GLOBAL EXISTENCE VERSUS BLOW UP
FOR SOME MODELS OF INTERACTING PARTICLES

BY

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Abstract. We study the global existence and space-time asymptotics of solutions for a class of nonlocal parabolic semilinear equations. Our models include the Nernst–Planck and Debye–Hückel drift-diffusion systems as well as parabolic-elliptic systems of chemotaxis. In the case of a model of self-gravitating particles, we also give a result on the finite time blow up of solutions with localized and oscillating complex-valued initial data, using a method due to S. Montgomery-Smith.

1. Introduction. In this paper we are concerned with semilinear parabolic systems of the form

$$
(1) \quad \partial_t u_j = \Delta u_j + \nabla \cdot \Big(\sum_{h,k=1}^{m} c_{j,h,k}\, u_h (\nabla E_d * u_k) \Big), \quad j = 1, \ldots, m,
$$

$$
u(0)(x) = u_0(x).
$$

Here the unknown is the vector field $u = (u_1, \ldots, u_m)$, defined on the whole space $\mathbb{R}^d$ (with $m \geq 1$ and $d \geq 2$), and $c_{j,h,k} \in L^\infty(\mathbb{R}^d)$, $j, h, k = 1, \ldots, m$, are given coefficients. Moreover, E_d denotes the fundamental solution of the Laplacian in $\mathbb{R}^d$.

Systems of the form (1) arise e.g. from plasma, semiconductors and electrolytes theories, biology (modelling of chemotaxis phenomena) and statistical mechanics. The basic example for us is the model for gravitating particles: in this case $m = 1$, $d \geq 2$, and the governing equations are usually written as

$$
(2) \quad \partial_t u = \Delta u + \nabla \cdot (u \nabla \varphi), \quad \Delta \varphi = u.
$$

Here $u = u(x, t)$ is the density of the particles and φ is the self-consistent gravitational potential generated by u. Related systems also appear in the theory of chemotaxis (see e.g. [8], [7], [3]). In this case the coefficients $c_{j,h,k}$ are constant and equal to 1. We do not require that $u \geq 0$ in our study, which is, however, relevant in physical applications; we even admit complex-valued

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solutions. Another important example is provided by the Debye system, in which the first equation of (2) is replaced with

$$(3) \qquad \partial_t u = \Delta u - \nabla \cdot (u \nabla \varphi).$$

A more general model, still belonging to the class (1), is the drift-diffusion system

$$(4) \qquad \begin{aligned} \partial_t v &= \Delta v - \nabla \cdot (v \nabla \phi), \\ \partial_t w &= \Delta w + \nabla \cdot (w \nabla \phi), \\ \Delta \phi &= v - w. \end{aligned}$$

In the theory by W. Nernst and M. Planck, v and w represent the density of positively and negatively charged particles, respectively.

A lot is known about the existence and nonexistence of real-valued solutions of these models; see e.g. [6], [4], [2], [1], and the references therein. For instance, if $d = 1$, then the models considered have global in time solutions. If $d \geq 2$, the Debye system (3) and the more general system (4) have global in time solutions and their asymptotics is described by suitable self-similar solutions (see [5] and [11]). These may be interpreted as a complete diffusion of charges to infinity due to repulsive interactions.

On the other hand, models describing either chemotaxis or gravitational interaction in $d \geq 2$ dimensions feature concentration phenomena which may eventually lead to a collapse of solutions. These phenomena manifest themselves by the formation of singularities of solutions like weak convergence either to Dirac point masses or to unbounded functions $\sim |x|^{-2}$.

One purpose in this paper is to show that a different kind of finite time blow up can occur for solutions of (2) (and for a few other particular cases of (1)). In particular, we will show that also *nonpositive* (in fact: *complex-valued*) *and oscillating* solutions can blow up. Our second purpose is to give a global existence result for "small" solutions of (1). Such a result will provide us with some decay profiles of solutions in space-time.

The global existence result for "well localized" solutions can be stated as follows (see Section 2 for a more general, and more precise, statement).

THEOREM 1.1. *Let $d \geq 3$. There exists $\eta > 0$ such that if*

$$(5) \qquad |u_0(x)| \leq \frac{\eta}{(1 + |x|)^2},$$

then there exists $C \geq 0$ and a unique solution u of (1) such that, for all $x \in \mathbb{R}^d$ and $t \geq 0$,

$$(6) \qquad |u(x,t)| \leq \frac{C}{(1 + |x|)^2} \quad and \quad |u(x,t)| \leq \frac{C}{1 + t}.$$

Moreover, if all the coefficients $c_{j,h,k}(x)$ are constant in $\mathbb{R}^d$ or homogeneous of degree zero, then the smallness assumption (5) can be replaced by the

weaker, scale invariant, condition

$$(7) \qquad \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |x|^2 |u_0(x)| \leq \eta.$$

It would be possible to establish similar decay profiles for the solution in space-time, with a spatial decay rate *larger* than two. In this case the decay rate as $t \to \infty$ is also increased, up to half the decay rate as $x \to \infty$ (or up to the rate $d/2$ if the space decay rate is larger than d). The exponent two is however the most interesting case, since it plays a special role in these models, for scaling reasons. For example, it corresponds to the expected decay rate of self-similar solutions (see e.g. [4]). It is also the decay of the well-known Chandrasekhar solution, $\widetilde{u}(x,t) = 2(d-2)|x|^{-2}$, which is a stationary solution of (2) for $d \geq 3$.

We now state our result on the blow up of solutions.

THEOREM 1.2. *There exists $u_0 \in \mathcal{S}_0(\mathbb{R}^d)$ (the space of functions belonging to the Schwartz class, with vanishing moments of all orders) such that the corresponding solution u of (2) blows up in finite time: there exists $t^* > 0$ such that $u(t^*) \notin \dot{B}_p^{s,q}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.*

We will restate this theorem in a more precise way in Section 3. Therein, we will also recall the definition of the Besov norm $\| \cdot \|_{\dot{B}_p^{s,q}(\mathbb{R}^d)}$. Here we only observe that this theorem tells us, in particular, that $\|u(t)\|_{L^p}$ blows up for all $1 \leq p \leq \infty$. The proof of Theorem 1.2 consists in proving suitable lower bound estimates for the Fourier transform $\widehat{u}$. We shall derive such estimates using an idea of Montgomery-Smith [10].

Our exploding solution u of Theorem 1.2 is in fact complex-valued since its Fourier transform enjoys some *positivity and nonsymmetry* properties. Of course, one can rewrite the scalar equation (2) for the real and imaginary part of u. This yields a blow up result for a real system of the form (1), which is formally close to (4).

Notations. In chains of inequalities, all the constants will be denoted by C even if they vary from line to line. We will simply write $\int$ instead of $\int_{\mathbb{R}^d}$.

2. Global existence for the general model. The proof of Theorem 1.1 relies on size estimates of the kernel ∇E_d. We have of course

$$|\nabla E_d(x)| \leq \frac{C}{|x|^{d-1}}.$$

Our method also applies if we replace ∇E_d with any kernel K such that K is a measurable function in $\mathbb{R}^d$, and

$$(8) \qquad |K(x)| \leq C|x|^{-d-1+\alpha}, \quad 1 < \alpha < d.$$

In what follows, we will consider this more general situation. For the applications that we have in mind, $\alpha = 2$, and this explains the restriction $d \geq 3$ in Theorem 1.1. The two-dimensional case is often a special case in these models, e.g. the Keller–Segel parabolic-elliptic model of chemotaxis.

Since we would obtain the same bounds for all the components of u, in the remaining part of this section we can assume that u is scalar and $c_{j,h,k} = c$. Moreover, without loss of generality, in the proof below we can assume that c is constant, essentially because multiplying K with an L^∞ function does not affect (8).

With this simplification, the models discussed above can be written in the following integral form:

$$(9) \qquad u(t) = e^{t\Delta} u_0 + \int\limits_0^t G(t-s) * (u(K*u))(s)\, ds,$$

where G behaves like a first order derivative of the Gaussian heat kernel, namely

$$(10) \qquad G(x,t) = t^{-(d+1)/2} \Psi(x/\sqrt{t}) \quad \text{with } \Psi \in \mathcal{S}(\mathbb{R}^d),$$

$$(11) \qquad e^{t\Delta/2} G(t/2 - s) = G(t-s).$$

To state our result in a precise way, we introduce a few useful spaces. For $\theta \geq 0$ we denote by L_θ^∞ the space of measurable functions f on $\mathbb{R}^d$ such that $(1 + |\cdot|)^\theta f \in L^\infty(\mathbb{R}^d)$. Let $\mathcal{E}_\theta$ be the space of all measurable functions $f = f(x,t)$ in $\mathbb{R}^d \times \mathbb{R}^+$ such that

$$(12) \qquad \operatorname*{ess\,sup}_{x \in \mathbb{R}^d,\, t \geq 0} (1 + |x|)^\theta |f(x,t)| < \infty,$$

$$(13) \qquad \operatorname*{ess\,sup}_{x \in \mathbb{R}^d,\, t \geq 0} (1 + t)^{\theta/2} |f(x,t)| < \infty,$$

and

$$(14) \qquad f \in \mathcal{C}((0,\infty), L_\theta^\infty).$$

The space $\mathcal{E}_\theta$ is equipped with its natural norm.

THEOREM 2.1. *Let $1 < \alpha < d$ and K be such that (8) holds. Assume that $u_0 \in L_\alpha^\infty$. Then we can find $\eta > 0$ such that if*

$$(15) \qquad \|u_0\|_{L_\alpha^\infty} < \eta,$$

then there exists a unique mild solution u of (9), such that $u \in \mathcal{E}_\alpha$ and $u(t) \xrightarrow{\mathcal{D}'} u_0$ as $t \to 0$. Moreover, if we assume, in addition, that K is homogeneous of degree $-d - 1 + \alpha$, then (15) can be replaced by the weaker, scale invariant condition

$$(16) \qquad \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |x|^\alpha |u_0(x)| < \eta.$$

Proof. The proof relies on the following simple lemma:

LEMMA 2.1. *Let K satisfy the assumption (8).*

1. *If $f \in L_\alpha^\infty$, then $K * f \in L_1^\infty$.*
2. *If $\phi \in \mathcal{E}_\alpha$, then $K * \phi \in \mathcal{E}_1$.*

Proof. Using the duality and the interpolation of Lorentz spaces, we get
$$\|K * f\|_{L^\infty} \leq \|K\|_{L^{d/(d+1-\alpha),\infty}} \|f\|_{L^{d/(\alpha-1),1}} \leq C\|f\|_{L^{d/\alpha,\infty}}^{1-1/\alpha} \|f\|_{L^\infty}^{1/\alpha}.$$
Thus,
$$\|K * f\|_{L^\infty} \leq C\|f\|_{L_\alpha^\infty}.$$
In particular, we may assume $|x| \geq 1$. Note that
$$|K * f(x)| \leq C \int |x-y|^{-d-1+\alpha} |f(y)|\, dy = I_1 + I_2 + I_3,$$
where $I_1 \equiv \int_{|y| \leq |x|/2} \ldots$, $I_2 \equiv \int_{|x|/2 \leq |y| \leq 3|x|/2} \ldots$, and $I_3 \equiv \int_{|y| \geq 3|x|/2} \ldots$. One easily checks that these three integrals are bounded by $C|x|^{-1}$. The first part of the lemma follows.

On the other hand, by the above inequality,
$$\|K * \phi(t)\|_{L^\infty} \leq C\|\phi\|_{L^\infty(\mathbb{R}^+, L^{d/\alpha,\infty})}^{1-1/\alpha} \|\phi(t)\|_{L^\infty}^{1/\alpha} \leq C(1+t)^{-1/2} \|\phi\|_{E_\alpha}.$$
Combining this with the first part of the lemma, applied to $\phi(t)$, yields the result. ∎

For $u \in \mathcal{E}_\theta$, the nonlinear term $u(K * u)$ belongs, by Lemma 2.1, to $\mathcal{E}_{\theta+1}$. Then it is natural to study the behavior of the linear operator

(17)
$$L(w)(t) = \int_0^t G(t-s) * w(s)\, ds$$

in such a space.

LEMMA 2.2. *Let $1 < \alpha < d$ and $w \in \mathcal{E}_{\alpha+1}$. Then $L(w) \in \mathcal{E}_\alpha$.*

Proof. We will use repeatedly the property
$$\|G(t-s)\|_{L^1} = C(t-s)^{-1/2},$$
which is a consequence of (10). A few estimates below bear some relations to those of Miyakawa [9], yielding space-time decay results for the Navier–Stokes equations. We start by observing that $L(w) \in L^\infty(\mathbb{R}^d \times \mathbb{R})$. Indeed,

$$\|L(w)(t)\|_{L^\infty} \leq \int_0^t \|G(t-s)\|_{L^1} \|w(s)\|_{L^\infty}\, ds$$

$$\leq C\|w\|_{\mathcal{E}_{\alpha+1}} \int_0^t (t-s)^{-1/2} s^{1/2}\, ds \leq C\|w\|_{\mathcal{E}_{\alpha+1}}.$$

Then we can assume in the following that $|x| \geq 1$ and $t \geq 1$.

We can write $L(w) = I_1 + I_2$, where

$$I_1 \equiv \int\limits_0^t \int\limits_{|y| \leq |x|/2} G(x-y)w(y,s)\, dy\, ds,$$

$$I_2 \equiv \int\limits_0^t \int\limits_{|y| \geq |x|/2} G(x-y)w(y,s)\, dy\, ds.$$

Now,

$$|I_1(x,t)| \leq |x|^{-d} \int\limits_0^t \int\limits_{|y| \leq |x|/2} (t-s)^{-1/2}(1+|y|)^{-\alpha}(1+s)^{-1/2}\, dy\, ds$$

$$\leq C|x|^{-d} \int\limits_{|y| \leq |x|/2} |y|^{-\alpha}\, dy \leq C|x|^{-\alpha}.$$

On the other hand,

$$|I_2(x,t)| \leq C|x|^{-\alpha} \int\limits_0^t \|G(t-s)\|_{L^1}\, s^{-1/2}\, ds \leq C|x|^{-\alpha}.$$

Thus, $|L(w)(x,t)| \leq C(1+|x|)^{-\alpha}\|w\|_{\mathcal{E}_{\alpha+1}}$ and, in particular,

$$\|L(w)\|_{L^\infty((0,\infty),L^{d/\alpha,\infty})} \leq C\|w\|_{\mathcal{E}_{\alpha+1}}.$$

To obtain a decay estimate as $t \to \infty$, we recall (11) and write

$$L(w)(t) = e^{t\Delta/2}L(w)(t/2) + \int\limits_{t/2}^t G(t-s) * w\, ds \equiv J_1 + J_2.$$

By duality (we denote by g_t the Gaussian kernel),

$$\|J_1\|_{L^\infty} \leq C\|g_{t/2}\|_{L^{d/(d-\alpha),1}}\|L(w)(t/2)\|_{L^{d/\alpha,\infty}} \leq Ct^{-\alpha/2}\|w\|_{\mathcal{E}_{\alpha+1}}.$$

Moreover,

$$\|J_2(t)\| \leq Ct^{-(\alpha+1)/2}\|w\|_{\mathcal{E}_{\alpha+1}} \int\limits_{t/2}^t \|G(t-s)\|_{L^1}\, ds \leq Ct^{-\alpha/2}\|w\|_{\mathcal{E}_{\alpha+1}}.$$

The decay estimates in space-time for $L(w)$ then follow. The continuity with respect to t being straightforward, the proof of Lemma 2.2 is finished. ∎

By Lemma 2.2, the bilinear operator

$$(18) \qquad B(u,v) = \int\limits_0^t G(t-s) * (u(K * v))(s)\, ds$$

is continuous from $\mathcal{E}_\alpha \times \mathcal{E}_\alpha$ to $\mathcal{E}_\alpha$. Note that our last lemma also implies that $\|u(t) - e^{t\Delta}u_0\|_{L^\infty} \leq C\sqrt{t}$, so that $u(t) \to u_0$ a.e. and in the distributional

sense. The existence (and uniqueness) of the solution of (9) under the assumption (15) now follows by a standard argument, i.e. the application of the contraction mapping theorem.

In order to finish the proof of Theorem 2.1 it only remains to show that the smallness assumption (15) can be relaxed when the kernel K is a homogeneous function. Consider the rescaling

$$u_\lambda(x, t) = \lambda^\alpha u(\lambda x, \lambda^2 t). \tag{19}$$

A direct computation shows that, if K is homogeneous of degree $-d-1+\alpha$, and u is a solution of (9), then u_λ is a solution of (9) as well. Now let $\eta > 0$ be the constant obtained in the first part of Theorem 2.1. Assume that the datum u_0 is such that (16) holds. Then we can choose a $\widetilde\lambda > 0$ such that

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \widetilde\lambda^\alpha (1 + |x|)^\alpha |u_0(\widetilde\lambda x)| < \eta.$$

We can apply the first part of Theorem 2.1 to the initial datum $\widetilde\lambda^\alpha u_0(\widetilde\lambda \,\cdot)$. If we denote by $\widetilde u$ the corresponding solution, we see that $\widetilde u_{\widetilde\lambda^{-1}}$ is the solution of (9) starting from u_0. Theorem 2.1 is now established. ∎

REMARK 2.1. With minor modifications of the decay exponents in the above proof, one sees that, for any finite $T > 0$, the bilinear operator (18) is bicontinuous in the space $\mathcal{C}([0, T], L_\theta^\infty)$, for all $\theta \geq 0$. The contraction mapping theorem guarantees that, if $u_0 \in L_\theta^\infty$, $\theta \geq 0$ (with arbitrary norm) and $T > 0$ is small enough, then there exists a unique solution $u \in \mathcal{C}((0, T], L_\theta^\infty)$ such that $u(t) \to u_0$ in the weak sense; we will write $u \in \mathcal{C}_{\mathrm{w}}([0, T], L_\theta^\infty)$ to express these properties.

3. Blow up for the model of gravitating particles.

In this section we show that there exist solutions of (2), with initial data u_0 in the Schwartz class and such that $\int x^\alpha u_0(x) \, dx = 0$ for all $\alpha \in \mathbb{N}^d$, which blow up in finite time. Here we adopt a quite general definition of solution: we ask that the Fourier transform $\widehat u(\cdot, t)$, also denoted $\widehat u_t$, satisfies for a.e. $\xi \in \mathbb{R}^d$ and all $t \in [0, T]$, $0 < T \leq \infty$, the integral equation

$$\widehat u_t(\xi) = e^{(s-t)|\xi|^2} \widehat u_0(\xi) + \frac{1}{(2\pi)^d} \int_0^t e^{(s-t)|\xi|^2} i\xi \cdot \left(\widehat u_s(\xi) * \frac{i\xi}{-|\xi|^2} \, \widehat u_s(\xi) \right) ds. \tag{20}$$

The definition of the Fourier transform for integrable functions that we adopt is $\widehat u_t(\xi) = \int u(x, t) e^{-i\xi \cdot x} \, dx$.

There are several ways to give a sense to the above integral and ensure the validity of (20). An obvious way is to consider the (local) solutions obtained in the setting of Remark 2.1, with $\theta > d$. But the above equality is true in more general settings. For example, it holds for the solutions $u \in \mathcal{C}_{\mathrm{w}}([0, T], \mathcal{PM}^{d-2})$ (with $0 < T \leq \infty$ and $d \geq 3$) constructed in [4], where

$\mathcal{PM}^a$ is the space of pseudomeasures

$$\mathcal{PM}^a = \{v \in \mathcal{S}'(\mathbb{R}^d) \colon \widehat{v} \in L^1_{\text{loc}}(\mathbb{R}^d), \ \|v\|_{\mathcal{PM}^a} \equiv \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^d} |\xi|^a |\widehat{v}(\xi)| < \infty\}.$$

As pointed out in [4], a distributional solution of the Cauchy problem for (2) also satisfies (20).

In this section we will consider initial data with nonnegative Fourier transform, and supported in $(\mathbb{R}_+)^d$. Under this condition, one immediately checks that the iteration scheme yielding a solution in $\mathcal{C}_{\mathrm{w}}([0,T], \mathcal{PM}^{d-2})$, or in $\mathcal{C}_{\mathrm{w}}([0,T], L^\infty_\theta)$, converges in the subset of functions u such that $\widehat{u}(\xi,t) \geq 0$ for all $t \in [0,T]$ and a.e. $\xi \in \mathbb{R}^d$. The crucial fact that will lead to the blow up is the following:

LEMMA 3.1. *Let*

$$H_j(\widehat{u})(\xi,t) \equiv \int\limits_0^t \int e^{(s-t)|\xi|^2} \xi_j \, \frac{\eta_j}{|\eta|^2} \, \widehat{u}_s(\xi-\eta)\widehat{u}_s(\eta)\, d\eta\, ds, \quad j = 1, \dots, d.$$

Then, if $0 \leq \widehat{u} \leq \widehat{v}$ and $\operatorname{supp}\widehat{u}$ and $\operatorname{supp}\widehat{v}$ are contained in $\{\xi \in \mathbb{R}^d \colon \xi_l \geq 0, l = 1, \dots, d\}$, then $0 \leq H_j(\widehat{u}) \leq H_j(\widehat{v})$ and their supports are still contained in $\{\xi \in \mathbb{R}^d \colon \xi_l \geq 0, \ l = 1, \dots, d\}$.

This simple observation allows us to adapt to our situation the argument introduced by Montgomery-Smith for the "cheap" Navier–Stokes equations (see [10]).

Let us first recall the definition of the Besov norm $\|\cdot\|_{\dot{B}^{a,\infty}_\infty}$, $a \in \mathbb{R}$. We consider a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\widehat{\psi} \geq 0$ in $\mathbb{R}^d$, $\widehat{\psi}(\xi) \geq 1$ for $1/2 \leq |\xi| \leq 1$, and $\widehat{\psi}(\xi) = 0$ for $|\xi| \leq 1/3$ or $|\xi| \geq 4/3$. Then, for a distribution f, we define

$$(21) \qquad\qquad \|f\|_{\dot{B}^{a,\infty}_\infty} = \sup_{k \in \mathbb{Z}} 2^{(a+d)k} \|\psi(2^k \cdot) * f\|_{L^\infty}.$$

REMARK 3.1. It is well known that any Besov space $\dot{B}^{s,q}_p(\mathbb{R}^d)$, as well as any Triebel–Lizorkin space $\dot{F}^{s,q}_p(\mathbb{R}^d)$ (so in particular the L^p-spaces, which are identified to $\dot{F}^{0,2}_p$), with $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, is embedded in $\dot{B}^{s-d/p,\infty}_\infty(\mathbb{R}^d)$. It is then sufficient to show that $u(t)$ blows up in the $\dot{B}^{a,\infty}_\infty$ norm, for all $a \in \mathbb{R}$, to deduce that all Besov and Triebel–Lizorkin norms of u must blow up. To be more precise, L^1 is not a Triebel–Lizorkin space, but we will see that $\widehat{u}_{t^*}$ becomes unbounded for a finite t^*, hence $\|u(t)\|_{L^1}$ does blow up.

A similar remark applies to pseudomeasure norms, since $\mathcal{PM}^a$ is continuously embedded in $\dot{B}^{a-d,\infty}_\infty(\mathbb{R}^d)$.

THEOREM 3.1. *Let $w_0 \in \mathcal{S}(\mathbb{R}^d)$ be such that $\widehat{w}_0$ is nonnegative and supported in the ball $B_{1/4}(3e_1/4)$, where e_1 is the unit vector, and $\|\widehat{w}_0\|_{L^1} = 1$. Let $A > 2^{19/3}(2\pi)^d$ and $u_0 = Aw_0$ (so in particular $u_0 \in \mathcal{S}_0(\mathbb{R}^d)$). Assume*

also that $u(\cdot, t)$ is a tempered distribution such that for all $t \geq 0$, $\widehat{u}_t \geq 0$ and (20) holds for a.e. $\xi \in \mathbb{R}^d$. Then, for all $a \in \mathbb{R}$,

$$(22) \qquad \|u(\cdot, t^*)\|_{\dot{B}_\infty^{a,\infty}} = \infty, \quad where \quad t^* = \log(2^{1/3}).$$

Proof. Set $t_0 = 0$, $t_k = \log 2 (\sum_{j=1}^k 2^{-2j})$ and $w_k = w_0^{2^k}$. Set also

$$\alpha_k(t) = 2^{2k+6-6\cdot 2^k} e^{-2^k t} \mathbf{1}_{t \geq t_k} \qquad (k \in \mathbb{N}).$$

We claim that, for $k = 0, 1, \ldots,$

$$(23) \qquad \widehat{u}_t(\xi) \geq A^{2^k} \alpha_k(t) \widehat{w}_k(\xi).$$

This is seen by induction. For $k = 0$ the claim follows from Lemma 3.1:

$$\widehat{u}_t(\xi) \geq A e^{-t|\xi|^2} \widehat{w}_0(\xi) \geq A e^{-t} \widehat{w}_0(\xi), \qquad t \geq 0.$$

Now assume that (23) holds for $k - 1$. Set

$$E_k = \{\xi \in \mathbb{R}^d : 2^{k-1} \leq \xi_1 \leq |\xi| \leq 2^k\}, \qquad k = 0, 1, \ldots,$$

Note that $\widehat{w}_k = (2\pi)^{-d} \widehat{w}_{k-1} * \widehat{w}_{k-1}$. Thus, $\operatorname{supp} \widehat{w}_k \subset E_k$.

But, for a.e. $\xi \in E_k$, estimating from below by zero all the terms on the right hand side of (20), except for the first term obtained after computing the scalar product, we get

$$\widehat{u}_t(\xi) \geq \frac{1}{(2\pi)^d} \int_0^t \int e^{(s-t)|\xi|^2} \frac{\xi_1 \eta_1}{|\eta|^2} \widehat{u}_s(\xi - \eta) \widehat{u}_s(\eta) \, d\eta \, ds$$

$$\geq \int_{t_{k-1}}^t \int_{\eta \in E_{k-1}} e^{(s-t)|\xi|^2} \frac{\xi_1 \eta_1}{|\eta|^2} (A^{2^{k-1}} \alpha_{k-1}(s))^2 \frac{\widehat{w}_{k-1}(\xi - \eta) \widehat{w}_{k-1}(\eta)}{(2\pi)^d} \, d\eta \, ds$$

$$\geq A^{2^k} 2^{4k+7-6\cdot 2^k} e^{-2^k t} \Big(\int_{t_{k-1}}^t e^{(s-t)2^{2k}} \, ds \Big) \widehat{w}_k(\xi).$$

In the second inequality we used our induction assumption. Now, for all $t \geq t_k$, we have $t - t_{k-1} \geq 2^{-2k} \log 2$, so that $1 - e^{(t_{k-1}-t)2^{2k}} \geq 1/2$. This in turn implies $\int_{t_{k-1}}^t e^{(s-t)2^{2k}} \, ds \geq 2^{-2k-1}$. Then, for all $t \geq t_k$, we get

$$\widehat{u}_t(\xi) \geq A^{2^k} 2^{2k+6-6\cdot 2^k} e^{-2^k t},$$

and (23) follows.

Moreover, $\|\widehat{w}_k\|_{L^1} = (2\pi)^{-d} \|\widehat{w}_{k-1}\|_{L^1}^2$. Since $\|\widehat{w}_0\|_{L^1} = 1$, by induction we get

$$\|\widehat{w}_k\|_{L^1} = (2\pi)^{-d(2^k-1)}.$$

Set $t^* = \lim_{k \to \infty} t_k = \log(2^{1/3})$. We have $\widehat{\psi}(2^{-k} \cdot) \geq 1$ in E_k. Hence,

$$\widehat{\psi}(2^{-k} \xi) \widehat{u}_{t^*}(\xi) \geq A^{2^k} \alpha_k(t^*) \widehat{w}_k(\xi) \geq 0.$$

Then

$$\|u_{t^*}\|_{\dot{B}_\infty^{a,\infty}} \geq \sup_{k\in\mathbb{N}} 2^{(a+d)k}|\psi(2^k\cdot)*u_{t^*}(0)| = (2\pi)^{-d}\sup_{k\in\mathbb{N}} 2^{ak}\|\widehat{\psi}(2^{-k}\cdot)\widehat{u}_{t^*}\|_{L^1}$$

$$\geq \sup_{k\in\mathbb{N}} A^{2^k}(2\pi)^{-d\,2^k}2^{ak+2k+6-6\cdot 2^k}e^{-2^k t^*}$$

$$= \sup_{k\in\mathbb{N}}\{(A\,2^{-19/3}(2\pi)^{-d})^{2^k}2^{(a+2)k+6}\} = \infty.\ \blacksquare$$

REMARK 3.2. The same proof goes through for solutions of (1), under the additional conditions $\widehat{c}_{j,h,k} \geq 0$ for all $j,h,k = 1,\ldots,m$, and, say, $\widehat{c}_{1,h,k} \geq c > 0$. In this case, one can obtain a solution $u = (u_1,\ldots,u_m)$ such that the first component blows up, starting from a datum u_0 such that $\widehat{u}_0 \geq 0$ and with the first component satisfying the conditions of Theorem 3.1.

REMARK 3.3. Analogous results can be obtained for space-periodic solutions of (1), i.e. those defined on the d-dimensional torus. Instead of the Fourier transform, we then consider the Fourier coefficients $\widehat{u}(\xi)$, $\xi \in \mathbb{Z}^d$.

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