)\|_\infty\Big)\|\omega\|\,,$$

where $\|\cdot\|$ denotes the natural norm in $\mathcal{C}([0,T], X[x_k,\gamma(k)])$.

P r o o f . To prove Lemma 2.3, instead of studying $A(u,\omega)$ we consider the linear operator

$$\tilde{A}(v)(x,t) \;=\; \int_0^t\!\!\int G(x-y,t-s)v(y,s)\,dy\,ds \quad \big(x\in\mathbb{R}^3,\ t\ge0\big)\,, \tag{2.4}$$

where $G(x,t)$ is the spatial gradient of the Gaussian $g_t(x) = (4\pi t)^{-3/2}e^{-|x|^2/(4t)}$. In particular,

$$\|G(t)\|_1 \;=\; c_0 t^{-1/2} \tag{2.5}$$

and

$$|G(x,t)| \;\le\; C_\alpha\,|x|^{-\alpha}t^{(\alpha-4)/2} \quad \text{for all} \quad \alpha \ge 0\,. \tag{2.6}$$

Let us show that $\tilde{A}$ is bounded in $L^\infty\big([0,T], L^\infty_\gamma\big)$, for all $\gamma\ge0$: we fix $\gamma\ge0$ and we take v such that

$$\sup_{t\in[0,T]}\|v(t)\|_{L^\infty_\gamma} \;<\; \infty\,.$$

Then we obviously have $\big\|\tilde{A}(v)(t)\big\|_\infty \le 2c_0T^{1/2}\sup_{t\in[0,T]}\|v(t)\|_\infty$. Therefore, it suffices to bound $\tilde{A}(v)(x,t)$ in the case $|x|\ge1$.

We now split the space integral of (2.4) into $|y|\le|x|/2$ and $|y|\ge|x|/2$. Let us bound the first term that we obtain using (2.6), and the second term using (2.5). It follows that,

$$\big|\tilde{A}(v)(x,t)\big| \;\le\; I_\alpha + J\,, \quad \text{for any} \quad \alpha \ge 0\,, \tag{2.7}$$

where

$$J \;\equiv\; c_0 \int_0^t (t-s)^{-1/2} \sup_{|y|\geq |x|/2} |v(y,s)|\, ds$$

and

$$I_\alpha \;\equiv\; 2^\alpha C_\alpha |x|^{-\alpha} \int_0^t \int_{|y|\leq |x|/2} (t-s)^{(\alpha-4)/2}\, |v(y,s)|\, ds\,.$$

For all $\gamma \geq 0$, we can simply bound $J(x,t)$ by $c_0 2^{\gamma+1} T^{1/2}\, |x|^{-\gamma} \sup_s \|v(s)\|_{L^\infty_\gamma}$. An obvious modification is needed in the case $\gamma = 1$.

It remains to bound I_α. If $1 \leq \gamma \leq 4$, then we complete our estimate of (2.7) by choosing $\alpha = 4$. Now,

$$I_4(x,t) \;\leq\; 16 C_4 T\, |x|^{-4} \left(\int_{|y|\leq |x|/2} (1+|y|)^{-\gamma}\, dy \right) \sup_s \|v(s)\|_{L^\infty_\gamma}\,.$$

Observe that for $\gamma = 1$ this gives, uniformly in $[0,T]$, $I_4(\cdot,t)\mathbf{1}_{|x|\geq 1} \in L^\infty_2 \subset L^\infty_1$. Thus, we can find a constant C' such that

$$\big\|\tilde{A}(v)(t)\big\|_{L^\infty_\gamma} \;\leq\; C' T \sup_{s\in[0,T]} \|v(s)\|_{L^\infty_\gamma}\,,$$

for all $t \in [0,T]$ and $0 \leq \gamma \leq 4$.

When $\gamma > 4$, we choose in (2.7) $\alpha = \Gamma$ (the constant of the statement of Theorem 2.2) and we may assume $\gamma \leq \Gamma$. Then

$$I_\Gamma(x,t) \;\leq\; C''_\Gamma T^{(\Gamma-2)/2}\, |x|^{-\Gamma} \sup_{s\in[0,T]} \|v(s)\|_{L^\infty_\gamma}\,.$$

Thus, there exists a constant $C(\Gamma,T)$, such that

$$\big\|\tilde{A}(v)(t)\big\|_{L^\infty_\gamma} \;\leq\; C(\Gamma,T) \sup_{s\in[0,T]} \|v(s)\|_{L^\infty_\gamma} \quad (t\in[0,T])\,,$$

for all $0 \leq \gamma \leq \Gamma$ and $v \in L^\infty\big([0,T], L^\infty_\gamma\big)$. Note also that $C(\Gamma,T) = O\big(T^{1/2}\big)$ as $T \to 0$.

The result for $A(u,\omega)$ now easily follows. Indeed, since $\tilde{A}(v)(x,t)$ is translation invariant, we can apply this result to $v(x,t) = \sum_{k=1}^\infty v_k(x-x_k,t)$, where

$$v_k \;=\; u(\cdot + x_k) \otimes \omega_k \in L^\infty_{\gamma(k)}$$

and $\omega(x,t) = \sum_{k=1}^\infty \omega_k(x-x_k,t)$.

We are allowed to do so, because we bounded the operator norm of $\tilde{A}$ in the space $L^\infty\big([0,T], L^\infty_\gamma\big)$ by a constant which is independent of γ, at least when γ varies in a compact interval $[1,\Gamma]$.

The continuity of $A(u,\omega)$ with respect to the time variable is easy to obtain. Indeed, one first checks that $\tilde{A}$ is bounded in $\mathcal{C}\big([0,T], L^\infty_\gamma\big)$. But this is immediate, since $\sqrt{t}\, G \in \mathcal{C}\big(]0,T], L^1(\mathbb{R}^3)\big)$ and $G \in \mathcal{C}\big(]0,T], L^\infty_\gamma\big)$, for all $\gamma \geq 0$.

The conclusion for A then follows arguing as above. This proves Lemma 2.3. □

We now come back to the proof of Theorem 2.2. Note that if $f \in L^\infty_\gamma$, for some $\gamma \geq 0$, then $\big\|e^{t\Delta} f\big\|_{L^\infty_\gamma}$ is uniformly bounded in any compact interval $[0,T]$. Hence, it follows from our assumptions that $\big\|e^{t\Delta}\mu\big\|_{X[x_k,\gamma(k)]}$ is uniformly bounded in $[0,T]$, for any fixed T $(0 < T < \infty)$.

Let us apply Lemma 2.3 with $u = K * \omega$: we get

$$\|A(u,\omega)\| \;\leq\; \big\|e^{t\Delta}\mu\big\| + C'(\Gamma,T)\, \|\omega\|^2\,,$$

where $C'(\Gamma,T) \leq c\, C(\Gamma,T)$, for some absolute constant c. Here we used that $\|u(t)\|_\infty \leq \sum_{k=1}^\infty \|K * \omega_k\|_\infty$ is bounded, up to a constant, by $\|\omega\|$, for all $t \in [0,T]$ (see the beginning of the proof).

Moreover,

$$\|A(u,\omega) - A(\widetilde{u},\widetilde{\omega}\,)\| \;=\; \|A(u,\omega-\widetilde{\omega}\,) + A(u-\widetilde{u},\widetilde{\omega}\,)\|$$

for all ω and $\widetilde{\omega}$ belonging to $\mathcal{C}([0,T],X[x_k,\gamma(k)])$ and $u = K*\omega$, $\widetilde{u} = K*\widetilde{\omega}$. Hence, again by Lemma 2.3,

$$\|A(u,\omega) - A(\widetilde{u},\widetilde{\omega}\,)\| \;\le\; C'(\Gamma,T)(\|\omega\| + \|\widetilde{\omega}\|)\,\|\omega-\widetilde{\omega}\|\,.$$

Then the standard iteration scheme (see e.g. [11])

$$\begin{aligned} \omega^{(j+1)} &= e^{t\Delta}\mu - A\big(u^{(j)},\omega^{(j)}\big)(t) \quad (j=0,1,\ldots)\,,\\ \omega^{(0)}(t) &= e^{t\Delta}\mu\,,\\ u^{(j)}(t) &= K*\omega^{(j)}(t)\,, \qquad\qquad (j=0,1,\ldots)\,, \end{aligned} \tag{2.8}$$

yields the existence of a solution $\omega = \lim_{j\to\infty}\omega^{(j)}$ in $\mathcal{C}([0,T],X[x_k,\gamma(k)])$ of (IE), for some $T = T(\Gamma,\mu) > 0$, which is unique in this space (recall that the continuity at $t=0$ is understood in the weak sense).

Our bound on the operator norm of A gives

$$T(\Gamma,\mu) \;\ge\; c'\,\|\mu\|^{-1/2}_{X[x_k,\gamma(k)]}\,, \tag{2.9}$$

for some constant $c' = c'(\Gamma) > 0$. Taking $\Gamma = 1$, the first statement of Theorem 2.2, with $T = T(1,\mu)$, is proved.

Let us now prove part (II). We show that, whenever $\mu \in X[x_k,\gamma(k)]$, with $1 \le \gamma(k) \le \Gamma$ for all positive integer k, then we can take $T(\Gamma,\mu) = T(1,\mu)$. To do this, we adapt an argument described in [17] (Proposition 25.1).

By (2.9), we just need to show that, if $T < T(1,\mu)$ is fixed so that

$$\omega \in \bigcap_{\tau<T}\mathcal{C}([0,\tau],X[x_k,\gamma(k)])\,,$$

then $\|\omega(t)\|_{X[x_k,\gamma(k)]}$ is uniformly bounded in $t\in[0,T[$. Indeed, for $0\le t<T$,

$$\|\omega(t)\|_{X[x_k,\gamma(k)]} \;\le\; C_1(T) + C_2(T)\int_0^t (t-s)^{-1/2}\,\|\omega(s)\|_{X[x_k,\gamma(k)]}\,ds\,. \tag{2.10}$$

The proof of (2.10) is fully similar to that of Lemma 2.3. If we bound $\|\omega(s)\|_{X[x_k,\gamma(k)]}$ using again (2.10) and we apply Fubini's theorem, we get

$$\|\omega(t)\|_{X[x_k,\gamma(k)]} \;\le\; C_3(T) + C_4(T)\int_0^t \|\omega(\sigma)\|_{X[x_k,\gamma(k)]}\,d\sigma\,.$$

Gronwall's lemma yields $\|\omega(t)\|_{X[x_k,\gamma(k)]} \le C_3(T)e^{C_4(T)t}$.

It remains to prove the localization property of the velocity field. From the previous conclusion and the Biot-Savart law, we may write

$$u(x,t) \;=\; \sum_{k=1}^{\infty} u_k(x-x_k,t) \quad (t\in[0,T])\,,$$

with $u_k = K*\omega_k$ and $\sum_{k=1}^{\infty}\|\omega_k(t)\|_{L^\infty_{\gamma(k)}} < \infty$.

Using again $|K(x-y)| \le C\,|x-y|^{-2}$, it is easy to see that we have, for any $k=1,2,\ldots$, and uniformly in $t\in[0,T]$,

$$u_k(t)\in L^\infty_{\gamma(k)-1}\,, \quad \text{if} \quad 1 \;\le\; \gamma(k) \;<\; 3 \quad \text{and} \quad \gamma(k) \;\ne\; 2\,.$$

We already proved this claim for $\gamma(k) = 1$ and the proof is identical in the other cases. The same argument shows that, if $\gamma(k)=2$, then $|u_k(x,t)| \le C(1+|x|)^{-1}\,\big(x\in\mathbb{R}^3,\ t\in[0,T]\big)$, so that $u_k \in \bigcap_{\beta<1}L^\infty_\beta$. Finally, $u_k(t)\in L^\infty_2$, if $\gamma(k)>3$.

Theorem 2.2 thus follows. □

Remark 2.4 Let k be a fixed positive integer. Under the assumptions of the second part of Theorem 2.2, if $\gamma(k)=2$ then the corresponding u_k term (of any suitable atomic decomposition of the velocity field), in general, does not belong to L^∞_1. We may get $u_k(t)\in L^\infty_1$ if, for example, $\mu_k\in L^\infty_2$ and $|\mu_k(x)|\le 2^{-2j}\epsilon_j$, for some $\epsilon_j\in\ell^1(\mathbb{N})$ and any x such that $2^j\le|x|<2^{j+1}$. A similar remark applies to the case $\gamma(k)=3$: if $\mu_k(x)\in L^\infty_3$ and $|\mu_k(x)|\le 2^{-3j}\epsilon_j$ $\big($with $\epsilon_j\in\ell^1$ and $2^j\le|x|<2^{j+1}$, which means $\int|\mu_k(x)|\,dx<\infty\big)$, then $u_k\in L^\infty_2$.

It is not difficult to see (by repeating the steps of the proof of Theorem 2.2) that these supplementary assumptions on the initial vorticity are conserved during the evolution.

2.1 Atomic decomposition with oscillating building blocks

Theorem 2.2 shows that if $\gamma(k)$ is large, then ω_k is much more localized than u_k. However, conclusion (2.3) can be improved by imposing some supplementary cancellations on the initial vorticity. This leads us to introduce a suitable subspace of $X[x_k,\gamma(k)]$.

Definition 2.5 Let $(x_k)\subset\mathbb{R}^3$ and $\gamma(k)\ge 0$ $(k=1,2,\ldots)$. We say that f is *oscillating and localized around x_k at rates $\gamma(k)$* if $f=\sum_{k=1}^\infty f_k(x-x_k)$ for some sequence $f_k\in L^\infty_{\gamma(k)}$ such that $\sum_{k=1}^\infty\|f_k\|_{L^\infty_{\gamma(k)}}<\infty$, and $\int f_k(x)\,dx=0$, for any k such that $\gamma(k)>3$.

In the vector valued case, $f=\big(f^1,f^2,f^3\big)$, we ask that the three components of f_k have a vanishing integral, whenever $\gamma(k)>3$. The space of oscillating and localized vector fields around (x_k) at rates $\gamma(k)$ will be denoted by $X_0[x_k,\gamma(k)]$. Taking the infimum over all decompositions of f as above, a Banach norm for such space can be defined.

We are now in position to complete the result of Theorem 2.2.

Theorem 2.6 *Let $(x_k)_1^\infty\subset\mathbb{R}^3$, $\Gamma\ge1$, $1\le\gamma(k)\le\Gamma$ and μ be a vector field, oscillating and localized around (x_k) at rates $\gamma(k)$. Let $\omega(x,t)$ $(0\le t\le T)$ be the solution to* (IE) *obtained in Theorem 2.2. Then, ω belongs to $\mathcal{C}([0,T],X_0[x_k,\gamma(k)])$. In this case, the localization results for $u(x,t)=\sum_{k=1}^\infty u_k(x-x_k,t)$ obtained in* (2.3) *are improved in the following way:*

$$\begin{aligned} &u_k(t)\in L^\infty_{\gamma(k)-1}\,, && \text{if}\quad 3<\gamma(k)<4\,,\\ &u_k(t)\in\bigcap_{\beta<\gamma(k)-1}L^\infty_\beta\,, && \text{if}\quad \gamma(k)=3\,,4\,,\\ &u_k(t)\in L^\infty_3\,, && \text{if}\quad \gamma(k)>4\,. \end{aligned}\tag{2.11}$$

Proof. The proof relies on the fact that the kernel of the bilinear operator $A(u,\omega)$ in (IE) has a vanishing integral.

It is easy to see that the fixed point argument applies also in the subspace $X_0[x_k,\gamma(k)]$ of oscillating and localized vector fields. Indeed, our assumption on μ obviously implies that $e^{t\Delta}\mu$ belongs to $X_0[x_k,\gamma(k)]$ for all $t\ge0$. On the other hand, we can come back to the linear operator $\tilde A$ (see the proof of Lemma 2.3): we already know that $\tilde A$ is bounded in $\mathcal{C}([0,T],X[x_k,\gamma(k)])$. Let us show that, if $v\in\mathcal{C}([0,T],X[x_k,\gamma(k)])$, then $\tilde A(v)$ belongs more precisely to $\mathcal{C}([0,T],X_0[x_k,\gamma(k)])$. This is due to the fact that

$$\tilde A(v)(x,t)\;=\;\sum_{k=1}^\infty\bigg(\int_0^t G(t-s)*v_k(s)\,ds\bigg)(x-x_k)\,.$$

Hence $\big($since $\int G(x)\,dx=0\big)$, for all k such that $\gamma(k)>3$ we have $v_k\in L^1\big(\mathbb{R}^3\big)$ and the corresponding "building blocks" of $\tilde A(v)$ have a vanishing integral.

This yields $\omega\in\mathcal{C}([0,T'],X_0[x_k,\gamma(k)])$, with $T'\ge c\,\|\mu\|^{-1/2}_{X_0[x_k,\gamma(k)]}$ for some positive constant c. Using again Gronwall's lemma we see that we may take $T'=T$.

Now, let $\omega=\sum_{k=1}^\infty\omega_k(x-x_k)$ be an oscillating decomposition of ω in $X_0[x_k,\gamma(k)]$ and $u=\sum_{k=1}^\infty u_k(x-x_k)$ be the corresponding decomposition of the velocity field. For all k such that $\gamma(k)>3$, we may write

$$u_k\;=\;-\frac{1}{4\pi}\int\big[(x-y)\,|x-y|^{-3}-x\,|x|^{-3}\big]\times\omega_k(y)\,dy\,.$$

Since the Biot-Savart kernel satisfies $|\nabla K(x-y)| \le C\,|x-y|^{-3}$, the Taylor formula and an integration by parts in the convolution integral, yield $|u_k(x,t)| \le C(1+|x|)^{-\gamma+1}$ (if $3<\gamma<4$) and $|u_k(x,t)| \le C(1+|x|)^{-3}$ (if $\gamma>4$), uniformly in $[0,T]$. This completes the proof of Theorem 2.6. □

Remark 2.7 Note that, if $\gamma(k)=4$ (for some fixed k), if $\int \mu_k(x)\,dx = 0$ and also $\int |\mu_k(x)|\,(1+|x|)\,dx < \infty$, then the conclusion $u_k \in \bigcap_{\beta<3} L^\infty_\beta$ can be improved by $u_k \in L^\infty_3$. It is easily seen that the supplementary condition $\int |\mu_k(x)|\,(1+|x|)\,dx < \infty$ is conserved during the evolution (see Remark 2.4).

2.2 Localization around one point

In this section we discuss the particular case of the constant sequences $x_k = 0$ and $\gamma(k) = \gamma > 4$ $(k=1,2,\ldots)$. In this situation, the series (1.1) reduces to a single term. By Theorem 2.2, we know that $u(x,t) = O\big(|x|^{-2}\big)$ at infinity, uniformly in a small interval $[0,T]$. But $\omega(t) \in L^1\big(\mathbb{R}^3\big)$ and $\mathrm{div}(\omega) = 0$ imply $\int \omega(x,t)\,dx = 0$ in $[0,T]$ (this is easily seen via the Fourier transform). Hence, by Theorem 2.6,

$$u(x,t) \;=\; O\big(|x|^{-3}\big) \quad \text{as} \quad |x| \;\to\; \infty\,.$$

In some cases, such decay rate for the velocity field can be improved. Indeed, a simple computation shows that, for $j = 1,2,3$,

$$\int x_j \sum_{h=1}^{3} \partial_h\big(u^h\omega - \omega^h u\big)(x,t)\,dx \;=\; -\int \big(u^j\omega - \omega^j u\big)(x,t) \;=\; 0 \tag{2.12}$$

$\big($the second equality follows from the definition of $\omega = \nabla \times u$ and integration by parts. All integrations by parts here are justified if μ decays at infinity faster than $|x|^{-2}\big)$. In particular, we see that the first order moments of $\omega(t)$ are invariant (this fact was well known, see e.g. [18]).

Now, let us assume that $\mu \in L^\infty_\gamma$, with $\gamma > 4$, and $\int x_j\mu(x)\,dx = 0$ $(j=1,2,3)$. A supplementary integration by parts in the Biot-Savart law yields, in this case, the following decay result for the velocity field (uniformly in a small interval $[0,T]$):

$$u(t) \in L^\infty_{\gamma-1}\,, \quad \text{if} \quad 4 \;<\; \gamma \;<\; 5\,,$$

$$u(t) \in L^\infty_4\,, \quad \text{if} \quad \gamma \;\ge\; 5 \quad \text{and} \quad \int |x|^2\,|\mu(x)|\,dx \;<\; \infty\,.$$

Such decay $u(x,t) = O\big(|x|^{-4}\big)$ for the velocity field at infinity can be obtained in many other different ways (see e.g. [21], [5]) and it is known to be optimal, at least in general, even if $u(\cdot,0) \in C_0^\infty\big(\mathbb{R}^3\big)$ and t is small. Indeed, the decay of u is closely related to the number vanishing moments of ω and we know that the conditions $\int x^\alpha\mu(x)\,dx = 0$ $(|\alpha| = 2)$ are not conserved, in general. We refer to [18] for some more details. See also [5] for a discussion on the instantaneous spatial spreading of the velocity field in a general setting.

Let us now assume that the initial vorticity is well localized and well oscillating around a fixed sequence (x_k). Say, $\mu = \sum_{k=1}^\infty \mu_k(x-x_k)$, where μ_k has a fast decay at infinity and $\int \mu_k(x)\,dx = \int x_j\mu_k(x)\,dx = 0$ for all k $(j=1,2,3)$. Then, it may be expected that $u(t)$ should belong to $X[x_k,\mathbf{4}]$ where $\mathbf{4}$ is the constant sequence $\gamma(k) = 4$ for all k. However, due to the non linearity of the equation, the cancellations (2.12), which we used in the case of the constant sequence $(x_k) \equiv 0$, cannot be decomposed into building blocks. On the other hand, it is not true that $A(u(\cdot + x_k),\omega_k)$ has vanishing first-order moments, if $x_k \neq 0$. Hence, the proof of Theorem 2.6 does not go through.

Thus, at later times, the building blocks $\omega_k(x,t)$ of the vorticity may have no vanishing moments other than $\int \omega_k(x,t)\,dx = 0$. This is why the last conclusion of (2.11) seems difficult to be improved.

3 Frequency estimates

We have seen that if μ decays at infinity at a sufficiently fast rate, then $\widehat{\omega}(0,t) = 0$ $(t \in [0,T])$ and we have also $\nabla\widehat{\omega}(0,t) = 0$ if the first order moments of μ vanish. Further, we will see in Section 5 that $\widehat{\omega}(\xi,t)$ vanishes at $\xi = 0$ even at larger orders for some particular flows. Combining these observations with the estimate

$$|\widehat{\omega}(\xi,t)| \;\le\; Ce^{-\sqrt{t}\,|\xi|} \quad \big(\xi \in \mathbb{R}^3,\ t \in [0,T]\big) \tag{3.1}$$

which is easily obtained under quite general assumptions (see below), implies that the vorticity is well localized also with respect to the frequency variable. Here, the Fourier transform of an integrable function f is defined as $\widehat{f}(\xi) = \int f(x)e^{-i\xi\cdot x}\,dx$.

We will often make use of the Lorentz spaces $L^{p,q}(\mathbb{R}^3)$ $(1 < p < \infty,\ 1 \le q \le \infty)$. We refer e.g. to [1] and [17] for a definition and the proofs of all the elementary properties of these spaces (Hölder-type inequalities, convolution estimates, interpolation results, etc.) that we use in this paper.

In this section we will prove the following simple statement and discuss some of its consequences.

Proposition 3.1 *Let μ be a bounded and divergence-free vector field with fast decay at infinity: $|\mu(x)| \le C(m)(1+|x|)^{-m}$ $(m = 1,2,\ldots)$. Then there exists $T > 0$ and a unique solution $\omega(t)$ of* (IE) *belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ for all $0 < t \le T$, such that $\omega \in \mathcal{C}(]0,T], \mathcal{S}(\mathbb{R}^3))$ and $\omega(t) \to \mu$ in the weak sense, as $t \to 0$.*

If, moreover, μ belongs to $\mathcal{S}(\mathbb{R}^3)$, then $\omega \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^3))$.

Proof. To prove that $\omega(t) \in \mathcal{S}(\mathbb{R}^3)$, we just have to show that $\omega(t) \in L^\infty_m$ and $\widehat{\omega}(t) \in L^\infty_m$ for all $m = 1,2,\ldots$.

The spatial decay of $\omega(t)$ follows (at least if t belongs to a small time interval $[0,T_1]$) applying Theorem 2.2, with the constant sequence $(x_k) \equiv 0$ and $\Gamma = \Gamma(m) = 1,2,\ldots$. This yields $\omega \in \mathcal{C}(]0,T_1], L^\infty_m)$ for all m and $\omega(t) \to \mu$ in the weak sense, as $t \to 0$.

On the other hand, the assumption of Proposition 3.1 implies that $\widehat{\mu} \in L^2 \cap L^\infty(\mathbb{R}^3)$. Let us show that this in turn gives (3.1).

We start with the following lemma.

Lemma 3.2 *Let $T > 0$, $3/2 < p < 3$, $1 \le q \le \infty$ and $\omega(t)$ be a tempered distribution on $\mathbb{R}^3$ $(0 \le t \le T)$ such that $\widehat{\omega} \in \mathcal{C}([0,T], L^{p,q})$. Let K and A be defined as in* (IE) *and $u = K * \omega$. Then,*

$$\widehat{A}(u,\omega) \in \mathcal{C}([0,T], L^{\alpha,1}), \quad \textit{for all} \quad \frac{3p}{6-p} < \alpha \le \frac{3p}{6-2p}.$$

Proof. The proof is straightforward: from $\frac{1}{|\cdot|} \in L^{3,\infty}(\mathbb{R}^3)$ it follows that $\widehat{u}(t) \in L^{3p/(3+p),q}$ $(t \in [0,T])$. By Young's inequality, $|\widehat{u}| * |\widehat{\omega}|(t) \in L^{3p/(6-2p),q/2}$ uniformly in t, for $3/2 < p < 3$ and $2 \le q \le \infty$. We now use

$$|\widehat{A}(u,\omega)|(\xi,t) \le \int_0^t |\xi|\, e^{-(t-s)\,|\xi|^2} |\widehat{u}| * |\widehat{\omega}|(\xi,s)\,ds \tag{3.2}$$

and

$$|\xi|\, e^{-(t-\cdot)\,|\xi|^2} \in L^1([0,t], L^{\alpha_1,\beta}) \cap L^1([0,t], L^\infty(\mathbb{R}^3)),$$

for all $3 < \alpha_1 < \infty$ and $1 \le \beta \le \infty$ $(t \in [0,T])$. Letting $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \frac{6-2p}{3p}$ $\left(\text{or } \frac{1}{\alpha} = \frac{6-2p}{3p}\right)$, we obtain

$$\left\|\widehat{A}(u,\omega)(t)\right\|_{\alpha,1} \le C(\alpha,p,q)\, \|\omega\|^2_{\mathcal{C}([0,T],L^{p,q})} t^{(6\alpha-3p-\alpha p)/(2\alpha p)}$$

(the exponent of t is positive). We skip the proof of the continuity of $\widehat{A}(u,\omega)$ with respect to t, since it follows from the same arguments that we used in the proof of Lemma 2.3. Lemma 3.2 thus follows. □

Inspired by a result of P. G. Lemarié-Rieusset, on the analyticity of solutions to the Navier-Stokes equations in a space of pseudomeasures type (see [17]), we now state a slightly modified version of the previous lemma.

Lemma 3.3 *Let p, q and $u = K * \omega$ be as above. Assume now that $e^{\sqrt{t}\,|\xi|}\widehat{\omega} \in \mathcal{C}([0,T], L^{p,q})$. Then*

$$e^{\sqrt{t}\,|\xi|}\widehat{A}(u,\omega) \in \mathcal{C}([0,T], L^{\alpha,1}),$$

for all $\frac{3p}{6-p} < \alpha \le \frac{3p}{6-2p}$.

Proof. We use in (3.2) an inequality due to Foias and Temam [10]. Such inequality reads

$$e^{-(t-s)\,|\xi|^2} e^{-\sqrt{s}\,|\xi-\eta|} e^{-\sqrt{s}\,|\eta|} \le e^2 e^{-\sqrt{t}\,|\xi|} e^{-(t-s)\,|\xi|^2/2}, \tag{3.3}$$

for all ξ and η in $\mathbb{R}^3$ and all $0 \le s \le t$ (see also [17] for a proof).

The proof of Lemma 3.3 is now identical to that of the previous lemma. $\square$

Let us come back to the proof of Proposition 3.1. Since $\widehat{\mu} \in L^2(\mathbb{R}^3)$, then we obviously have $e^{\sqrt{t}\,|\xi|}e^{-t\,|\xi|^2}\widehat{\mu} \in \mathcal{C}\big([0,T], L^2(\mathbb{R}^3)\big)$ for all $T > 0$. Hence, if we apply Lemma 3.3 with $p = q = \alpha = 2$ we see that the approximation scheme (2.8) yields the existence and the unicity of a solution ω to (IE), such that $e^{\sqrt{t}\,|\xi|}\widehat{\omega} \in \mathcal{C}\big([0,T_2], L^2(\mathbb{R}^3)\big)$.

Observe that Lemma 3.3 also implies that $e^{\sqrt{t}\,|\xi|}\widehat{A}(u,\omega) \in \mathcal{C}\big([0,T_2], L^{3,1}\big)$. Since $\widehat{\mu} \in L^{3,1}$ (this follows by interpolation), we deduce that

$$e^{\sqrt{t}\,|\xi|}\widehat{\omega} \in \mathcal{C}\big([0,T_2], L^{3,1}\big)\,.$$

But, if $\widehat{f} \in L^{3,1}(\mathbb{R}^3) \subset L^{3,2}(\mathbb{R}^3)$, then $\big(\widehat{f}/|\cdot|\big) * \widehat{f}$ belongs to $L^\infty(\mathbb{R}^3)$, because of the duality between $L^{3/2,2}(\mathbb{R}^3)$ and $L^{3,2}(\mathbb{R}^3)$. Using this observation, (3.2) and (3.3) we deduce $e^{\sqrt{t}\,|\xi|}\widehat{A}(u,\omega) \in \mathcal{C}\big([0,T_2], L^\infty\big)$.

Recalling that $\widehat{\mu} \in L^\infty(\mathbb{R}^3)$, we see that $e^{\sqrt{t}\,|\xi|}\widehat{\omega} \in \mathcal{C}([0,T_2], L^\infty)$.

This implies that, if $T = \min\{T_1, T_2\}$, then $\omega \in \mathcal{C}\big(]0,T], \mathcal{S}(\mathbb{R}^3)\big)$. Moreover, if we know that $\mu \in \mathcal{S}(\mathbb{R}^3)$, then the continuity obviously holds true in the closed interval $[0,T]$. $\square$

Remark 3.4 Proposition 3.1 provides an example of a situation in which the flow is given by the sum of a sparse wavelet series.

Indeed, let $\psi_\epsilon(2^j x - k)$ $(j \in \mathbb{Z}, k \in \mathbb{N}^n)$ an orthogonal wavelet basis of $L^2(\mathbb{R}^3)$, where ϵ belongs to a finite set of indexes and $\psi_\epsilon \in \mathcal{S}(\mathbb{R}^3)$ for all ϵ. Let also f be a function defined in $\mathbb{R}^3$, and

$$f \;=\; \sum_\epsilon \sum_{j,k} \alpha_\epsilon(j,k)\psi_\epsilon\big(2^j x - k\big)$$

be its wavelet expansion. We refer to [19], [20] for generalities on wavelets and discussions on their possible applications to the Navier-Stokes equations.

Elementary computations (integration by parts and decay estimates) show that, if $f \in \mathcal{S}(\mathbb{R}^3)$ then its wavelet coefficients

$$\alpha_\epsilon(j,k) \;=\; 2^{3j}\int f(x)\psi_\epsilon\big(2^j x - k\big)\,dx$$

form a sparse matrix, in the following sense:

$$S_p(f) \;\equiv\; \sum_\epsilon \sum_{j,k} |\alpha_\epsilon(j,k)|^p \;<\; \infty \quad \text{for all} \quad p \;>\; 0\,. \tag{3.4}$$

This means that the non-increasing rearrangement $c_1^* \ge c_2^* \ge \ldots$ of $|\alpha_\epsilon(j,k)|$ has a fast decay at infinity:

$$\sup_{m\in\mathbb{N}} (1+m)^k c_m^* \;\le\; C_k \;<\; \infty \quad (k = 1, 2, \ldots)\,.$$

It is worth remarking that $(S_p(f))^{1/p}$ turns out to be a norm for the homogeneous Besov space $\dot{B}_p^{3/p,p}(\mathbb{R}^3)$ (quasi-norm, if $0 < p \le 1$). See [14] and the bibliography therein contained for equivalent definitions and a detailed study of Besov spaces. Thus, the property that f has a sparse wavelet expansion is equivalent to

$$f \in \bigcap_{p>0} \dot{B}_p^{3/p,p}(\mathbb{R}^3)\,.$$

Proposition 3.1 implies that, if $\mu \in \mathcal{S}(\mathbb{R}^3)$, then

$$\sup_{t\in[0,T]} S_p(\omega(t)) \;<\; \infty \quad \text{for all} \quad p \;>\; 0\,,$$

thus, the solution has a sparse wavelet series, uniformly in $[0,T]$. It is not difficult to see that in this case also the velocity field admits a sparse wavelet expansion. It is not difficult to see that in this case also the velocity field admits a sparse wavelet expansion. This assumption on the initial datum, of course, is by no means optimal.

On the other hand if, for example, μ is discontinuous, then μ does not have a sparse wavelet expansion since, necessarily, $S_1(\mu)=\infty$. But if such a μ is well localized, then it follows from Proposition 3.1 that an instantaneous reorganization occurs in the flow, and $\omega(t)$ $(0<t\le T)$ turns out to be the sum of a sparse wavelet series.

4 Large time behavior

The solutions constructed in Section 2 turn out to be defined globally in time, under suitable smallness assumptions on the initial vorticity. The aim of this section is the study of the asymptotic profiles as $|x|+t\to\infty$ of such solutions.

We will consider only the special case where ω is localized around a single point, say, the origin. We start with a lemma inspired by [21] where conclusion (4.2) below was obtained under slightly different assumptions.

Lemma 4.1 *Let μ be a soleinoidal vector field in $\mathbb{R}^3$, such that $\operatorname{div}(\mu)=0$ and $|\mu(x)|\le c\,(1+|x|)^{-2}$. There exists an absolute constant $\eta>0$ such that if $\sup_{\mathbb{R}^3}|x|^2\,|\mu(x)|<\eta$, then there exists a global solution ω of* (IE), *such that*

$$|\omega(x,t)|\ \le\ C\,(1+|x|)^{-2}\,,\quad |\omega(x,t)|\ \le\ C\,(1+t)^{-1}\,, \tag{4.1}$$

$$|u(x,t)|\ \le\ C\,(1+|x|)^{-1}\,,\quad |u(x,t)|\ \le\ C(1+t)^{-1/2}\,, \tag{4.2}$$

and $\omega(0)=\mu$ in the distributional sense.

Proof. If we show (4.1), then (4.2) immediately follows. Indeed, from the decay of the Biot-Savart kernel K and the interpolation of Lorentz spaces, we have

$$\|u(t)\|_\infty\ \le\ C\,\|K\|_{L^{3/2,\infty}}\,\|\omega(t)\|^{1/2}_{L^{3/2,\infty}}\,\|\omega(t)\|^{1/2}_\infty\ \le\ Ct^{-1/2}\,.$$

On the other hand, we already observed at the end of the proof of Theorem 2.2 that $\omega\in L^\infty_2$ (uniformly in t) implies $|u(x,t)|\le C(1+|x|)^{-1}$.

By a simple rescaling argument $\big($recall that if $\omega(x,t)$ is a solution of (IE), then the same is true for $\lambda^2\omega\big(\lambda x,\lambda^2 t\big)$, we can assume that $\sup_x\big(1+|x|^2\big)\,|\mu(x)|$ is small. Hence, we can find a constant $C>0$ such that $|e^{t\Delta}\mu(x)|\le C\eta(1+|x|)^{-2}$ and $|e^{t\Delta}\mu(x)|\le C\eta t^{-1}$. The conclusion of our lemma will be an immediate consequence of the iteration scheme (2.8), if we can show that $\omega\mapsto A(K*\omega,\omega)$ is bounded in the space given by (4.1).

Equivalently, we have to show that if $v=v(x,t)$ is a function such that $\sup_{x,t}(1+|x|)^3\,|v(x,t)|$ and $\sup_{x,t}(1+t)^{3/2}\,|v(x,t)|$ are finite, then $\tilde A(v)$ satisfies (4.1) (here $\tilde A$ is the linear operator introduced in (2.4)).

This is easy, if we use the arguments of [21]: (2.5) implies that $\big\|\tilde A(v)(t)\big\|_\infty\le Ct^{1/2}$. Thus, we just consider the case $|x|\ge 1$ and $t\ge 1$. The estimate $\sup_{x,t}|x|^2\big|\tilde A(v)(x,t)\big|<\infty$ follows from the bounds on v and the properties of G (see (2.5), (2.6)), writing $\tilde A(v)\equiv I_1+I_2$, where

$$I_1\ \equiv\ \int_0^t\!\!\int_{|y|\le|x|/2}G(x-y,t-s)v(y,s)\,dy\,ds \tag{4.3}$$

and

$$I_2\ \equiv\ \int_0^t\!\!\int_{|y|\ge|x|/2}G(x-y,t-s)v(y,s)\,dy\,ds\,. \tag{4.4}$$

Next, $\tilde A(v)(t)=J_1+J_2$, with

$$J_1\ \equiv\ e^{t\Delta/2}\tilde A(v)(t/2) \tag{4.5}$$

and

$$J_2 \;\equiv\; \int_{t/2}^{t} G(t-s) * v(s)\, ds\,. \tag{4.6}$$

Since $\sup_t \big\|\tilde{A}(v)(t)\big\|_{L^{3/2,\infty}} < \infty$, by duality $\|J_1(\cdot,t)\|_\infty \le C\,\|g_t\|_{L^{3,1}} \le Ct^{-1}$. Moreover, by (2.5) and the time decay of v, we get $\|J_2(\cdot,t)\|_\infty \le Ct^{-1}$. This ends the proof of Lemma 4.1. □

If the initial vorticity decays at a faster rate, we can obtain more general space-time profiles for ω. If $\mu \in L^\infty_\gamma$, with $\gamma \ge 2$, then we will obtain bounds of the form $\sup_{x,t}(1+|x|)^\alpha\,|\omega(x,t)| \le C_\alpha(t)$, for all $0 \le \alpha \le \gamma$.

Before giving a more detailed statement, let us point out that we cannot expect for $\omega(x,t)$ to have a fast decay both in space and time, in general. Indeed, even if $\mu \in \mathcal{S}(\mathbb{R}^3)$ and all the moments of μ vanish, a fast space-time decay would not hold true for the linear evolution $e^{t\Delta}\omega(x,t_0)$, for any fixed $t_0 > 0$. This is relied to the instantaneous loss of the cancellations of ω, which we mentioned at the end of Section 2.

Anyhow, we can bound the explosion at infinity of the constants $C_\alpha(t)$, as the following theorem shows.

Theorem 4.2 *Let $\gamma \ge 2$, $\gamma \ne 3,4$, μ be a soleinoidal vector field, such that $|\mu(x)| \le C\,(1+|x|)^{-\gamma}$. There exists an absolute constant ϵ $(0<\epsilon\le\eta)$ such that, if $\sup |x|^2\,|\mu(x)| < \epsilon$, then*

i) *The solution obtained in Lemma* 4.1 *satisfies*

$$|\omega(x,t)| \;\le\; C_\alpha(1+|x|)^{-\alpha}(1+t)^{-\beta/2} \quad (0\le\alpha\le\gamma,\ \alpha+\beta=\min\{\gamma,4\})\,. \tag{4.7}$$

ii) *If $\gamma > 4$ $(\gamma \ne 5)$ and $\int x_j\mu(x)\,dx = 0$ $(j=1,2,3)$, then* (4.5) *is improved by*

$$|\omega(x,t)| \;\le\; C_\alpha(1+|x|)^{-\alpha}(1+t)^{-\beta/2} \quad (0\le\alpha\le\gamma,\ \alpha+\beta=\min\{\gamma,5\})\,. \tag{4.8}$$

Such conclusions remain true for $\gamma = 3,4,5$, if the solution of the heat equation with initial data μ satisfies $\sup_{x,t}(1+|x|)^\gamma\,|e^{t\Delta}\mu(x)| < \infty$ *and* $\sup_{x,t}(1+t)^{\gamma/2}\,|e^{t\Delta}\mu(x)| < \infty$.

Proof. We can limit ourselves to $|x| \ge 1$ and $t \ge 2$.

Step 1. Let us start by treating the case $2<\gamma<3$. We show that the approximate solutions $\omega^{(j)}$ constructed as in (2.8) are localized uniformly in j, if the initial vorticity is localized. More precisely, we are going to prove that $(1+|x|)^\gamma\,|\omega^{(j)}(x,t)| \le M_{j,\gamma}$ $(j=0,1,\ldots)$ and $\sup_j M_{j,\gamma} \equiv M_\gamma < \infty$. This argument is similar to that of [15], in which it is treated the case of the velocity field and the localization properties are measured by means of the L^p spaces $(p>1)$.

We obviously have $|\omega^{(0)}(x,t)| = |e^{t\Delta}\mu(x)| \le M_{0,\gamma}(1+|x|)^{-\gamma}$. We now proceed by induction on j. Assume that $M_{j,\gamma}$ is finite.

By the proof of the previous lemma, we can find a constant $c>0$ such that $|u^{(j)}(x,t)| \le c\,\epsilon t^{-1/2}$ for all $j=0,1,\ldots$. Then, from (2.5), (2.6), we see that $\big|A(u^{(j)},\omega^{(j)})(x,t)\big|$ is bounded by

$$\begin{aligned}
&C_\gamma\epsilon M_{j,\gamma}\int_0^t\!\!\int_{|y|\le|x|/2} |x-y|^{-3}(t-s)^{-1/2}s^{-1/2}(1+|y|)^{-\gamma}\,ds\,dy\\
&+ C_\gamma\epsilon M_{j,\gamma}\bigg(\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\bigg)(1+|x|)^{-\gamma}\,.
\end{aligned}$$

This implies $M_{j+1,\gamma} \le M_{0,\gamma} + C'_\gamma\epsilon M_{j,\gamma}$.

If ϵ is small enough, in a such way that $C'_\gamma\epsilon < 1$, we obtain $M_\gamma < \infty$ and $|\omega(x,t)| \le M_\gamma(1+|x|)^{-\gamma}$. Note that $C'_\gamma \to \infty$ as $\gamma \to 3$ but, as we shall see hereafter, we need to apply this argument only for a fixed γ_0 $(2<\gamma_0<3)$. Therefore, the smallness assumption will actually be independent of γ. By the Biot-Savart law we get

$$\sup_{x,t}\,(1+|x|)^{\gamma-1}\,|u(x,t)| \;<\; \infty\,.$$

To improve the two estimates $\|\omega(t)\|_\infty \le C\,(1+t)^{-1}$ and $\|u(t)\|_\infty \le C\,(1+t)^{-1/2}$ we observe that $\|u\omega(t)\|_1 \le C_\delta(1+t)^{-\delta}$, for all $0\le\delta<(3\gamma-6)/(2\gamma-1)$. We now fix a real a $(0<a<1)$, which will be chosen later. Next, $\|A(u,\omega)(t)\|_\infty$ is bounded, up to an absolute constant, by

$$\int_0^{t-t^a} (t-s)^{-2}\,\|u\omega(s)\|_1\,ds + \int_{t-t^a}^{t} (t-s)^{-1/2}\,\|u\omega(s)\|_\infty\,ds\,.$$

The first integral is bounded by $C_\delta t^{-2a+1-\delta}$ and the second by $Ct^{-3/2+a/2}$ (recall that $t \ge 2$). Let us choose $a = 1 - 2\delta/5$, with $0 \le \delta < (3\gamma - 6)/(2\gamma - 1)$. Hence,

$$\|A(u,\omega)(t)\|_\infty \;\le\; C_\beta t^{-\beta/2}\,, \quad \text{for all} \quad 2 \;\le\; \beta \;<\; (26\gamma - 22)/(10\gamma - 5)\,.$$

On the other hand, $\left\|e^{t\Delta}\mu\right\|_\infty \le C_\gamma t^{-\gamma/2}$ and this estimate is better than that we just obtained for the non-linear term. Therefore, $\|\omega(t)\|_\infty \le C_\beta(1+t)^{-\beta/2}$ $(2 \le \beta < (26\gamma - 22)/(10\gamma - 5))$. This in turn implies $\|u(t)\|_\infty \le C'_\beta(1+t)^{-(\beta-1)/2}$.

So far we proved the following: there exists γ_0 $(2 < \gamma_0 \le \gamma < 3)$ such that

$$|\omega(x,t)| \;\le\; C\,(1+|x|)^{-\gamma_0} \quad \text{and} \quad \|\omega(t)\|_\infty \;\le\; C\,(1+t)^{-\gamma_0/2}\,.$$

Further, we obtain the same estimates for $u(x,t)$, provided that we replace γ_0 with $\gamma_0 - 1$.

It is now easy, coming back to the terms I_1, I_2, J_1 and J_2 which we introduced in the proof of Lemma 4.1, to see that such estimates hold for $A(u,\omega)$, with an exponent $\gamma_1 > \gamma_0$. Using a boot-strap argument, we find after finitely many iterations

$$\|\omega(t)\|_\infty \;\le\; C_\gamma t^{-\gamma/2}\,.$$

Step 2. We now consider the case $3 \le \gamma < 4$. We come back once again to I_1 and I_2: from (2.5), (2.6) and the result of the previous step we immediately get $I_2 \le C_\delta(1+|x|)^{-\delta}$ for all $0 \le \delta < 4$. Hence, $I_2 \le C_\gamma(1+|x|)^{-\gamma}$. Similarly, we obtain $I_1 \le C_\gamma|x|^{-\gamma}$.

It follows that $|v(x,t)| = |u \otimes \omega|(x,t)$ is bounded by $C_\alpha(1+|x|)^{-\alpha}$, and for any $\alpha < 5$. Using this fact and applying (2.5)–(2.6) to $\tilde{A}(v)(t) = J_1 + J_2$, we get $\|\omega(t)\|_\infty \le C_\beta(1+t)^{-\beta}$ for any $\beta < 2$. The proof of (4.7) is now complete in the case $3 \le \gamma < 4$.

The proof in the case $\gamma \ge 4$ makes use of a boot-strap argument: using estimates of the same kind as before, it is easily seen that if we have a bound of the form $|\omega(x,t)| \le C_\alpha\,(1+|x|)^{-\alpha}(1+t)^{-(4-\alpha)/2}$, for some $\delta \ge 3$ and all α such that $0 \le \alpha \le \delta$, then we have also $|A(u,\omega)(x,t)| \le C_\alpha\,(1+|x|)^{-\alpha}(1+t)^{-(4-\alpha)/2}$, for any $0 \le \alpha \le \delta + 1$. After finitely many iterations, conclusion (i) follows.

Step 3. The proof of the second part of the theorem is easy. Indeed, under the assumptions of (ii) we immediately see that $\left|e^{t\Delta}\mu(x)\right|$ is bounded by the right-hand side of (4.8). Moreover, from $\omega = \nabla \times u$, $\operatorname{div}(u) = 0$ and the decay at infinity of ω and u obtained in (i), we get $\int x_j \sum_{h=1}^{3} \partial_h\big(u^h\omega - \omega^h u\big)\,dx = 0$ $(j = 1,2,3)$. This supplementary cancellation allows us to integrate by parts in the convolution integral of $A(u,\omega)$. Hence, $A(u,\omega)$ essentially equals $\int_0^t\!\int G'(x-y,t-s)v(y,s)\,dy\,ds$, where v is given by the product of components of u with components of ω, and G' behaves as the second order derivatives of the Gaussian. In particular, $|G'(x,t)| \le C_\alpha|x|^{-\alpha}t^{-(5-\alpha)/2}$, for all $\alpha \ge 0$. Conclusion (ii) now follows from straightforward modifications of the previous proof. $\square$

5 A class of more oscillating vorticities

The bound (4.8) (and in particular the restriction $\alpha + \beta \le \min\{\gamma, 5\}$) is optimal, at least in the generic case (this is a consequence of the results on the spatial spreading of the velocity field (see [5]). However, the proof of Theorem 4.2 shows that we could improve on this estimate if we had the supplementary cancellations

$$\int x^\alpha \mu(x)\,dx \;=\; 0$$

and

$$\int x^\alpha \sum_{h=1}^{3} \partial_h\big(u^j\omega^h - \omega^j u^h\big)(x,t)\,dx \;\equiv\; 0$$

for all $t \ge 0$, $|\alpha| = 2$, and $j = 1,2,3$. In this case, the moments of μ would be invariant up to the order two.

Actually, such a special class of solutions does exist. Indeed, we simply consider vorticities associated with symmetric velocity fields, introduced in [2]. A symmetric flow is characterized by the following properties:

(i) $\omega^1(x_1,x_2,x_3,t) = \omega^2(x_3,x_1,x_2,t) = \omega^3(x_2,x_3,x_1,t)$,
(ii) $\omega^j(x,t)$ is even with respect to x_j and odd with respect to x_k $(j,k = 1,2,3$ and $j \neq k)$.
In this case, $u = K * \omega$ also satisfies condition (i), and the condition (ii) with a reversed parity.

If we come back to the construction of the solution ω obtained in Lemma 4.1, it is easily seen that if μ satisfies (i) and (ii) at $t = 0$, then it is the same for $\omega(t)$, for all $t > 0$.

Let us now make a suitable localization assumption for μ, say $|\mu(x)| \leq C(1+|x|)^{-\gamma}$, with $\gamma > 5$. If μ satisfies (ii) at $t = 0$, then $\int x^\alpha \mu(x)\,dx = 0$ $(0 \leq |\alpha| \leq 1)$. On the other hand, since $\mathrm{div}\mu = 0$, we have $\int \big(x_2x_3\mu^1(x) + x_3x_1\mu^2(x) + x_1x_2\mu^3(x)\big)\,dx = 0$ (see, e.g. [12]). If μ satisfies also (i), it follows that the moments of μ vanish up to the second order. Moreover, if $\gamma > 6$, also the third order moments vanish.

Thus, for localized symmetric flows we see that $\left|e^{t\Delta}\mu(x)\right|$ is bounded by

$$C_\alpha(1+|x|)^{-\alpha}(1+t)^{-\beta/2} \quad (0 \leq \alpha \leq \gamma,\ \alpha+\beta = \min\{\gamma,6\})\,, \tag{5.1}$$

and, if $\gamma > 6$, also by

$$C_\alpha(1+|x|)^{-\alpha}(1+t)^{-\beta/2} \quad (0 \leq \alpha \leq \gamma,\ \alpha+\beta = \min\{\gamma,7\})\,. \tag{5.2}$$

In order to see that the solution $\omega(x,t)$ itself is bounded by (5.1) or (5.2), we just have to show that

$$\int x^\alpha \sum_{h=1}^{3} \partial_h\big(u^h\omega - \omega^h u\big)(x,t)\,dx \;\equiv\; 0\,, \tag{5.3}$$

respectively, for $|\alpha| = 2$ or $|\alpha| \leq 3$. Indeed, it then suffices to proceed as in the last step of the proof of Theorem 2.6.

But cancellations (4.3) are trivial for $|\alpha| = 3$, or for $|\alpha| = 2$ and $x^\alpha = x_j^2$ $(j = 1,2,3)$. For the other cases a short calculation is needed: we can choose, for example, $j = 1$ and $\alpha = (0,1,1)$. Then we have to show that $\int x_2x_3\partial_2\big(u^1\omega^2 - u^2\omega^1\big)\,dx + \int x_2x_3\partial_3\big(u^1\omega^3 - u^3\omega^1\big)\,dx = 0$. Indeed, by the definition of ω and $\mathrm{div}(u) = 0$, this expression equals $\int \frac{1}{2}\big(\big(u^2\big)^2 - \big(u^3\big)^2\big)\,dx + \int\big(x_3u^2\partial_3u^2 + x_3u^1\partial_3u^1 - x_2u^3\partial_2u^3 - x_2u^1\partial_2u^1\big)\,dx$. By (i), such two integrals vanish.

This discussion leads us to the following result:

Theorem 5.1 i) *Let μ be a symmetric vector field, satisfying the conditions of Theorem* 4.2 *with* $\gamma > 5$ $(\gamma \neq 6)$. *Then estimate* (4.7) *holds for any α and β such that* $0 \leq \alpha \leq \gamma$ *and* $\alpha + \beta = \min\{\gamma,6\}$.

ii) *If* $\gamma > 6$ $(\gamma \neq 7)$, *then we can take* $\alpha+\beta = \min\{\gamma,7\}$. *Moreover, such conclusions hold true also for* $\gamma = 6, 7$, *if* $\sup_{x,t}(1+|x|)^\gamma\left|e^{t\Delta}\mu(x)\right| < \infty$.

It should be noted that when the cancellations (5.3) are satisfied for all $t \geq 0$, and for all α, with $|\alpha| \leq m$ $(m \in \mathbb{N})$, then the moments of μ are conserved during the evolution, up to the order m. This situation applies to symmetric flows for $m = 0,1,2,3$. We can obtain solutions with even more vanishing moments by putting more stringent symmetries on μ (see [4]).

We finally observe that Gallay and Wayne, applying the invariant manifold theory to the Navier-Stokes equations, succeeded in proving the existence of solutions such that $\|\omega(t)\|_\infty$ decays at arbitrarily high (algebraic) decay rates (see [12]). However, their methods do not provide explicit examples of the corresponding initial data.

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