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ASYMPTOTIC BEHAVIOR OF THE ENERGY AND POINTWISE ESTIMATES FOR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

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1. Introduction

In this paper we deal with the asymptotic behavior, in the space-time variables, of weak and strong solutions to the Navier–Stokes system. For an incompressible viscous fluid which fills the whole space $\mathbb{R}^n$, in the absence of external forces, the Navier–Stokes equations read

$$\text{(NS)} \qquad \begin{cases} \partial_t u + \nabla \cdot (u \otimes u) = \Delta u - \nabla p \\ u(x,0) = a(x) \\ \operatorname{div}(u) = 0. \end{cases}$$

Here $u : \mathbb{R}^n \times [0,\infty[\to \mathbb{R}^n$ $(n \geq 2)$ denotes the velocity field and $p(x,t)$ is the pressure.

Starting with the pioneering work of Leray [21], a considerable number of papers is concerned with questions related to the large-time behavior of the L^2-norm of $u(t)$. The problem of finding optimal decay rates for the energy of generic weak solutions is now well understood.

Indeed, Wiegner [41] showed that $||u(t)||_2 \leq C(1+t)^{-\alpha}$ $(0 < \alpha \leq (n+2)/4)$, if such decay holds for the solution $e^{t\Delta}a$ of the heat equation starting with the same data. This improved previous results by Kato [17], Schonbek [29] and Kaijkiya-Miyakawa [19]. The bound on α is now known to be optimal: optimality was first discussed in [30] and, more recently, in [28], [13], [14] with different methods.

However, exceptional flows which decay much faster do exist. For example, it is known since a long time that, in dimension $n = 2$, there exists a very particular and explicit solution of the Navier–Stokes equations with radial vorticity. This condition on the vorticity implies that the nonlinearity has the potential form (*i.e.* $\nabla \cdot (u \otimes u) = -\nabla p$), so that u is also a solution of the homogeneous heat equation. It was pointed out by Majda and Schonbek that for such flow $||u(t)||_2$ has an exponential decay at infinity (see *e.g.* [30], [10], [28]). In dimension 2, no other examples with such a property seem to be known.

Similar flows with exponential decay exist in higher *even* dimension and a general method for their construction is described in [32]. All these solutions, sometimes called generalized Beltrami flows, turn out to solve simultaneously (NS) and the heat equation. As discussed in [32], it seems impossible to adapt these examples to the $n = 3$ case or for general *odd* dimensions.

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Beside generalized Beltrami flows, a few other exact solutions of the Navier–Stokes equations are known (see *e.g.* [12], [38] and the review [40]). But, at best of our knowledge, no examples of solutions in $\mathbb{R}^3$ with fast decay of the energy have been constructed so far.

The aim of this paper is to construct, in *any dimension* $n \geq 2$, a class of solutions such that the energy norm decays at infinity faster that $t^{-(n+2)/4}$. Such construction will be achieved by imposing some special symmetries on the initial data, which are preserved by the Navier–Stokes evolution.

As an application of our results in the $n = 3$ case, we will provide examples of initial data a such that the corresponding weak solutions u are non-trivial and satisfy $||u(t)||_2 = O(t^{-9/4})$. These examples answer a question raised by M. Schonbek in [30].

Another important feature of our construction is that it provides solutions such that the nonlinear term $\mathbb{P}\nabla \cdot (u \otimes u)$ *does not* vanish identically (here $\mathbb{P}$ is the Leray-Hopf projector onto the field of soleinoidal vectors). Thus, in general, our symmetric solutions are not merely generalized Beltrami flows.

We refer to [13], [14] for an interesting geometrical insight of such rapidly decaying solutions. We also would like to observe that some of these results can be adapted in the case of flows in the half-space (see [11]).

In this paper we also study the asymptotic behavior in the space-time variables of global strong solutions. Pointwise estimates on the decay of u, as $|x| + t \to \infty$, have been obtained by Takahashi [35] and then improved in [26], [1] and [16], using different methods.

Miyakawa [26] showed that the Navier–Stokes equations admit a unique strong solution u which behaves as $|u(x,t)| \sim |x|^{-\alpha} t^{-\beta/2}$, for all $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + \beta \leq n + 1$, if the initial data satisfies

$$\sup_{x,t}(1+|x|)^{n+1}|e^{t\Delta}a(x)| < \epsilon \qquad \text{and} \qquad \sup_{x,t}(1+t)^{(n+1)/2}|e^{t\Delta}a(x)| < \epsilon, \tag{1.1}$$

with $\epsilon > 0$ small enough. Examples of flows satisfying (1.1) are provided in [26] and [27], but this condition seems somewhat too stringent.

The bound $||u(t)||_\infty \leq C(1+t)^{-(n+1)/2}$ was obtained (among other things) also by Amrouche *et al.* [1] and He, Xin [16] under more natural assumptions on the initial data: while [1] deals with data belonging to L^2-Sobolev spaces, in [16] a is supposed to belong to wheighted-L^p spaces. However, it seems difficult to obtain Miyakawa's estimate $|u(x,t)| \leq C(1+|x|)^{-(n+1)}$ under this type of assumptions. Let us stress the fact that such space decay rate is optimal for generic flows (see [3], [4], [8]).

In this paper we will deduce the optimal decay rates of [26], under a smallness assumption slightly more general than (1.1). Furthermore we show that, inside the class of "symmetric flows" in $\mathbb{R}^n$ ($n \geq 2$), we can find strong solutions with an *over-critical* space-time decay. This last result was announced in [3]. We will present it in a slightly sharper form.

The rest of the paper is organized as follows. In sections 2 we deal with weak solutions: we start recalling a theorem by Miyakawa and Schonbek [28], which relates the long time behavior of the energy of u with the following non-linear integral identities:

$$\int_0^\infty \int u_h(x,t) u_k(x,t)\,dx\,dt = c\delta_{h,k} \qquad (h,k = 1,\dots,n), \tag{1.2}$$

with $\delta_{h,k} = 1$ if $h = k$, else $\delta_{h,k} = 0$.

Then we introduce the class of *symmetric vector field* in $\mathbb{R}^n$ and we show that, starting with initial data inside this class, allows us to obtain weak solutions which are symmetric for all $t > 0$. Thus, we will be able to prove that nontrivial flows satisfying (1.2) do exist (this was left open in [28], for $n = 3$). We finally show that the existence of solutions such that $||u(t)||_2 = o(t^{-(n+2)/4})$ at infinity ($n \geq 2$) is an immediate consequence of our construction and of the result of [28].

In section 3, we study in some more detail the asymptotic behavior of symmetric weak solutions ($n = 3, 4$). We shall obtain the bound $||u(t)||_2 \leq Ct^{-(n+6)/4}$, which seems to be optimal within the class of symmetric solutions. Our method relies on the Fourier transform, and more precisely on Schonbek's Fourier splitting device [29]. We also make use of the energy inequality and of some recent estimates (see [31], [16], [10]) on the decay at infinity of the first order moments of $|u(\cdot,t)|^2$.

In section 4 we deal with strong solutions. After treating the case of generic flows, for symmetric solutions we will obtain the improved decay rates

$$|u(x,t)| \sim |x|^{-\alpha} t^{-\beta/2} \qquad \alpha \geq 0, \quad \beta \geq 0, \qquad \alpha + \beta \leq n + 3 \tag{1.3}$$

as $|x| + t \to \infty$. A similar result (in the case $\alpha + \beta = n + 3$) has been recently proved also by Miyakawa in [27], using the ideas of [3], after the first version of this paper was completed. However, Miyakawa's assumptions on the initial data are slightly more stringent than ours.

We refer to [5] for a more systematic study of the space-time decay in dimension two and three and the computation of the spatial decay rates of solutions which are left invariant under the action of subgroups of the orthogonal group of $\mathbb{R}^2$ and $\mathbb{R}^3$

Notations. Throughout this paper we shall use the following usual notations. For any multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set

$$\gamma! = \gamma_1!\gamma_2!\cdots\gamma_n! \qquad |\gamma| = \gamma_1 + \cdots + \gamma_n$$

$$\partial_i = \frac{\partial}{\partial x_i} \quad (i = 1,\dots,n) \qquad \partial^\gamma = \frac{\partial^{|\gamma|}}{\partial_1^{\gamma_1}\cdots\partial_n^{\gamma_n}}$$

and

$$x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}. \tag{1.4}$$

Further, $\nabla = (\partial_1, \dots, \partial_n)$, Δ denotes the laplacian on $\mathbb{R}^n$ and

$$e^{t\Delta} a(x) = \int (4\pi t)^{-n/2} e^{-|x-y|^2/4t} a(y)\,dy$$

is the heat semigroup.

2. Symmetric weak solutions

We start recalling some known facts about the large time behavior of weak solutions (see *e.g.* [7], [19], [34] for their construction). By definition, a weak solution to (NS) is a function

$$u \in \mathrm{C}_w([0,\infty[, \mathrm{L}^2(\mathbb{R}^n)) \cap \mathrm{L}^2_{loc}(\mathbb{R}^+, \mathrm{H}^1(\mathbb{R}^n)), \tag{2.1}$$

such that $\mathrm{div}(u) = 0$, satisfying the integral equation

$$\begin{aligned} &\langle u(t), \phi(t)\rangle - \langle a, \phi(0)\rangle \\ &= \int_0^t \left\{ \langle u(s), \frac{\partial \phi}{\partial s}(s)\rangle - \langle \nabla u(s), \nabla\phi(s)\rangle - \langle (u(s)\cdot\nabla)u(s), \phi(s)\rangle \right\} ds \qquad (t>0) \end{aligned} \tag{2.2}$$

for all smooth, compactly supported and soleinoidal vector fields $\phi \in \mathrm{C}^\infty(\mathbb{R}^n \times [0,\infty[)$. Here $\langle\cdot,\cdot\rangle$ is the L^2-inner product and C_w is the space of weakly continuous functions.

For $2 \le n \le 4$, we shall assume that weak solutions satisfy the strong energy inequality:

$$||u(t)||_2^2 + 2\int_s^t ||\nabla u(r)||_2^2\, dr \le ||u(s)||_2^2, \tag{2.3}$$

for $s = 0$, almost $s > 0$ and all $t \ge s$. Unicity of weak solutions is still an open problem for $n \ge 3$.

Wiegner's results [41] states that $||u(t)||_2 \to 0$ at infinity, whenever (2.3) holds (see also [17], [22], [29], [19] for previous results). Moreover, if the solution of the heat equation satisfies

$$||e^{t\Delta}a||_2 \le c(1+t)^{-\alpha_0} \qquad (t \ge 0) \tag{2.4}$$

then

$$||u(t)||_2 \le C(1+t)^{-\bar\alpha}, \qquad \bar\alpha = \min\{\alpha_0, \tfrac{n+2}{4}\} \tag{2.5}$$

and $||u(t) - e^{t\Delta}a||_2 = o(t^{-\alpha_0})$ if $0 \le \alpha_0 < \frac{n+2}{4}$. Thus, (2.5) is optimal for such low decay rates. As pointed out in [41], (2.5) holds even if u does not satisfy (2.3) (and it may be the case, if $n \ge 5$), but can be suitably approximated by functions satisfying (2.3). Since this is true in any space dimension, Wiegner's result is meaningful also for $n \ge 5$.

A simple consequence of [41] is the following: if $a \in \mathrm{L}^2(\mathbb{R}^n)$ $(n \ge 2)$ satisfies

$$\int |a(x)|(1+|x|)\, dx < \infty, \tag{2.6}$$

then there exists a weak solution of (NS) such that $u(0) = a$ and $||u(t)||_2 \le C(1+t)^{-(n+2)/4}$. As it is easy to check, this relies on the fact that the diverge-free condition implies the cancellation $\int a(x)\, dx = 0$.

We now recall a characterization by Miyakawa and Schonbek of the exceptional solutions which decay faster than predicted by Wiegner.

Theorem 2.1 ([28]). *Let $a \in \mathrm{L}^2(\mathbb{R}^n)$ $(n \ge 2)$ a soleinoidal vector field satisfying (2.6) and let u be a weak solution to the Navier–Stokes equations such that*

$||u(t)||_2 \leq C(1+t)^{-(n+2)/4}$. *We set*

$$b_{h,k} = \int x_h a_k(x)\,dx \qquad \text{and} \qquad \Lambda_{h,k} = \int_0^\infty \int (u_h u_k)(x,t)\,dx\,dt \tag{2.7}$$

$(h,k = 1\ldots,n)$. *Then we have*

(1) *If* $(b_{h,k}) \equiv 0$ *and if there exists* $C \in \mathbb{R}$ *such that* $\Lambda_{h,k} = C\delta_{h,k}$, *then*

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} ||u(t)||_2 = 0. \tag{2.8}$$

(2) *Conversely, if* $(b_{h,k}) \not\equiv 0$ *or* $(\Lambda_{h,k})$ *is not scalar, then*

$$\liminf_{t\to\infty} t^{\frac{n+2}{4}} ||u(t)||_2 > 0. \tag{2.9}$$

As the authors themselves observe, nothing is known about the solutions satisfying $\Lambda_{h,k} = C\delta_{h,k}$. In particular, they provided as an example just the classical two-dimensional flow with radial vorticity: let us recall that such flow is defined in the phase space by $\widehat{u}(\xi,t) = (-\mathrm{i}\xi_2, \mathrm{i}\xi_1)|\xi|^{-2}e^{-t|\xi|^2}\widehat{\omega}_0(\xi)$, where $\widehat{\omega}_0$ is a smooth and compactly supported radial function, such that $\xi = 0$ does not belong to $\mathrm{supp}(\widehat{\omega}_0)$.

In this section we will show that part 1 of Theorem 2.1 is non-vacuous in any space dimension $n \geq 2$.

It seems natural to look for solutions with fast decay by studying the class of initial data which satisfy the following orthogonality relations:

$$\int a_h(x)a_k(x)\,dx = c\delta_{h,k} \qquad (h,k = 1,\ldots,n). \tag{2.10}$$

The main difficulty arises from the fact that (2.10), in general, instantaneously brakes down during the evolution. We refer to [4] for an example.

This leads us to consider the following sub-class of (2.10).

Definition 2.2. We say that a vector field $a = (a_1,\ldots,a_n) : \mathbb{R}^n \to \mathbb{R}^n$ is *symmetric* if the the following conditions are satisfied, for all $j,k = 1,\ldots,n$.

$$a_j \text{ is odd with respect to } x_j \text{ and even with respect to } x_k,\ j \neq k \tag{2.11}$$

$$a_1(x) = a_2(\sigma x) = \ldots = a_n(\sigma^{n-1}x), \tag{2.12}$$

where σ is the cycle $\sigma(x_1,\ldots,x_n) = (x_n,x_1,\ldots,x_{n-1})$.

In the case of two and three-dimensional periodic flows, similar symmetries have been considered by S. Kida [18], with completely different motivations. See also [5] for a geometric interpretation of (2.11)-(2.12) and more general class of symmetries.

A simple three dimensional example of a symmetric and soleinoidal vector field is given by

$$a(x_1,x_2,x_3) = \begin{pmatrix} x_1(x_3^2 - x_2^2)e^{-|x|^2} \\ x_2(x_1^2 - x_3^2)e^{-|x|^2} \\ x_3(x_2^2 - x_1^2)e^{-|x|^2} \end{pmatrix}. \tag{2.13}$$

This example generalizes in an obvious manner to any dimension $n \geq 3$:

$$a_h(x_1,\ldots,x_n) = x_h(x_{h-1}^2 - x_{h+1}^2)e^{-|x|^2}, \qquad h = 1,\ldots,n \qquad (n \geq 3). \tag{2.14}$$

Here we posed $x_0 = x_n$ and $x_{n+1} = x_1$.

A simple two-dimensional example is:

$$a(x_1,x_2) = \begin{pmatrix} (x_1^3 - 3x_1x_2^2)e^{-|x|^2} - 2(x_1^3x_2^2 - x_1x_2^4)e^{-|x|^2} \\ (x_2^3 - 3x_1^2x_2)e^{-|x|^2} - 2(x_1^2x_2^3 - x_1^4x_2)e^{|x|^2} \end{pmatrix}. \tag{2.15}$$

These examples were first considered in [4], in order to show that there is no *instantaneous spatial spreading* of the velocity field, in general, when (2.10) hold.

We now state a simple but important result concerning the existence of symmetric weak solutions to the Navier–Stokes equations.

Theorem 2.3. *Let $a \in \mathrm{L}^2(\mathbb{R}^n)$ $(n \geq 2)$ be a soleinoidal vector field. If a satisfies (2.11) (respectively, (2.12)) then there exists a weak solution $u(t)$ to (NS) which satisfies (2.11) (respectively, (2.12)) for all $t \geq 0$.*
In particular, for any $n \geq 2$ we can find $u(x,t) \not\equiv 0$ satisfying (2.8).

The proof is straightforward and we will only sketch it. We will just follow the retarded mollifier method of Caffarelli, Kohn and Nirenberg [7] (see also [19], for the general case $n \geq 2$) with slight modifications, in order to ensure that the symmetries are conserved at any step of their construction. We would obtain the same result by following other constructions of weak solutions, such as that of [34] or [16].

Proof. We start by stating simple algebraic properties of symmetric vector fields in the Hilbert spaces $\mathrm{L}^2(\mathbb{R}^n)$ and $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ (the homogeneous Sobolev space). We denote by $S_{(i)}$ the class of functions $f : \mathbb{R}^n \to \mathbb{R}^n$ which satisfy (2.11), and by $S_{(ii)}$ the class of functions for which (2.12) holds.
Then we have:

Lemma 2.4. *Let S be $S_{(i)}$ or $S_{(ii)}$. All functions $f : \mathbb{R}^n \to \mathbb{R}^n$ can be decomposed into*

$$f = f_S + (f - f_S),$$

where $f_S \in S$, and

$$f_S \perp (f - f_S) \qquad \textit{in } \dot{\mathrm{H}}^1(\mathbb{R}^n), \quad \textit{if } f \in \dot{\mathrm{H}}^1(\mathbb{R}^n) \tag{2.16}$$

$$f_S \perp (f - f_S) \qquad \textit{in } \mathrm{L}^2(\mathbb{R}^n), \quad \textit{if } f \in \mathrm{L}^2(\mathbb{R}^n) \tag{2.17}$$

Proof. The proof in the case $S = S_{(i)}$ is trivial. Let us consider the case $S = S_{(ii)}$. If $n = 2$, the decomposition simply reads:

$$\begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_1(x_1,x_2) + f_2(x_2,x_1) \\ f_1(x_2,x_1) + f_2(x_1,x_2) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} f_1(x_1,x_2) - f_2(x_2,x_1) \\ -f_1(x_2,x_1) + f_2(x_1,x_2) \end{pmatrix}.$$

In the general case the decomposition can be performed by choosing:

$$f_S(x) = \frac{1}{n}\begin{pmatrix} f_1(x) + f_2(\sigma x) + \cdots + \cdots + f_n(\sigma^{n-1}x) \\ f_1(\sigma x) + f_2(\sigma^2 x) + \cdots + f_n(x) \\ \cdots \\ f_1(\sigma^{n-1}x) + f_2(x) + \cdots + f_n(\sigma^{n-2}x) \end{pmatrix}. \tag{2.18}$$

The orthogonality of f_S and $f - f_S$, both in $\mathrm{L}^2(\mathbb{R}^n)$ and in $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ is simple to check. $\square$

In the next lemma we show the symmetries (2.11)-(2.12) are invariant under the linearized Navier–Stokes evolution. We shall adopt the classical notations of [36]: we thus denote by V the closure in $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ of the set $\mathcal{V}$ of all smooth, compactly supported and divergence-free vector fields. The closure of $\mathcal{V}$ in $\mathrm{L}^2(\mathbb{R}^n)$ will be denoted by H.

Lemma 2.5. *Let $T > 0$ and $a \in \mathrm{L}^2(\mathbb{R}^n)$ a soleinoidal vector field. Let $W = (W_1, \dots, W_n)$ a bounded and divergence-free vector field in $\mathrm{C}^\infty(\mathbb{R}^n \times]0,T[)$. Then we know that there exists a unique function u and a distribution p such that*

$$u \in \mathrm{C}([0,T], H) \cap \mathrm{L}^2(]0,T[, V) \tag{2.19}$$

$$\partial_t u + (W \cdot \nabla)u - \Delta u = -\nabla p \tag{2.20}$$

in the distributional sense and such that $u(0) = a$. If both a and $W(t)$ belong to S ($S = S_{(i)}$ or $S_{(ii)}$) for all $t \in]0,T[$, than this property holds true also for $u(t)$.

In the setting of Lemma 2.5, (2.20) means that

$$\frac{d}{dt}(u,v) + (W \cdot \nabla u, v) + ((u,v)) = 0$$

for all $v \in V$, where $(u,v) = \int u(x) \cdot v(x)\,dx$ and $((u,v)) = \sum_{j=1}^n (\partial_j u, \partial_j v)$.

Proof. The solution u is obtained by the Faedo-Galerkin scheme. We refer to [7] for a proof. Let $V_S = V \cap S$ and $\tilde{V}_S$ the complement of V_S in V. Let us denote by H_S and $\tilde{H}_S$, respectively, the closure of V_S and $\tilde{V}_S$ in H. Thus,

$$V = V_S \oplus \tilde{V}_S \qquad \text{and} \qquad H = H_S \oplus \tilde{H}_S, \tag{2.21}$$

where the orthogonality in H follows from Lemma 2.4.

In constructing the Galerkin approximations of $u(t)$, we may choose an orthonormal basis of V such that all vectors belong to V_S or $\tilde{V}_S$. By our hypotheses, the operator $W \cdot \nabla$ is bounded from V_S to $\mathrm{L}^2(\mathbb{R}^n) \cap S$. Thus, by (2.21), when $a \in H_S$, the Galerkin approximations $(u_m(t))$ $(m = 1, 2 \dots)$ still belong to S for all $t \in]0,T[$. Passing to the limit for $m \to \infty$ yields $u(t) \in S$ for all t. □

We can now outline the proof of Theorem 2.3. As before, we just indicate the necessary modifications of the classical construction in order to get the symmetries. For more details on the classical construction we refer to [7] and [19].

For each integer $N \geq 1$ and $u \in \mathrm{L}^\infty(]0,T[, \mathrm{L}^2(\mathbb{R}^n))$ we consider the *retarded mollification* of u:

$$\Psi_N(u) = \frac{1}{\delta^{n+1}} \iint \psi(\tfrac{y}{\delta}, \tfrac{\tau}{\delta}) \tilde{u}(x-y, t-\tau)\,dy\,d\tau, \qquad \delta = \tfrac{1}{N}.$$

As in [7] and [19], ψ is a smooth, non-negative function, such that

$$\iint \psi(x,t)\,dx\,dt = 0$$

and $\operatorname{supp}\psi \subset \{(x,t) : |x|^2 < t,\ 1 < t < 2\}$. Moreover, $\tilde{u}(x,t) = u(x,t)$ if $t \geq 0$ and $\tilde{u}(x,t) = 0$ if $t < 0$.

We now define the *approximate solution* u_N of (NS) $(N = 1, 2 \ldots)$ as the solution of the system

$$(\mathrm{NS}_N) \qquad \begin{cases} \partial_t u_N + (\Psi_N(u_N)\cdot\nabla)u_N - \Delta u_N = -\nabla p_N \\ \mathrm{div}(u_N) = 0 \\ u_N(0) = a, \end{cases} \qquad (x,t) \in \mathbb{R}^n \times [0,T[.$$

Note that (NS_N) can be solved subsequentely on the intervals $[(k-1)\delta, k\delta]$ $(k = 1,\ldots,N)$. Thus, solving (NS_N), is equivalent to solve N linear equations of the form (2.20).

We now would like to show that u_N is symmetric. But this seems not to be the case under the previous assumptions on ψ. However, we can take $\psi(x,t)$ which, in addition, is *radial* in $\mathbb{R}^n$ for all t. Thus, a trivial computation shows that if $u(s) \in S$ for all $s \in]t-2\delta, t-\delta[$, then $\Psi(u)(t)$ also belons to S, for all t. By Lemma 2.5, we get $u_N(t) \in S$ for all $t \in]0,T[$.

Usual *a priori* estimates and embedding theorems imply the existence of a subsequence $u_{N'}$ of u_N which converges (*e.g.,* weakly in $\mathrm{L}^2(]0,T[,V))$ to a weak solution of the Navier–Stokes equations. Hence $u(t) \in S$ for almost all $t > 0$. But, since weak solutions are weakly continuous in $\mathrm{L}^2(\mathbb{R}^n)$, $u(t)$ is symmetric for all $t \geq 0$.

The second conclusion of Theorem 2.3 is now an immediate consequence of Theorem 2.1. Indeed, let us start with the symmetric initial data (2.14) or (2.15). Obviously, (2.6) holds and $\int x_h a_k(x)\,dx = 0$ $(h,k = 1,\ldots,n)$. On the other hand, the solution constructed in the first part of the proof of Theorem 2.3 satisfies $\int (u_h u_k)(x,t)\,dx = c(t)\delta_{h,k}$ for all $t \geq 0$ (because of the symmetries). All the assumptions of Theorem 2.1 are satisfied and (2.8) follows. □

Remark 2.6. Recall that, for a n-dimensional vector field a, the corresponding vorticity field is given by the $n\times n$ antisymmetric matrix $\Omega = \nabla a - (\nabla a)^*$. In the symmetric case, $\Omega_{h,k}(x)$ is an odd function with respect to x_h and x_k $(h,k = 1,\ldots,n)$. It follows that nontrivial symmetric flows have nonradial vorticity. Thus, symmetric solutions are not the natural generalization of the two-dimensional solution described in [30], [10], [28], but they are quite different flows.

Moreover, a tedious but elementary computation shows that if a is defined by (2.15) or (2.13), then $\nabla\cdot(a\otimes a)$ has not the potential form. Thus, the corresponding solution of (NS) cannot solve simultaneously the heat equation.

3. Decay estimates for symmetric solutions

In this section we obtain some explicit decay rates for the L^2-norm of symmetric solutions. More generally, se shall study the class of solutions such that

$$\int u_h(x,t)u_k(x,t)\,dx = c(t)\delta_{h,k} \qquad \text{for almost all } t > 0. \tag{3.1}$$

Let us observe that we are not able to characterize the initial data such that (3.1) holds true, at least for a suitable weak solution.

We prefer to deal with (3.1), and not with the slightly more general assumption

$$\int_0^\infty\!\!\int u_h(x,t)u_k(x,t)\,dx\,dt = C\delta_{h,k} \tag{3.2}$$

contained in Theorem 2.1, for the following reason: conditions (3.1) are invariant under the translations $\tau \mapsto u(t+\tau)$. This is obviously the case also for the condition

$||u(t)||_2 = o(t^{-(n+2)/4})$, but, on the other hand, (3.2) is not invariant. For a detailed discussion on this point, and the description of a method (very different from ours) to overcome this type of difficulties, we refer to [13], [14].

We now state our main result of this section:

Theorem 3.1. *(1) Let* $a \in \mathrm{L}^2(\mathbb{R}^n)$ $(n = 3, 4)$ *a divergence-free vector field such that* $\int |x|^2|a(x)|^2\,dx < \infty$ *and* $||e^{t\Delta}a||_2 \le C(1+t)^{-\alpha_0}$*, with* $\alpha_0 > \frac{n+2}{4}$*. Then there exists a weak solution* u *of (NS), such that* $u(0) = a$ *and* $\int |x|^2|u(x,t)|^2\,dx < \infty$ *for all* $t \ge 0$*. If*

$$\int u_h(x,t)u_k(x,t)\,dx = c(t)\delta_{h,k}, \qquad a.e.\ in\ ]0,\infty[, \tag{3.3}$$

then $||u(t) - e^{t\Delta}a||_2 = O(t^{-(n+4)/4})$ *as* $t \to \infty$*. In particular,*

$$||u(t)||_2 \le C(1+t)^{-\bar\alpha}, \qquad with \quad \bar\alpha = \min\{\alpha_0, \tfrac{n+4}{4}\}. \tag{3.4}$$

(2) Furthermore, if $\int |x|^3|a(x)|^2\,dx < \infty$*, then* $\int |x|^3|u(x,t)|^2\,dx < \infty$ *for all* $t \ge 0$*. In this case, if* $\alpha_0 > \frac{n+4}{4}$ *and if* u *also satisfies*

$$\int x_k u_k(x,t)u_k(x,t)\,dx = \int x_k u_h(x,t)u_h(x,t)\,dx \tag{3.5}$$

$$\int x_j u_h(x,t)u_k(x,t)\,dx = 0, \qquad h \ne k \tag{3.6}$$

for all $j, h, k = 1, \dots, n$ *and almost all* $t > 0$*, then* $||u(t) - e^{t\Delta}a||_2 = O(t^{-(n+6)/4})$ *as* $t \to \infty$*. Thus, (3.4) is improved by*

$$||u(t)||_2 \le C(1+t)^{-\bar\alpha}, \qquad with \quad \bar\alpha = \min\{\alpha_0, \tfrac{n+6}{4}\}. \tag{3.7}$$

A few comments are in order:

Remark 3.2. The result about the persistence of the conditions $|x|a \in \mathrm{L}^2(\mathbb{R}^n)$ and $|x|^{3/2}a \in \mathrm{L}^2(\mathbb{R}^n)$ is due to [31] and [16]. The simplest examples of data such that $||e^{t\Delta}a||_2 \le C(1+t)^{-\alpha_0}$ are obtained by taking $\widehat{a}(\xi) \in \mathrm{L}^2(\mathbb{R}^n)$, with a suitable vanishing condition for $\xi = 0$.

Let us also observe that, if $a \in \mathrm{L}^2(\mathbb{R}^n)$, then the assumption $||e^{t\Delta}a||_2 \le C(1+t)^{-\alpha_0}$ is equivalent to

$$a \in \mathrm{L}^2(\mathbb{R}^n) \cap \dot{\mathrm{B}}_2^{-2\alpha_0,\infty}(\mathbb{R}^n) \qquad (\alpha_0 > 0) \tag{3.8}$$

where $\dot{\mathrm{B}}_2^{-2\alpha_0,\infty}(\mathbb{R}^n)$ is the homogeneus Besov space. We refer to the appendix for a definition of Besov spaces and to [6] for a proof this characterization. Here α is a parameter which is essentially related to the oscillations of a.

Remark 3.3. We observe that if $u(t)$ is a symmetric vector field, then for all $(j,h,k) \in \{1,\dots,n\}^3$, $x \mapsto x_j u_h u_k(x,t)$ is an odd function with respect to at least one variable. Thus, all the integrals contained in (3.5)-(3.6) vanish and the second part of the theorem is non-vacuous. As we will see below, however, the vanishing of $\int x_k u_k^2(x,t)\,dx$ $(k = 1,\dots,n)$ would not be necessary.

Remark 3.4. We point out that this theorem implies also the existence of solutions u such that $||u(t)||_2 = o(t^{-(n+2)/4})$, but which *do not satisfy* the conditions of the theorem of Miyakawa and Schonbek (the same remark is done in [14]). Indeed, Theorem 2.1, relies on the assumption

$$\int |a(x)|(1+|x|)\,dx < \infty, \tag{3.9}$$

which is slightly more restrictive than $|x|a \in \mathrm{L}^2(\mathbb{R}^n)$, at least from the localization point of view.

An important difference between this last condition and (3.9) is that the former is conserved for all t (see below, or [16], [31]), while the second, in general, is not. More exactly, as shown in [3], (3.9) instantaneously brakes down, unless the orthogonality relations

$$\int a_h(x)a_k(x)\,dx = c\delta_{h,k} \qquad (h,k=1,\ldots,n) \tag{3.10}$$

hold.

Proof of Theorem 3.1. The finiteness and the decay of the first order moments of $|u(x,t)|^2$ was studied *e.g.* in [31], [16] and [10]. In particular, it is now well known that the moments of $|u(x,t)|^2$ are finite up to the order 3 (at least when $n=3,4$), if the datum belongs to the corresponding wheighted L^2-spaces. To be more precise, we recall that if $(1+|x|)a \in \mathrm{L}^2(\mathbb{R}^n)$, then there exists a weak solution u which satisfies the energy inequality (2.3) and such that $\sup_{t\ge 0} ||(1+|x|)u(t)||_2 < \infty$. Moreover, if we also have $(1+|x|^{3/2})a \in \mathrm{L}^2(\mathbb{R}^n)$, then there exists a constant C such that

$$\int (1+|x|^3)|u(x)|^2\,dx \le C\log(2+t). \tag{3.11}$$

We refer to [16] for a detailed proof in the case $n=3$, and to [10] for $n=3,4$ (see also [31]). There it is also shown that it would be possible to get rid of the logarithmic factor in (3.11), under some supplementary assumptions on the localization of the data, which we will not need in the sequel.

By Hölder's inequality,

$$\int |x|\,|u(x,t)|^2\,dx \le \left(\int |u(x,t)|^2\,dx\right)^{1/2}\left(\int |x|^2|u(x,t)|^2\,dx\right)^{1/2}.$$

Thus, by (2.5) we get

$$\int_0^\infty\!\!\int |x|\,|u(x,t)|^2\,dx\,dt < \infty. \tag{3.12}$$

We now follow Schonbek's approach [29] and we bound the energy of u by splitting $||\widehat{u}(\cdot,t)||_2^2$ into two time dependent domains, namely $\int_{|\xi|\le g(t)} |\widehat{u}(\xi,t)|^2\,d\xi$ and $\int_{|\xi|\ge g(t)} |\widehat{u}(\xi,t)|^2\,d\xi$. As in [29], we will choose

$$g(t) = \sqrt{\alpha}(1+t)^{-1/2}, \tag{3.13}$$

where $\alpha > 0$ is a large constant.

The first of these two terms is treated in the next lemma.

Lemma 3.5. *Under the assumption (3.3) we have:*

$$\int_{|\xi|\le g(t)} |\widehat{u}(\xi,t)|^2\,d\xi \le C\big[\,||e^{t\Delta}a||_2^2 + g(t)^{n+4}\,\big]. \tag{3.14}$$

If (3.5)-(3.6) also hold, then

$$\int_{|\xi|\le g(t)} |\widehat{u}(\xi,t)|^2\,d\xi \le C\big[\,||e^{t\Delta}a||_2^2 + g(t)^{n+6}\,\big]. \tag{3.15}$$

Proof. Taking the Fourier transform of the j-component ($j = 1, \dots, n$) of the Navier-Stokes equation in its integral form yields

$$\widehat{u}_j(\xi,t) = e^{-t|\xi|^2}\widehat{a}_j(\xi) - \mathrm{i}\int_0^t e^{-(t-s)|\xi|^2} \sum_{h,k=1}^n \xi_h \left(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\widehat{u_h u_k}(\xi,s)\,ds. \tag{3.16}$$

This is justified by the application of the Plancherel theorem and a simple limiting argument, choosing in (2.2) ϕ in the following way:

$$\widehat{\phi}(\xi,s) = e^{-(t^*-s)|\xi|^2}\widehat{\phi}_0(\xi) \qquad (0 \le s \le t), \tag{3.17}$$

for any fixed t and t^* ($0 < t < t^*$), where $\phi_0 \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ is an arbitrary soleinoidal smooth vector field. We refer to [41] for the details of this argument (see also [30]).

By (3.3), we have

$$\widehat{u_h u_k}(\xi,t) = c(t)\delta_{h,k} + \xi \cdot \int_0^1 \nabla\widehat{u_h u_k}(\xi\theta,t)\,d\theta, \qquad a.e. \text{ in }]0,\infty[, \tag{3.18}$$

for all $h,k = 1,\dots,n$. Here the application of the Taylor formula is justified by (3.12).

But, for any fixed j ($j = 1,\dots,n$),

$$\sum_{h,k=1}^n \xi_h(\delta_{j,k} - \xi_j\xi_k|\xi|^{-2})\delta_{h,k} = \sum_{k=1}^n \xi_k\delta_{j,k} - \xi_j \equiv 0$$

Hence, (3.16) and (3.18) yield

$$\begin{aligned} \widehat{u}_j(\xi,t) &= e^{-t|\xi|^2}\widehat{a}_j(\xi) \\ &\quad - \mathrm{i}\int_0^t\!\!\int_0^1 e^{-(t-s)|\xi|^2} \sum_{h,k=1}^n \xi_h\left(\delta_{j,k} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\xi\cdot\nabla\widehat{u_h u_k}(\xi\theta,s)\,ds\,d\theta. \end{aligned} \tag{3.19}$$

Thus, $|\widehat{u}(\xi,t)| \le e^{-t|\xi|^2}|\widehat{a}(\xi)| + C|\xi|^2\int_0^\infty\!\!\int |x|\,|u(x,s)|^2\,dx\,ds$ and (3.14) immediately follows.

The proof of (3.15) is very similar, but we have to postpone it, since it is a consequence of the first part of Theorem 3.1 (namely, estimate (3.4)). □

As in [41], Schonbek's Fourier splitting idea will be used with the integrated equation. We have

$$||\nabla u(t)||_2^2 \ge g(t)^2 \int_{|\xi|\ge g(t)} |\widehat{u}(\xi,t)|^2\,d\xi = g(t)^2||u(t)||_2^2 - \beta(t), \tag{3.20}$$

where

$$\beta(t) = g(t)^2\int_{|\xi|\le g(t)} |\widehat{u}(\xi,t)|^2\,d\xi. \tag{3.21}$$

Integrating this inequality and (2.3) yield

$$||u(t)||_2^2 + \int_s^t g(r)^2||u(r)||_2^2\,dr \le ||u(s)||_2^2 + \int_s^t \beta(r)\,dr \tag{3.22}$$

for $s = 0$, almost all $s > 0$ and all $t \ge s$.

We now state the following lemma (we use the same notations as in [41]).

Lemma 3.6. *Let $g(t)$, $y(t)$ and $\beta(t)$ be three functions defined on $[0,\infty[$, such that g is smooth, $y(t) \geq 0$, $\beta(t) \geq 0$, y is locally bounded and β is locally integrable. Assume that (after suitable modification of the values of $y(t)$ on a set of measure zero)*

$$y(t) + \int_s^t g(r)^2 y(r)\,dr \leq y(s) + \int_s^t \beta(r)\,dr \tag{3.23}$$

holds for $s = 0$, almost all $s > 0$ and all $t \geq s$. Let also $e(t) = \exp(\int_0^t g(r)^2\,dr)$. Then,

$$y(t)e(t) \leq y(0) + \int_0^t e(r)\beta(r)\,dr \qquad \textit{for all } t \geq 0.$$

Proof. This lemma is implicit in Wiegner's paper [41], but we give the proof for reader's convenience.

Let us fix $T > 0$ and consider t and h such that $h \geq 0$, $t - h \geq 0$ and $t \leq T$. We start observing that, by Taylor's formula,

$$e(t) - e(t-h) = e(t-h)\int_{t-h}^t g(r)^2\,dr + \epsilon_t(h),$$

where $|\epsilon_t(h)| \leq C_0(T)h^2$, for some positive constant $C_0(T)$ and all $t \in [0,T]$. Then, for almost all h,

$$\begin{aligned} &y(t)e(t) - y(t-h)e(t-h) \\ &\quad = y(t)e(t-h)\int_{t-h}^t g(r)^2\,dr + e(t-h)(y(t) - y(t-h)) + y(t)\epsilon_t(h). \end{aligned} \tag{3.24}$$

Using $y(t) - y(t-h) \leq -\int_{t-h}^t g(r)^2 y(r)\,dr + \int_{t-h}^t \beta(r)\,dr$, we see that the right hand side of (3.24) is bounded by

$$e(t-h)\int_{t-h}^t (y(t) - y(r))g(r)^2\,dr + e(t-h)\int_{t-h}^t \beta(r)\,dr + C_1(T)h^2.$$

But, $\int_{t-h}^t (y(t) - y(r))g(r)^2\,dr \leq C_2(T)h\int_{t-h}^t \beta(\tau)\,d\tau$. Thence,

$$y(t)e(t) - y(t-h)e(t-h) \leq C_3(T)h\int_{t-h}^t \beta(r)\,dr + \int_{t-h}^t e(r)\beta(r)\,dr + C_1(T)h^2.$$

The conclusion of Lemma 3.6 immediately follows by summation, letting $h \to 0$. □

Choosing $y(t) = ||u(t)||_2^2$, g as in (3.13) and β as in (3.21), we get

$$||u(t)||_2^2(1+t)^\alpha \leq ||a||_2^2 + \int_0^t (1+s)^\alpha \beta(s)\,ds. \tag{3.25}$$

By (3.14), $\beta(s)$ is bounded by $C_\alpha(1+s)^{-1}[(1+s)^{-2\alpha_0} + (1+s)^{-(n+4)/2}]$. Now (3.4) immediately follows, by choosing $\alpha > (n+4)/2$.

We now come to the proof of (3.15) and (3.7). Let us take $\alpha_0 > \frac{n+4}{4}$. Since

$$\int |x|^2|u(x,t)|^2\,dx \leq \left(\int |u(x,t)|^2\,dx\right)^{1/3}\left(\int |x|^3|u(x,t)|^2\,dx\right)^{2/3},$$

by (3.4) and (3.11) we get

$$\int_0^\infty \int |x|^2\,|u(x,t)|^2\,dx\,dt < \infty. \tag{3.26}$$

Then, by Taylor's formula,

$$\begin{aligned}\widehat{u_h u_k}(\xi,t) &= c(t)\delta_{h,k} + \sum_{i=1}^n \xi_i \partial_i \widehat{u_h u_k}(0,t)\\ &\quad + \sum_{|\gamma|=2} \xi^\gamma \int_0^1 \frac{2(1-\theta)}{\gamma!} \partial^\gamma \widehat{u_h u_k}(\theta\xi,t)\,d\theta.\end{aligned} \tag{3.27}$$

We claim that under the assumptions (3.5)-(3.6) the identities

$$\sum_{i,h,k=1}^n \xi_i\xi_h(\delta_{j,k} - \xi_j\xi_k|\xi|^{-2})\partial_i \widehat{u_h u_k}(0,t) \equiv 0 \qquad (j=1,\ldots,n) \tag{3.28}$$

hold true, for almost all $t \in]0,\infty[$. Then, (3.15) follows from (3.16), (3.27) and (3.28), by the same argument that we used to get (3.14).

Our claim is an immediate consequence of the following very simple lemma.

Lemma 3.7. *Let $\mu_{ihk} \in \mathbb{R}$ $(i,h,k = 1,\ldots,n)$, such that*

$$\mu_{ihk} = \mu_{ikh} \qquad \text{for all } \ i,h,k = 1,\ldots,n.$$

Then the two following conditions are equivalent:

(1)

$$\sum_{i,h,k=1}^n \mu_{ihk}\xi_i\xi_j\xi_h\xi_k = \sum_{i,h=1}^n \mu_{ihj}\xi_i\xi_h|\xi|^2 \qquad \text{for all } \ \xi\in\mathbb{R}^n \text{ and } j=1,\ldots,n \tag{3.29}$$

(2) *the coefficients μ_{ihk} satisfy*

$$\mu_{kkk} = \mu_{khh} \qquad (h,k=1,\ldots,n) \tag{3.30}$$

$$\mu_{ihk} = 0 \qquad \text{for all } h \neq k \quad (i,h,k=1,\ldots,n). \tag{3.31}$$

Proof. This is elementary: indeed, if (3.29) holds, then for any j the polynomial $\sum_{i,h=1}^n \mu_{ihj}\xi_i\xi_h$ is divisible by ξ_j. Hence, $\mu_{ihj} = 0$ if $i \neq j$ and $h \neq j$. Further, $\mu_{hhj} = \mu_{hjh} = 0$ if $h \neq j$. This gives (3.31). But from (3.31) and (3.29) we immediately get (3.30).

Proving that (3.30) and (3.31) imply (3.29) is an obvious computation. □

We are now in position to prove (3.7). From (3.15) and (3.21) we get an improved bound for β. Indeed, $\beta(s) \le C_\alpha(1+s)^{-1}[(1+s)^{-2\alpha_0} + (1+s)^{-(n+6)/2}]$. Conclusion (3.7) now follows from (3.25).

To finish the proof of Theorem 3.1 we now have to show that the decay rates (3.4) and (3.7) are optimal. This will be done by showing that, under the assumptions of the first part of Theorem 3.1, we have

$$||u(t) - e^{t\Delta}a||_2 = O(t^{-(t+4)/4}) \qquad \text{as } \ t\to\infty.$$

Further, we have to show that

$$||u(t) - e^{t\Delta}a||_2 = O(t^{-(t+6)/4}), \qquad (t\to\infty)$$

under the supplementary conditions (3.5)-(3.6).

To do this, we will use a well known strategy (see *e.g.* [29], [19], [41]): we replace $u(t)$ with $D(t) = u(t) - e^{t\Delta}a$ and we proceed along the same lines. Note that D satisfies an energy inequality (in slightly modified form):

$$||D(t)||_2^2 + \int_s^t ||\nabla D(r)||_2^2 \, dr \le ||D(s)||_2^2 + \int_s^t ||u(r)||_2^2 \, ||e^{r\Delta}a||_\infty^2 \, dr$$

for almost all $s \ge 0$ and all $t \ge s$ (see [41]).

Let us choose g as in (3.13). Next we observe that

$$\int_{|\xi|\le g(t)} |\widehat{D}(\xi,t)| \, d\xi \le Cg(t)^{n+4}$$

(and the better bound $\int_{|\xi|\le g(t)} |\widehat{D}(\xi,t)| \, d\xi \le Cg(t)^{n+6}$ holds, under the assumptions of the second part of Theorem 3.1).

Then, conclusion $||D(t)||_2 \le C'(1+t)^{-(t+4)/4}$ (respectively, $||D(t)||_2 \le C'(1+t)^{-(t+6)/4}$) follows applying Lemma 3.6 with

$$y(t) = ||D(t)||_2^2 \qquad \text{and} \qquad \beta(t) = ||u(t)||_2^2 \, ||e^{t\Delta}a||_\infty^2 + \int_{|\xi|\le g(t)} |\widehat{D}(\xi,t)| \, d\xi.$$

This completes the proof of Theorem 3.1.

4. Pointwise estimates of strong solutions

In this section, motivated by previous results of Takahashi [35], Miyakawa [26], He and Xin [16] and Amrouche *et. al.* [1] we want construct global strong solutions u to the Navier–Stokes equations with profiles at infinity of the form

$$|u(x,t)| \sim |x|^{-\alpha} t^{-\beta/2}, \tag{4.1}$$

for any $\alpha \ge 0$, $\beta \ge 0$ such that $\alpha + \beta = \gamma$ and $1 \le \gamma \le n+3$.

The existence of such solutions is already known in many important situations, namely the *under-critical case* $1 \le \gamma \le n+1$ (see the the previously cited papers and the remarks below). To deal with the case $\gamma > n+1$, we will use symmetric initial data.

It should be emphasized that the existence of solutions with profile (4.1), in the special situation $\alpha = 0$ and $n+1 < \beta \le n+3$ immediately follows from Theorem 3.1 and the results of [1]. Indeed, Amrouche *et al.* prove that, under suitable assumptions on the data, $||u(t)||_\infty \le C(1+t)^{-n/4}||u(t)||_2$ $(2 \le n \le 5)$. Moreover, decay estimates for the spatial derivatives of u would also follow from [1] but, for sake of simplicity, we will not discuss such estimates in this paper.

However, in the case $\alpha > 0$, profiles (4.1) are not a consequence of the results of [1]. Further, we want to derive (4.1) in any space dimension.

To prove (4.1) we will apply the fixed point theorem to the integral form of (NS), in some L^∞-weighted subspaces of $\mathrm{C}([0,\infty[, \mathrm{L}^n_w(\mathbb{R}^n))$ (here, $\mathrm{L}^n_w = \mathrm{L}^{n,\infty}$ denotes the weak L^n space). This approach have already been used in [26].

Recall that the integral formulation of (NS) reads

$$u(t) = e^{t\Delta}a - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds. \tag{IE}$$

Here $e^{t\Delta}\mathbb{P}\nabla$ is a convolution operator and the components of its kernel $F(x,t)$ are given by

$$\widehat{F}_{jhk}(\xi,t) = \mathrm{i}\xi_h e^{-t|\xi|^2}(\delta_{j,k} - \xi_j\xi_k|\xi|^{-2}). \tag{4.2}$$

We will prove the following:

Theorem 4.1. *Let $1 \le \gamma \le n+3$ and let a be a divergence-free vector field such that*

$$(1+|x|)^\gamma a(x) \in \mathrm{L}^\infty(\mathbb{R}^n). \tag{4.3}$$

If $\gamma = n$, $n+1$, $n+2$ or $n+3$ we also assume

$$\sup_{x\in\mathbb{R}^n, t\ge 0} (1+|x|)^\gamma |e^{t\Delta}a(x)| < \infty. \tag{4.4}$$

In the case $\gamma = n$, we also suppose that $t^{n/2}|e^{t\Delta}a(x)|$ is uniformly bounded in x and t.

When $n+1 < \gamma \le n+3$, we assume a to be symmetric.

Then, there exists an absolute constant $\eta > 0$ with the following property. If

$$\sup_{x\in\mathbb{R}^n} |x|\,|a(x)| < \eta, \tag{4.5}$$

then there exist a constant C and a solution u of (IE) such that $u(0) = a$ (e.g. in the distributional sense) and

$$|u(x,t)| \le C(1+|x|)^{-\gamma}, \qquad |u(x,t)| \le C(1+t)^{-\gamma/2}. \tag{4.6}$$

Remark 4.2. Conclusion (4.6) is due to [26] in the case $1 \le \gamma \le n+1$ (see also [35], for the case $1 \le \gamma \le n$ and [16], for $\gamma = n, n+1$). The result in the over-critical case $\gamma > n+1$ was announced in [3], but the proof was only sketched. A recent proof in the case $\gamma = n+3$ (under slightly more stringent assumptions), based on the ideas of [3], is contained in [27].

We also recall that if $u(x,t)$ is a solution to the Navier–Stokes equations, then the same is true for $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ $(\lambda > 0)$. But the smallness assumptions of [26], [16], [1], [3] and [27] are not invariant under this natural scaling. On the other hand, (4.5) is invariant.

Remark 4.3. The solution is unique in $\mathrm{C}([0,\infty[, \mathrm{L}^\infty_\gamma(\mathbb{R}^n))$, where $\mathrm{L}^\infty_\gamma(\mathbb{R}^n)$ is the space of all functions f such that $(1+|x|)^\gamma f(x) \in \mathrm{L}^\infty(\mathbb{R}^n)$ and the continuity in $t=0$ is defined in the distributional sense (as it is usually done in non-separable spaces). The proof of the continuity with respect to the time variable is straightforward and will be omitted (the standard argument is described *e.g.* in [24]).

The proof of Theorem 4.1 will be based on three lemmata, which are useful to describe the space-time decay of the linear evolution $e^{t\Delta}a(x)$. The first lemma gives examples of initial data satisfying (4.4) and explain why the assumptions of Theorem 4.1 are slightly more stringent in the particular cases $\gamma = n$, $n+1$, $n+2$, and $n+3$.

Lemma 4.4. *Let a such that $(1+|x|)^\gamma a(x) \in \mathrm{L}^\infty(\mathbb{R}^n)$. We set $I(r) = [r]$ (the integer part), for any noninteger real number r, and $I(r) = r-1$ if r is integer.*

(1) *If $1 \le \gamma < n$ then, for some constant $C > 0$,*

$$\sup_{t\ge 0} |e^{t\Delta}a(x)| \le C(1+|x|)^{-\gamma} \tag{4.7}$$

Estimate (4.7) holds true when $\gamma \geq n$, provided that $(1+|x|)^{\gamma-n} a \in L^1(\mathbb{R}^n)$ and the moments of a vanish up to the order $I(\gamma - n)$.

(2) *For $\gamma \geq 1$ and $\gamma \neq n$, let a be such that (4.7) holds. Then*

$$\sup_{x\in\mathbb{R}^n} |e^{t\Delta} a(x)| \leq C'(1+t)^{-\gamma/2} \tag{4.8}$$

Conclusion 1 is well known: see *e.g.* [26]. Conclusion 2 is slightly more deeper and follows from the theory of weak Hardy and Besov spaces. In the appendix we will provide a proof of this second statement, which is a straightforward adaptation of an argument due to [25] and [26].

In Theorem 4.1 we made no assumptions on the moments of a. Actually, these assumptions are implicit for localized symmetric and soleinoidal vector fields. Indeed the divergence-free condition implies many conditions on the higher-order moments of a:

Lemma 4.5. *Let $m \in \mathbb{N}$ and let a be a soleinoidal vector field such that $(1+|x|)^m a \in \mathrm{L}^1(\mathbb{R}^n)$. Then, for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ such that $1 \leq |\alpha| \leq m+1$, we have*

$$\alpha_1 \int x_1^{\alpha_1 - 1} x_2^{\alpha_2} \dots x_n^{\alpha_n}\, a_1(x)\,dx + \dots + \alpha_n \int x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n - 1}\, a_n(x)\,dx = 0. \tag{4.9}$$

In particular,

$$\int a_j(x)\,dx = \int x_j a_j(x)\,dx = \ldots = \int x_j^m a_j(x)\,dx = 0 \tag{4.10}$$

and, if $m \geq 1$,

$$\int x_h a_k(x)\,dx = -\int x_k a_h(x)\,dx \qquad (h, k = 1, \dots, n). \tag{4.11}$$

Conditions (4.10) and (4.11) are due to Truesdell (see [37]), at least for three-dimensional vector fields. The general conditions (4.9) are not so much known, but they can be probably deduced from the slightly more difficult Truesdell's identities after some computations (and conversely). Here we give a more direct and elementary proof of (4.9), using the Fourier transform.

Proof. We start observing that, for $\ell = 0, 1 \dots, m$ we have

$$\sum_{h=1}^{n} \sum_{|\gamma|=\ell} \frac{\xi^\gamma \xi_h}{\gamma!} \partial^\gamma \widehat{a}_h(\xi) \equiv 0 \tag{4.12}$$

(here we use the notations introduced in (1.4)). Indeed, this is obvious for $\ell = 0$. The general case follows by induction on m: for any fixed $k = 1, \dots, n$, let $e_k = (0, \dots, 1, \dots, 0)$ be the k-th vector of the canonical base in $\mathbb{R}^n$. If $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ and $|\gamma| \leq m-1$ then we have:

$$\partial_k\Bigl(\sum_{h=1}^{n} \xi^\gamma \xi_h \partial^\gamma \widehat{a}_h(\xi)\Bigr) = \xi^\gamma \partial^\gamma \widehat{a}_k(\xi) + \sum_{h=1}^{n} \gamma_k \xi^{\gamma - e_k} \xi_h \partial^\gamma \widehat{a}_h(\xi) + \sum_{h=1}^{n} \xi^\gamma \xi_h \partial^\gamma \partial_k \widehat{a}_h(\xi).$$

Multiplying this expression by $\xi_k/\gamma!$ and summing on k and γ yields, by induction,

$$0 = \sum_{k=1}^{n} \sum_{|\gamma|=\ell} \frac{\xi^\gamma \xi_k}{\gamma!} \partial^\gamma \widehat{a}_k(\xi) + \sum_{h,k=1}^{n} \sum_{|\gamma|=\ell} \frac{\gamma_k \xi^\gamma \xi_h}{\gamma!} \partial^\gamma \widehat{a}_h(\xi) + \sum_{h,k=1}^{n} \sum_{|\gamma|=\ell} \frac{\xi^\gamma \xi_h \xi_k}{\gamma!} \partial^\gamma \partial_k \widehat{a}_h(\xi). \tag{4.13}$$

If (4.12) holds true for all ℓ ($0 \leq \ell \leq m-1$), then we see that the two first terms on the right hand side of (4.13) vanish. But,

$$\sum_{k=1}^{n} \sum_{|\gamma|=\ell} \frac{\xi_k \xi^\gamma}{\gamma!} \partial^\gamma \partial_k \widehat{a}_h(\xi) \equiv (\ell+1) \sum_{|\beta|=\ell+1} \frac{\xi^\beta \partial^\beta}{\beta!} \widehat{a}_h(\xi).$$

Thus, (4.12) holds true also for $\ell = m$.

Now, if $a \in \mathrm{L}^1(\mathbb{R}^n, (1+|x|^m)dx)$, the left hand side in (4.12) is a continuous function in ξ, for all $\ell = 0, \ldots, m$. Taking $\xi = r\eta$, $\eta \in \mathbb{R}^n$, $|\eta| = 1$ and $r \to 0$, we get

$$\sum_{h=1}^{n} \sum_{|\gamma|=\ell} \frac{\partial^\gamma \widehat{a}_h(0)}{\gamma!} \eta^\gamma \eta_h = 0, \qquad |\eta| = 1, \quad \ell = 0, \ldots, m$$

and the vanishing of this homogeneous polynomial is equivalent to (4.9).

Choosing $\alpha = (0, \ldots, \ell+1, \ldots, 0)$ (with $\ell = 0, \ldots, m$) yields (4.10). In order to see that $\int x_h a_k(x)\, dx$ is skew-symmetric, we just take $\alpha = e_h + e_k$ ($h, k = 1, \ldots, n$). □

The following lemma is now immediate:

Lemma 4.6. *Under the assumptions of Theorem 4.1 we have, for any* $1 \leq \gamma \leq n+3$,

$$\sup_{t \geq 0} |e^{t\Delta} a(x)| \leq C(1+|x|)^{-\gamma}, \qquad \text{and} \tag{4.14}$$

$$\sup_{x \in \mathbb{R}^n} |e^{t\Delta} a(x)| \leq C(1+t)^{-\gamma/2}. \tag{4.15}$$

Proof. Indeed, this is obvious for $\gamma = n$ and it is a consequence of Lemma 4.4 for $\gamma \neq n$. To apply Lemma 4.4, observe that the vanishing of $\int a(x)\, dx$ (when $\gamma > n$), as well as the vanishing of $\int x_j a_j(x)\, dx$ (when $\gamma > n+1$) comes from Lemma 4.5. On the other hand, the vanishing of the integrals $\int x_j a_k(x)\, dx$ ($j \neq k$) comes from the symmetry of a. When $n+2 < \gamma \leq n+3$, the vanishing of $\int x_j x_h a_k(x)\, dx$ ($j, h, k = 1, \ldots, n$) is an immediate consequence of the symmetries as well. □

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. We first consider the case $\gamma = 1$. By (4.3), (4.5) and a rescaling argument, we can assume that $\sup_x (1+|x|)|a(x)|$ is small (if necessary, we replace $a(x)$ by $\lambda_0 a(\lambda_0 x)$, with a suitable $\lambda_0 > 0$).

Our solution to (IE) will be constructed applying the standard fixed point argument in the space defined by (4.16) below. To do this, let us introduce the bilinear operator

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes v)\, ds.$$

Then we define the approximate solutions $u^{(0)} = e^{t\Delta} a$, $u^{(k+1)} = e^{t\Delta} a - B(u^{(k)}, u^{(k)})$ ($k = 1, 2, \ldots$).

Let us show that B is bounded on the space of functions such that

$$\sup_{x,t} (1+|x|)|u(x,t)| < \infty \qquad \text{and} \qquad \sup_{x,t} (1+t)^{1/2} |u(x,t)| < \infty. \tag{4.16}$$

This is due to [26] and it follows from some well known decay properties of the kernel F. Indeed, by (4.2) we have

$$|F(x,t)| \le C|x|^{-(n+1)}, \qquad |F(x,t)| \le Ct^{-(n+1)/2}, \tag{4.17}$$

$$||F(\cdot,t)||_1 \le Ct^{-1/2}, \qquad F(x,t) = t^{-(n+1)/2}F(xt^{-1/2},1) \tag{4.18}$$

(see *e.g.* [26]). Then the bound

$$\sup_{x,t}(1+|x|)\,|B(u,v)(x,t)| < \infty \tag{4.19}$$

easily follows by splitting $B(u,v)(t)$ into $\int_0^t\int_{|y|\le|x|/2} F(x-y,t-s)(u\otimes v)(y,s)\,dy\,ds$ and $\int_0^t\int_{|y|\ge|x|/2} F(x-y,t-s)(u\otimes v)(y,s)\,dy\,ds$.

On the other hand, we can write

$$B(u,v)(t) = e^{t\Delta/2}B(u,v)(t/2) + \int_{t/2}^t F(t-s)*(u\otimes v)(s)\,ds. \tag{4.20}$$

Because of (4.19), applying Lemma 4.6 to $B(u,v)$, we see that the first term on the right hand side is bounded by $Ct^{-1/2}$. The estimate of the second term is obvious, by (4.17) and (4.18).

On the other hand, applying again Lemma 4.6 with $\gamma=1$, we see that the linear term $e^{t\Delta}a$ belongs to the space given by (4.16) (with a small norm). Theorem 4.1 is thus proved for $\gamma=1$.

To prove the theorem in the case $\gamma>1$ we could apply the fixed point theorem in the space of functions such that $|u(x,t)|\le C(1+|x|)^{-\gamma}$ and $|u(x,t)|\le C(1+t)^{-\gamma/2}$. This was done in [26], [3], [27]. We will not use the approach, since it leads to too stringent assumptions on the data

We will obtain the conclusion in the case $\gamma>1$ by means of some *boot-strap* arguments. A first essential step is the following lemma.

Lemma 4.7. *Let $\gamma>1$ and a as in Theorem 4.1. Then there exists δ ($1<\delta\le\gamma$) and a constant C such that the solution u obtained for $\gamma=1$ satisfies*

$$|u(x,t)| \le C(1+|x|)^{-\delta} \qquad \text{and} \tag{4.21}$$

$$|u(x,t)| \le C(1+t)^{-\delta/2}, \tag{4.22}$$

for all $x\in\mathbb{R}^n$ and $t\ge 0$.

Proof. We first show that u satisfies

$$||\nabla u(\cdot,t)||_{n/\beta} \le ct^{-1+\beta/2}, \qquad 0<\beta<1, \tag{4.23}$$

for some small constant $c>0$ and all $t>0$. The proof of (4.23) is almost the same as in [17] and relies on two simple facts. Firstly,

$$||\nabla e^{t\Delta}a||_{n/\beta} \le C||\nabla g_t||_{\mathrm{L}^{n/(n-1+\beta),1}}||a||_{\mathrm{L}^{n,\infty}} \le \eta Ct^{-1+\beta/2},$$

(here and below, $g_t(x)=(4\pi t)^{-n/2}e^{-t|x|^2/(4t)}$ is the gaussian) by Young's inequality for Lorentz spaces (see [2]).

Secondly, $B(u,v)$ is bounded in the subspace of $\mathrm{C}([0,\infty[,\mathrm{L}^{n,\infty}(\mathbb{R}^n))$ given by all soleinoidal vector fields satisfying (4.16) and (4.23). To see this, we first observe that, by the divergence-free condition, $\nabla F(t-s)*(u\otimes v)(s)$ can be written as $\nabla e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla)v$. Next, the kernel $\tilde F$ of $\nabla e^{(t-s)\Delta}\mathbb{P}$ essentially equals F, and satisfies the same bounds (4.17) and (4.18). By Young and Hölder's inequality, and since $||u(s)||_r \le C||u(s)||^{n/r}_{\mathrm{L}^{n,\infty}} ||u(s)||^{1-n/r}_\infty$ $(n < r < \infty)$, we have, if $0 < \beta < 1$ and $n < r < n/(1-\beta)$,

$$\begin{aligned}
||\nabla B(u,v)(t)||_{n/\beta} &\le \int_0^t ||\tilde{F}(t-s)||_{\frac{r}{r-1}} ||u(s)||_r ||\nabla v(s)||_{n/\beta}\, ds \\
&\le K(u,v) \int_0^t (t-s)^{-1/2-n/2r} s^{-3/2+n/2r+\beta/2}\, ds \\
&\le K(u,v)\, t^{-1+\beta/2}.
\end{aligned}$$

Here,

$$K(u,v) \le C \sup_{t>0} t^{1-\beta/2}\, ||\nabla v(t)||_{n/\beta} \biggl(\sup_{t>0} ||u(t)||_{\mathrm{L}^{n,\infty}} + \sup_{t>0} t^{1/2}\, ||u(t)||_\infty \biggr),$$

where C is an absolute constant,

It is now easy to show that $u(t)$ belongs to $\mathrm{L}^p(\mathbb{R}^n)$, for all p such that $\max\{1, \frac{n}{\gamma}\} < p \le \infty$, uniformly in time. This is obviously true for $t = 0$, by (4.3). Moreover, the approximate solutions $u^{(k)}$ (see the first step of the proof) converge in the space of functions which satisfy all the previous conditions and which, in addition, belong to $\mathrm{L}^\infty(]0,\infty[, \mathrm{L}^p(\mathbb{R}^n))$. Indeed, $||e^{t\Delta}a||_p \le ||a||_p$ and

$$\begin{aligned}
||B(u,v)(t)||_p &\le C \int_0^t (t-s)^{-\beta/2} ||\nabla u(s)||_{n/\beta} ||v(s)||_p\, ds \\
&\le C \sup_{s>0} ||\nabla u(s)||_{n/\beta} \sup_{s>0} ||v(s)||_p,
\end{aligned}$$

for any $0 < \beta < 1$. Thus, $||u^{(k+1)}(t)||_p \le ||a||_p + c' \sup_{s>0} ||u^{(k)}(s)||_p$ where, by (4.5) and (4.23), c' is a small constant independent of t. This in turn gives

$$\sup_{t>0} ||u(t)||_p < \infty.$$

We are now in position to improve estimates (4.16). Let us choose $p = \frac{n}{2}(1 + \frac{1}{\gamma})$. Then we write $u(t) = e^{t\Delta}a - B(u,u)(t)$ and we apply Young's inequality to the linear evolution. This yields

$$||e^{t\Delta}a||_\infty \le Ct^{-n/2p} = Ct^{-\gamma/(\gamma+1)}.$$

Moreover, if $\beta > 0$ is small enough, so that $\frac{n}{2p} + \frac{\beta}{2} < 1$, we have

$$\begin{aligned}
||B(u,u)(t)||_\infty &\le C \int_0^t (t-s)^{-n/2p-\beta/2} ||u(s)||_p ||\nabla u(s)||_{n/\beta}\, ds \\
&\le C \int_0^t (t-s)^{-n/2p-\beta/2} s^{-1+\beta/2}\, ds \\
&\le Ct^{-n/2p} = Ct^{-\gamma/(\gamma+1)}.
\end{aligned}$$

Thus,

$$|u(x,t)| \le C(1+t)^{-\gamma/(\gamma+1)}, \qquad x \in \mathbb{R}^n,\ t \ge 0, \tag{4.24}$$

which is slightly better than the second of (4.16).

By the first of (4.16) and (4.24), we get

$$|u \otimes u|(y,s) \le C(1+|y|)^{-(1+(\gamma-1)/(2\gamma))}(1+s)^{-1/2}. \tag{4.25}$$

We now split $B(u,u)(x,t)$ as before into

$$\int_0^t \int_{|y|\le|x|/2} F(x-y,t-s)\,(u\otimes u)(y,s)\,dy\,ds$$

and

$$\int_0^t \int_{|y|\ge|x|/2} F(x-y,t-s)(u\otimes u)(y,s)\,dy\,ds.$$

Using $|F(x,t)|\le C|x|^{-n}t^{-1/2}$ and (4.25), we see that the first integral is bounded by $c|x|^{-(1+(\gamma-1)/(2\gamma))}$. But this estimate holds true also for the second integral, because of $||F(\cdot,t)||_1 = Ct^{-1/2}$ and (4.25).

Since $1 < 1+(\gamma-1)/(2\gamma) < \gamma$ (when $\gamma>1$), Lemma 4.4 and $u(t)=e^{t\Delta}a - B(u,u)(t)$ imply

$$|u(x,t)|\le C(1+|x|)^{-(1+(\gamma-1)/(2\gamma))}, \qquad x\in\mathbb{R}^n,\ t\ge 0.$$

Lemma 4.7 is thus proved. □

Proof of Theorem 4.1 in the case $1<\gamma\le 2$. Let δ be as in Lemma 4.7. We may assume $1<\delta<\gamma\le 2$. By (4.21) and (4.22), we have

$$|u\otimes u|(y,s)\le C(1+|y|)^{-2\delta+1}(1+s)^{-1/2} \qquad (y\in\mathbb{R}^n,\ s\ge 0). \tag{4.26}$$

Let us first treat the case $n=2$ and $\frac{3}{2}<\delta<\gamma\le 2$: arguing as in Lemma 4.7, (4.26) yields $|B(u,u)(x,t)|\le C|x|^{-2}$. Thus, $|u(x,t)|\le(1+|x|)^{-\gamma}$. If $n=2$ and $\delta=\frac{3}{2}$, the bilinear term is bounded at infinity by $|x|^{-2}\log(x)$. But this allows us to improve the rate of decay in (4.26). Thus, we can obtain the bound $|u(x,t)|\le C(1+|x|)^{-\gamma}$ as we did for $n=2$ and $\delta>\frac{3}{2}$.

We now consider the other cases, *i.e.* $n=2$ and $1<\delta<\min\{\frac{3}{2},\gamma\}$, or $n\ge 3$ and $1<\delta<\gamma\le 2$. The same argument as above implies that $|B(u,u)(x,t)|$ is uniformly bounded by $c|x|^{-2\delta+1}$. Thus,

$$|u(x,t)|\le C(1+|x|)^{-\delta_1}, \qquad \delta_1=\min\{\gamma, 2\delta-1\}.$$

This is slightly better than (4.21).

Let us show that we have also

$$|u(x,t)|\le C(1+t)^{-\delta_1/2}, \qquad x\in\mathbb{R}^n,\ t\ge 0. \tag{4.27}$$

We start by splitting $B(u,u)$ as in (4.20). For the second term, we obviously have

$$\int_{t/2}^t ||F(t-s)*(u\otimes u)(s)||\,ds \le t^{-\delta+1/2}.$$

For the first term, we have

$$||e^{t\Delta/2}B(u,u)(t/2)||_\infty \le t^{-\delta_1/2}.$$

This can be seen as follows. If $\delta_1<n$ we simply use $||B(u,u)(t/2)||_{\mathrm{L}^{n/\delta_1,\infty}}\le C$ and the duality between $\mathrm{L}^{n/(n-\delta_1),1}(\mathbb{R}^n)$ and $\mathrm{L}^{n/\delta_1,\infty}(\mathbb{R}^n)$. If $\delta_1\ge n$ (since $\delta<2$, this happens only if $\delta_1=n=\gamma=2$), we have

$$\begin{aligned} ||e^{t\Delta/2}B(u,u)(t/2)||_\infty &\le ||g_{t/2}||_\infty ||B(u,u)(t/2)||_1 \\ &\le Ct^{-1}\int_0^{t/2} ||F(t/2-s)||_1 ||u(s)||_2^2\,ds \le Ct^{-1}, \end{aligned} \tag{4.28}$$

where g is the gaussian and $g_t(x)=t^{-n/2}g(x/\sqrt{t})$.

This implies (4.27). If $\delta_1 = \gamma$ the proof is finished. If $\delta_1 < \gamma$ the conclusion follows after finitely many iterations.

Proof in the case $2 < \gamma \leq n$. By the above proof, in (4.21) and (4.22) we can choose δ such that $2 < \delta < n$. We easily obtain *e.g.* $|B(u,u)|(x,t)| \leq C|x|^{-\delta-1}$. Hence,

$$|u(x,t)| \leq C(1+|x|)^{-\tilde{\delta}}, \qquad \tilde{\delta} = \min\{\gamma, \delta+1\}.$$

If $\tilde{\delta} < \gamma \leq n$, (4.20) and (4.22) imply $||B(u,u)(t)||_\infty \leq Ct^{-\tilde{\delta}/2} + Ct^{-\delta+1/2}$. Thus, $|u(x,t)| \leq C(1+t)^{-\tilde{\delta}/2}$. After finitely many iterations of this argument, we find $\tilde{\delta} = \gamma \leq n$. Hence, $|u(x,t)| \leq C(1+|x|)^{-\gamma}$.

On the other hand, if $\gamma < n$ the estimate $|u(x,t)| \leq C(1+t)^{-\gamma/2}$ now follows from (4.20) and the duality of Lorentz spaces. If $\gamma = n$, the conclusion follows by arguing exactly as in the case $\gamma = n = 2$.

Proof in the case $n < \gamma \leq n+1$. In this case we may choose $\delta > n$ in (4.21) and (4.22). In particular, $\int_0^\infty\int |u(y,s)|^2\,dy\,ds < \infty$. By (4.17) and (4.18) we get $|B(u,u)(x,t)| \leq C|x|^{-(n+1)}$. Thus, $|u(x,t)| \leq C(1+|x|)^{-\gamma}$.

The time decay estimate of the bilinear term is also immediate. Indeed,

$$||B(u,u)(t)||_\infty \leq \int_0^{t/2} ||F(t-s)||_\infty ||u(s)||_2^2\,ds + \int_{t/2}^t ||F(t-s)||_1 ||u(s)||_\infty^2\,ds.$$

The first term is bounded by $t^{-(n+1)/2}$ and the second by $t^{-\delta+1/2}$. Thus, $||u(t)||_\infty \leq C(1+t)^{-\gamma/2}$.

Proof in the case $n+1 < \gamma \leq n+3$ *(a symmetric)*. Let us first observe that approximate solutions $u^{(0)} = e^{t\Delta}a$, $u^{(m+1)} = e^{t\Delta}a - B(u^{(m)}, u^{(m)})$ are symmetric for all t. This can be seen by induction, in the following way: we first introduce the vector fields $\theta(x,t)$ and $\phi(x,t)$ which are defined in the phase space by

$$\widehat{\theta}(\xi,t) \equiv \sum_{h=1}^n \xi_h \widehat{u_h^{(m)} u^{(m)}}(\xi,t)$$

and

$$\widehat{\phi}(\xi,t) = \xi|\xi|^{-2} \sum_{h,k=1}^n \xi_h\xi_k \widehat{u_h^{(m)} u_k^{(m)}}(\xi,t).$$

If $u^{(m)}(x,t)$ is a symmetric vector field for all t, then the same is true for $\widehat{u^{(m)}}(\xi,t)$. Thus, $\widehat{\theta}(\xi,t)$ and $\widehat{\phi}(\xi,t)$ are symmetric for all t, with respect to the ξ variable. Then,

$$\widehat{B}(u^{(m)}, u^{(m)})(\xi,t) = \mathrm{i}\int_0^t e^{-(t-s)|\xi|^2}(\widehat{\theta}(\xi,s) - \widehat{\phi}(\xi,s))\,ds$$

is symmetric.

Passing to the limit for $m \to \infty$, implies that the solution $u(t)$ obtained in the first step of the proof is symmetric for all t.

Before going further, let us recall some known decay estimates for the spatial derivatives of F (see *e.g.* [10]). For any $\alpha \in \mathbb{N}^n$ we have

$$|\partial^\alpha F(x,t)| \leq C|x|^{-n-1-|\alpha|}, \tag{4.29}$$

$$|\partial^\alpha F(x,t)| \leq Ct^{-(n+1+|\alpha|)/2}. \tag{4.30}$$

The conclusion of Theorem 4.1, at least in the case $n \geq 3$, will be an immediate consequence of the following lemma.

Lemma 4.8. *Let $n \geq 3$ and $u(x,t)$ a symmetric vector field such that (4.21) and (4.22) hold with $\delta = n+1$. Then*

$$|B(u,u)(x,t)| \leq C(1+|x|)^{-(n+3)} \qquad \textit{and} \tag{4.31}$$

$$|B(u,u)(x,t)| \leq C(1+t)^{-(n+3)/2}. \tag{4.32}$$

Proof. For $h,k = 1,\ldots,n$, we introduce the functions

$$r_{h,k} = u_h u_k(x,t) - \lambda(t) g(x) \delta_{h,k},$$

where $\lambda(t) = \int u_1^2(x,t)\,dx = \ldots = \int u_n^2(x,t)\,dx$ and g is the gaussian, normalized by $\int g(x)\,dx = 1$. By the parity conditions on u, we have

$$\int r_{h,k}(x,t)\,dx = \int x_l r_{h,k}(x,t)\,dx = 0,$$

for all $h,k,l = 1,\ldots,n$ and all $t \geq 0$. Moreover, by (4.2) we have $\sum_{h=1}^n F_{jhh}(\cdot,t) * g \equiv 0$. Thus,

$$B(u,u)_j(t) = \sum_{h,k=1}^n \int_0^t F_{jhk}(\cdot,t-s) * r_{h,k}(s)\,ds \qquad (j=1,\ldots,n).$$

Using (4.18), we see that $\int_0^t \int_{|y|\geq|x|/2} |F(x-y,t-s)|\,|r(y,s)|\,dy\,ds$ is bounded by $(1+|x|)^{-2n-1}$. Since the moments of $r_{h,k}$ vanish (up to the order 1), we can split $\int_0^t\int_{|y|\leq|x|/2} F_{jhk}(x-y,t-s) * r_{h,k}(s)\,dy\,ds$ into

$$\begin{aligned} I_1 \equiv \int_0^t \int_{|y|\leq|x|/2} [F_{jhk}(x-y,&t-s) - F_{jhk}(x,t-s) \\ &- \textstyle\sum_l y_l \partial_l F_{jhk}(x,t-s)] r_{h,k}(y,s)\,dy\,ds \end{aligned} \tag{4.33}$$

and

$$I_2 \equiv -\int_0^t \int_{|y|\geq|x|/2} [F_{jhk}(x,t-s) + \textstyle\sum_l y_l \partial_l F_{jhk}(x,t-s)] r_{h,k}(y,s)\,dy\,ds.$$

The first integral is treated by means of the Taylor formula. By (4.29) (with $|\alpha| = 2$), and the fact that $\int_0^\infty\int |y|^2 |r_{h,k}(y,s)|\,dy\,ds < \infty$ (here we use $n \geq 3$), we get $I_1 \leq |x|^{-n-3}$.

The second integral is treated by means of (4.29) and (4.30). We immediately find that $|I_2|$ is bounded at infinity by $|x|^{-2n-1}$. This ends the proof of (4.31).

To obtain (4.32), we split $\int_0^t\int F_{jhk}(x-y,t-s) * r_{h,k}(s)\,dy\,ds$ into

$$\begin{aligned} J_1 \equiv \int_0^{t/2} \int [F_{jhk}(x-y,t-s) &- F_{jhk}(x,t-s) \\ &- \textstyle\sum_l y_l \partial_l F_{jhk}(x,t-s)] r_{h,k}(y,s)\,ds \end{aligned} \tag{4.34}$$

and

$$J_2 \equiv \int_{t/2}^t \int F_{jhk}(x-y,t-s) r_{h,k}(y,s)\,dy\,ds.$$

By the Taylor formula and (4.30) we get $|J_1| \leq Ct^{-(n+3)/2}$. On the other hand, (4.18) yields $|J_2| \leq Ct^{-n-1/2}$ and (4.32) follows. □

Theorem 4.1 is completely proved for $n \geq 3$. An obvious modification is necessary for $n = 2$. Indeed, under the assumption of Lemma 4.8, we only have $\int_0^\infty\int |y|\,|u(y,s)|^2\,dy\,ds < \infty$. But the same argument as before gives $|B(u,u)|(x,t) \leq C(1+|x|)^{-(n+2)}$ and $|B(u,u)|(x,t) \leq C(1+t)^{-(n+2)/2}$.

The theorem follows for $n = 2$ and $3 < \gamma \leq 4$. If $4 < \gamma \leq 5$, we get $\int_0^\infty\int |y|^2\,|u(y,s)|^2\,dy\,ds < \infty$. Thus, the same arguments above now applies also in the $n = 2$ case.

Appendix

Besov and weak-Hardy Spaces. The aim of this section is to recall the basic definitions and properties of the Besov and weak Hardy spaces. Next we recall an useful injection by Miyakawa which will play an important role in the proof of the second part of Lemma 4.4. More details can be found *e.g.* in [2], [9], [23] or [25].

Recall that a tempered distribution f is in the *Hardy–Lorentz* space $H^{p,q}$ ($0 < p < \infty$ and $0 < q \leq \infty$) if and only if:

$$f^*(x) = \sup_{t>0} |e^{t\Delta} f(x)| \in \mathrm{L}^{p,q}(\mathbb{R}^n),$$

where $\mathrm{L}^{p,q}(\mathbb{R}^n)$ is the usual Lorentz space (see [2]). Moreover, $||f||_{H^{p,q}} = ||f^*||_{\mathrm{L}^{p,q}}$. In particular, $H^p = H^{p,p}$ is the usual Hardy space and $H^p_w = H^{p,\infty}$ is the weak Hardy space.

This space can be also defined by real interpolation. Indeed, for $0 < p_0 < p_1 < 1$,

$$H^p_w = (H^{p_0}, H^{p_1})_{\theta,\infty}, \qquad 1/p = (1-\theta)/p_0 + \theta/p_1 \qquad (0 < \theta < 1) \tag{4.35}$$

(see [15]).

Besov spaces can be easily defined by means of a Littlewood–Paley decomposition: let us choose a scalar function $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi}$ is supported by $\{1/2 \leq |\xi| \leq 2\}$ and $|\widehat{\psi}(\xi)| \geq c > 0$, if $1 \leq |\xi| \leq 2$. We next define $\psi_j(x) = 2^{nj}\psi(2^j x)$ ($j \in \mathbb{Z}$). The homogeneous Littlewood-Paley decomposition of a tempered distribution u is the series $\sum_{j=-\infty}^{\infty} \Delta_j u$, where Δ_j is the convolution with ψ_j. As it is well known, this series converges in the distributional sense, modulo polynomials.

For any $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$, the homogeneous Besov (quasi-)norm is defined by

$$||u||_{\dot{\mathrm{B}}^{s,q}_p} = \left(\sum_{j=-\infty}^{\infty} (2^{js}||\Delta_j u||_p)^q \right)^{1/q}. \tag{4.36}$$

Now, let $m = [s - n/p]$ (the integer part) if $s - n/p \notin \mathbb{Z}$ and $m = s - n/p - 1$ if $s - n/p \in \mathbb{Z}$. Let us denote by $\mathcal{P}_m$ the set of polynomials of degree $\leq m$ ($\mathcal{P}_m = \emptyset$ if $m \leq -1$). The homogeneous Besov space are then defined in the following way: $f \in \dot{\mathrm{B}}^{s,q}_p(\mathbb{R}^n)$ if and only if $f = \sum_{j=-\infty}^{\infty} \Delta_j u$, with $u \in \mathcal{S}'/\mathcal{P}_m$, where the series converges in $\mathcal{S}'/\mathcal{P}_m$ and $||f||_{\dot{\mathrm{B}}^{s,q}_p} < \infty$.

Since $\Delta_j\Delta_{j'} \equiv 0$, for $|j - j'| \geq 2$, and $\Delta_{j'}\Delta_j$ is essentially equal to Δ_j, for $|j - j'| \leq 1$, the space $\dot{\mathrm{B}}^{s,q}_p(\mathbb{R}^n)$ is independent of the particular choice of the test function ψ.

A basic interpolation result on Besov spaces reads

$$\dot{\mathrm{B}}_1^{s,\infty} = (\dot{\mathrm{B}}_1^{s_0,1}, \dot{\mathrm{B}}_1^{s_1,1})_{\theta,\infty}, \qquad 1/s = (1-\theta)/s_0 + \theta/s_1, \quad 0<\theta<1 \tag{4.37}$$

(see [2]).

We refer to [9] for a Littlewood–Paley characterization of the Hardy spaces. Such characterization immediately gives the injection

$$H^p \subset \dot{\mathrm{B}}_1^{-n(1/p-1),q}(\mathbb{R}^n), \qquad 0<p<1, \quad 1\le q\le\infty. \tag{4.38}$$

Using this, with $s_0=-n(1/p_0-1)$, $s_1=-n(1/p_1-1)$, (4.35) and (4.37), we get Miyakawa's injection (see [25])

$$H^p_w \subset \dot{\mathrm{B}}_1^{-n(1/p-1),\infty}(\mathbb{R}^n), \qquad (0<p<1). \tag{4.39}$$

Time decay for the heat equation. We can now prove the following statement, which we used in the proof of Theorem 4.1.

Lemma 4.9. *Let $\gamma\ge 1$, $\gamma\neq n$ and $a\in\mathcal{S}'(\mathbb{R}^n)$, such that $\sup_{t\ge0}|e^{t\Delta}a(x)|\le c(1+|x|)^{-\gamma}$. Then, there exists a constant $C>0$ such that, for all $t\ge0$,*

$$\sup_{x\in\mathbb{R}^n}|e^{t\Delta}a(x)|\le C(1+t)^{-\gamma/2}. \tag{4.40}$$

Proof. Let us first consider the case $\gamma>n$. Then $(1+|x|)^{-\gamma}$ belongs to the Lorentz space $\mathrm{L}^{n/\gamma}_w(\mathbb{R}^n)$. Thus, $a\in\mathrm{H}^{n/\gamma}_w$ and, by (4.39), $a\in\mathrm{B}_1^{-(\gamma-n),\infty}(\mathbb{R}^n)$. We now consider the Littlewood-Paley decompositions $\sum_j\Delta_j g_t$ and $\sum_j\Delta_j a$, respectively, of the gaussian and a. The two series obviously converge in $\mathcal{S}'(\mathbb{R}^n)$. Then (4.40) follows from the straightforward duality argument below:

$$\begin{aligned}
\|e^{t\Delta}a\|_\infty &\le C\sum_{j=-\infty}^{\infty}\|(\Delta_j a)*(\Delta_j g_t)\|_\infty\\
&\le C\sup_j 2^{-j(\gamma-n)}\|\Delta_j a\|_1\Big(\sum_{j=-\infty}^{\infty}2^{j(\gamma-n)}\|\Delta_j g_t\|_\infty\Big)\\
&\le C\|a\|_{\dot{\mathrm{B}}_1^{-(\gamma-n),\infty}}\|g_t\|_{\dot{\mathrm{B}}_\infty^{(\gamma-n),1}}\le Ct^{-\gamma/2},
\end{aligned}$$

In the case $1\le\gamma<n$ we do not need Besov spaces. We simply observe that $a\in\mathrm{L}^{n/\gamma}_w(\mathbb{R}^n)$ and the conclusion follows from the duality of Lorentz spaces. □

When $|a(x)|\le C(1+|x|)^{-\gamma}$, the estimate $\sup_{t\ge0}|e^{t\Delta}a(x)|\le c(1+|x|)^{-\gamma}$ follows easily if *e.g.* $\int(1+|x|)^{[\gamma-n]}|a(x)|\,dx<\infty$ and the moments of a vanish, up to the order $[\gamma-n]$ (or $\gamma-n-1$, if γ is integer). We point out that in this slightly more restrictive case, the estimate $|e^{t\Delta}a(x)|\le C(1+t)^{-\gamma/2}$ can be proved by simple computations (without the using of Hardy or Besov spaces). We refer to [27] for more details.

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Abstract. We study the large time behavior of the energy of a class of Navier–Stokes flows in $\mathbb{R}^n$ $(n \geq 2)$ with special symmetries. Inside this class, we construct examples of solutions such that the energy norm decays faster than $t^{-(n+2)/4}$.

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