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L^p-Solutions of the Steady-State Navier–Stokes Equations with Rough External Forces

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In this paper we address the existence, the asymptotic behavior and stability in L^p and $L^{p,\infty}$, $\frac{3}{2} < p \leq \infty$, for solutions to the steady state 3D Navier–Stokes equations with possibly very singular external forces. We show that under certain smallness conditions of the forcing term there exists solutions to the stationary Navier–Stokes equations in L^p spaces, and we prove the stability of these solutions. Namely, we prove that such small steady state solutions attract time dependent solutions with large initial velocity driven by the same forcing. We also give non-existence results of stationary solutions in L^p, for $1 \leq p \leq \frac{3}{2}$.

Keywords Steady Navier–Stokes.

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1. Introduction

In this paper we consider the solutions to the three-dimensional steady state Navier–Stokes equations in the whole space $\mathbb{R}^3$,

$$\begin{cases} \nabla \cdot (U \otimes U) + \nabla P = \Delta U + f \\ \nabla \cdot U = 0. \end{cases} \tag{1.1}$$

Here $U = (U_1, U_2, U_3)$ is the velocity, P the pressure and $f = (f_1, f_2, f_3)$ a given time independent external force. Equation (1.1) will be complemented with a boundary condition at infinity of the form $U(x) \to 0$ in a weak sense: typically, we express this condition requiring that U belongs to some L^p spaces. Three problems will be addressed.

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We will first establish the existence of solutions $U \in L^p$, with $\frac{3}{2} < p \leq \infty$, to equations (1.1) for (small) functions f as general as possible, and non-existence results in the range $1 \leq p \leq \frac{3}{2}$.

Next we will study the asymptotic properties as $|x| \to \infty$ for a relevant subclass of the solutions obtained.

The third problem at hand is the stability of the solutions in the sense of solutions to (1.1) being "fixed point" in L^p to the non-stationary incompressible Navier–Stokes equations in $\mathbb{R}^3$

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + f \\ \nabla \cdot u = 0 \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where u, p are the time dependent velocity and pressure of the flow. We assume f to be constant in time, but our methods could also be applied to the more general case of time dependent forces suitably converging to a steady state forcing term. We will show that *small* stationary solutions U of (1.1) will attract all global non-stationary solutions u to (1.2) verifying mild regularity conditions, and emanating from possibly large data u_0. This will be achieved by first proving that a wide class of global solutions of (1.2) must become small in $L^{3,\infty}$ after some time, and then applying the stability theory of small solutions in $L^{3,\infty}$ as developed, e.g., in [9, 19, 29]. In addition, for small solutions, we will extend the results on the stability in the existing literature by giving necessary and sufficient conditions to have $u(t) \to U$ in L^p as $t \to \infty$.

The existence and stability of stationary solutions is well understood in the case of bounded domains. See for example [10]. For related results in exterior domains we refer the reader to [11–13, 16]. A wider list of references regarding connected literature can be found in [3]. For example, the existence and the stability of stationary solutions in L^p with $p \geq n$, where n is the dimension of the space, is obtained in [23], under the condition that the Reynolds number is sufficiently small, and in [19, 29] under the assumption that the external force is small in a Lorentz space. Similar results in the whole domain $\mathbb{R}^n$, always for $p \geq n$, have been obtained also in [8, 9, 18].

On the other hand, not so much can be found in the literature about the existence and stability of stationary solutions in $\mathbb{R}^n$ with $p < n$. This problem have been studied recently in the case $n = 3$ and $p = 2$ in [3]. In this paper we extend the results of [3] to the range $\frac{3}{2} < p \leq \infty$, and improve such results also in the case $p = 2$ by considering a more general class of forcing functions. The methods in this paper differ completely from the ones used in [3]. In the former paper the construction of solutions with finite energy was based on a well known formal observation: if Φ is the fundamental solution for the heat equation then $\int_0^\infty \Phi(t, \cdot)dt$ is the fundamental solution for Poisson's equation. Using that idea it was possible to make a time dependent PDE similar to the Navier–Stokes equation with f as initial data with a solution that can be formally integrated in time to find a solution of (1.1).

As we shall see, the conditions on f in the present paper which yield that $U \in L^p$ are, essentially, necessary and sufficient. This will be made possible by a systematic use of suitable function spaces.

One could also complement the system (1.1) with different type of boundary condition at infinity. For example, conditions of the form $U(x) \to U_\infty$ as $|x| \to \infty$, where $U_\infty \in \mathbb{R}^3$ and $U_\infty \neq 0$ are also of interest. However the properties of stationary solutions satisfying such condition are already quite well understood. We refer to the treatise of Galdi [14] for a comprehensive study of this question.

On the other hand, the understanding of the problem in the case $U_\infty = 0$ is less satisfactory. For example, the construction of solutions obeying to the natural energy equality (obtained multiplying the equation (1.1) by U and formally integrating by parts), without putting any smallness assumption on f, is still an open problem. The main difficulty, for example when $\Omega = \mathbb{R}^3$ (or when Poincaré's inequality is not available), is that the usual *a priori* estimate on the Dirichlet integral

$$\|\nabla U\|_{L^2} \leq \|f\|_{\dot{H}^{-1}}$$

ensures only that $U \in \dot{H}^1 \subset L^6$: but to give a sense to the integral in the formal equality

$$\int \left[\nabla \cdot (U \otimes U)\right] \cdot U \, dx = 0$$

one would need, e.g., that U belongs also to L^4.

More generally, one motivation for developing the L^p theory (especially for *low values* of p) of stationary solutions is that this provides additional information on the asymptotic properties of U in the far field. On the other hand, condition like $U \in L^p$ for large p are usually easily recovered via the standard regularity theory, as bootstrapping procedures show that weak solutions $U \in \dot{H}^1$ are regular if f is so. See also [23] for this case.

The paper will be organized as follows. After the introduction we have a section of general notation, where we recall definitions of several function spaces which will be needed in the sequel.

Section 2 deals with the existence of solutions in L^p, $\frac{3}{2} < p \leq \infty$. Section 3 addresses the pointwise behavior in $\mathbb{R}^3$ of the solutions and the asymptotic profiles. We note that the study of the asymptotic profiles has been largely dealt with in the literature, starting with the well known results of Finn [13] in exterior domains. Our results being in the whole domain are simpler, but we are able to get them with weaker conditions. Non-existence results of (generic) solutions in $U \in L^p$, $p \leq \frac{3}{2}$ will also follow from such analysis.

Section four handles the stability of stationary solutions. More precisely in the setting of the Navier–Stokes equation we investigate the stability of the stationary solution U in the L^p and the Lorentz $L^{p,\infty}$-norms. We consider a possibly large $L^{3,\infty}$ non-stationary solution and a stationary solution $U \in L^{3,\infty} \cap L^p$ or $U \in L^{3,\infty} \cap L^{p,\infty}$ which is small in $L^{3,\infty}$. We show that the non-stationary solution eventually becomes small in $L^{3,\infty}$ (but does not converge to 0 in this space), we prove some decay estimates for it and we give a necessary and sufficient condition to have that $u(t) \to U$ in L^p or $L^{p,\infty}$.

The fact that small steady state solutions U attract small non-stationary solutions was proved by several authors in different functional settings, see, e.g. [8, 9, 18, 19, 29]. The main novelty of our approach is that we can prove the same result for a class of large solutions. At best of our knowledge, this was known only in the particular case $U = 0$ (see [1, 15]). Our main tool will be a decomposition criterion for functions in Lorentz-spaces.

1.1. *Notations*

1.1.1. Function Spaces. We recall that the fractional Sobolev spaces (or Bessel potential spaces) are defined, for $s \in \mathbb{R}$ and $1 < p < \infty$, as

$$H_p^s = \{f \in \mathcal{S}'(\mathbb{R}^3) : \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}}\hat{f} \in L^p\},$$

and their homogeneous counterpart is

$$\dot{H}_p^s = \{f \in \mathcal{S}'(\mathbb{R}^3) : \mathcal{F}^{-1}|\xi|^s\hat{f} \in L^p\}.$$

Their differential dimension is $s - \frac{3}{p}$. We will only deal with the case $s - \frac{3}{p} < 0$, so that the elements of $\dot{H}_p^s$ can indeed be realized as tempered distributions. As usual, we will simply write H^s and $\dot{H}^s$ instead of H_2^s and $\dot{H}_2^s$ for the classical Sobolev spaces.

The fractional Sobolev spaces can be identified with particular Triebel-Lizorkin spaces, namely $F_p^{s,2}$ and $\dot{F}_p^{s,2}$. This identification will be useful, because it allows us to handle the limit case for $p = 1$: the corresponding spaces are defined as above, but replacing L^1 with its natural substitute, *i.e.*, the Hardy space $\mathcal{H}^1$. Similarly, in the limit case $p = \infty$ one replaces L^∞ space with BMO. The classical reference for function spaces is [28].

We will make extensive use of the Lorentz spaces $L^{p,q}$, with $1 < p < \infty$ and $1 \leq q \leq \infty$. For completeness we recall their definition.

Let (X, λ) be a measure space. Let f be a scalar-valued λ-measurable function and

$$\lambda_f(s) = \lambda\{x : f(x) > s\}.$$

Then re-arrangement function f^* is defined as usual by:

$$f^*(t) = \inf\{s : \lambda_f(s) \leq t\}.$$

By definition, for $1 < p < \infty$,

$$L^{p,q}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C}, \text{ measurable} : \|f\|_{L^{p,q}} < \infty\},$$

where

$$\|f\|_{L^{p,q}} = \begin{cases} \dfrac{q}{p}\left[\displaystyle\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q\right]^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup\limits_{t>0}\left\{t^{\frac{1}{p}} f^*(t)\right\}, & \text{if } q = \infty. \end{cases}$$

We note that it is standard to use the above as a norm even if it does not satisfy the triangle inequality since one can find an equivalent norm that makes the space into a Banach space.

In particular, $L^{p,\infty}$ agrees with the weak L^p space (or Marcinkiewicz space)

$$L^{p*} = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ measurable}, \|f\|_{L^{p*}} < \infty\}.$$

The quasi-norm

$$\|f\|_{L^{p*}} = \sup_{t>0} t[\lambda_f(t)]^{\frac{1}{p}}$$

is equivalent to the norm on $L^{p,\infty}$, for $1 < p < \infty$.

Our measure λ will be chosen to be the Lebesgue measure. The Lebesgue measure of a set A will be denoted by mes(A). For basic properties of these spaces useful reference are also [20, 30]. It is well-known that the space $L^{p,q}$, $1 < p < \infty$ and $1 \le q \le \infty$, is the interpolated space $L^{p,q} = [L^1, L^\infty]_{1-\frac{1}{p},q}$. Here $[\cdot,\cdot]_{1-\frac{1}{p},q}$ denotes the interpolated space by the real interpolation method. Using the reiteration theorem for interpolation, see [20, Theorem 2.2], one has that $L^{p,q} = [L^{p_1,q_1}, L^{p_2,q_2}]_{\theta,q}$ for all $1 < p_1 < p_2 < \infty$, $1 \le q, q_2, q_2 \le \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. In particular, one has that $L^{p_1,q_1} \cap L^{p_2,q_2} \subset L^{p,q}$ for all $1 < p_1 < p < p_2 < \infty$ and $1 \le q, q_2, q_2 \le \infty$. The Hölder inequality in Lorentz spaces can be stated in the following form.

Proposition 1.1. *Suppose that*

$$1 < p, p_1, p_2 < \infty, \quad 1 \le q, q_1, q_2 \le \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then the pointwise product is a bounded bilinear operator from $L^{p_1,q_1} \times L^{p_2,q_2}$ to $L^{p,q}$, from $L^{p,q} \times L^\infty$ to $L^{p,q}$ and from $L^{p,q} \times L^{p',q'}$ to L^1 where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

The proof of this proposition can be found in [20, Proposition 2.3]. The similar property for convolution is proved in [20, Proposition 2.4] and reads as follows.

Proposition 1.2. *Assume that*

$$1 < p, p_1, p_2 < \infty, \quad 1 \le q, q_1, q_2 \le \infty, \quad 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then the convolution is a bounded bilinear operator from $L^{p_1,q_1} \times L^{p_2,q_2}$ to $L^{p,q}$, from $L^{p,q} \times L^1$ to $L^{p,q}$ and from $L^{p,q} \times L^{p',q'}$ to L^∞ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

We also recall the definition of the Morrey–Campanato spaces. In their homogeneous version, for $1 \le q \le p$, their elements are all the $L^q_{\mathrm{loc}}(\mathbb{R}^3)$ functions f satisfying

$$\|f\|_{\mathcal{M}_{p,q}} = \sup_{x_0 \in \mathbb{R}^3} \sup_{R>0} R^{\frac{3}{p}-\frac{3}{q}} \left(\int_{|x-x_0|<R} |f(x)|^q \, dx \right)^{\frac{1}{q}} < \infty.$$

We recall that

$$L^p = L^{p,p} = \mathcal{M}_{p,p} \subset L^{p,\infty} \subset \mathcal{M}_{p,q}, \quad 1 \le q < p < \infty, \tag{1.3}$$

with continuous injections. The $\mathcal{M}_{p,q}$ spaces are of course increasing in the sense of the inclusion as q decreases. On the other hand, the $L^{p,q}$- spaces increase with q.

For $\theta \geq 0$, we introduce the space $\dot{E}_\theta$ of all measurable functions (or vector field) f in $\mathbb{R}^3$, such that

$$\|f\|_{\dot{E}_\theta} \equiv \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} |x|^\theta |f(x)| < \infty.$$

1.1.2. Other Notations. We denote by $\mathbb{P} = \mathrm{Id} - \nabla\Delta^{-1}\mathrm{div}$ the Leray projector onto the divergence-free vector field. Notice that $\mathbb{P}$ is a pseudodifferential operator of order zero, which is bounded in H^s_p, $\dot{H}^s_p$ and $L^{p,q}$, for $1 < p < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$. Thus, when f belongs to those spaces, the validity of an Helmholtz decomposition $f = \mathbb{P}f + \nabla g$ implies that one could assume, without restriction, that f is divergence-free.

However, we will *not make* this assumption in order to avoid unpleasant restrictions, especially when working in weighted spaces (notice that $\mathbb{P}$ is not bounded in $\dot{E}_\theta$) or in L^1. Indeed, it has some interest to consider integrable external forces with non-zero mean, which prevents $\mathrm{div}\, f = 0$.

2. Solutions in $L^p(\mathbb{R}^3)$

The equations (1.1) are invariant by the natural scaling $(U, p, f) \mapsto (U_\lambda, p_\lambda, f_\lambda)$ for all $\lambda > 0$ and $U_\lambda = \lambda U(\lambda\cdot)$, $P_\lambda = \lambda^2 P(\lambda\cdot)$ and $f_\lambda = \lambda^3 f(\lambda\cdot)$. Following a well established procedure, not only for Navier–Stokes, we consider the following program:

(1) Existence: first construct (rough) solutions U in a scaling invariant setting, *i.e.* in a functional space with the same homogeneity of L^3 assuming that the norm of f is small in a function space (as large as possible) with the same homogeneity of L^1.
(2) Propagation: deduce from additional properties of f (oscillations, localization,...) additional properties for U (localization, asymptotic properties,...).

We will not discuss the propagation of the regularity since this issue is already well understood (see [14]). For example for, not necessarily small, external forces belonging to $\dot{H}^{-1} \cap H^s$, with $s > \frac{3}{2}$, one deduces that solutions with finite Dirichlet integral are twice continuously differentiable and solve (1.1) in the classical sense.

Concerning the first part of this program, in order to give a sense to the nonlinearity one wants to have $U \in L^2_{\mathrm{loc}}$. As noticed in [21], the largest Banach space X, continuously included in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, which is invariant under translations and such that $\|U_\lambda\|_X = \|U\|_X$, is the Morrey–Campanato space $\mathcal{M}_{3,2}$. Therefore, the weakest possible smallness assumption under which one can hope to apply the first part of the program should be

$$\|\Delta^{-1} f\|_{\mathcal{M}_{3,2}} < \varepsilon.$$

However, it seems impossible to prove the existence of a solution under this type of condition. Indeed, $U \otimes U$ would belong to $\mathcal{M}_{\frac{3}{2},1}$, and the singular integrals involved in equivalent formulations of (1.1) are badly behaved in Morrey spaces of L^1_{loc} functions (see the analysis of Taylor [26] and in particular equation (3.37) of his paper).

Here the situation is less favorable than for the free non-stationary Navier–Stokes equations, where the existence of a global in time solution can be ensured if the initial datum of the Cauchy problem is small in $\mathcal{M}_{3,2}$ (or even under more general smallness assumptions, see [20]). The complication, in our case, arises from the lack of the regularizing effect of the heat kernel.

On the other hand, the above difficulty disappears in the slightly smaller spaces $\mathcal{M}_{3,q}$. Indeed, Kozono and Yamazaki established the following result:

Theorem 2.1 (See [18]). *Let $2 < q \leq 3$. Then there exists a positive number δ_q and a strictly monotone function $\omega_q(\delta)$ on $[0, \delta_q]$ satisfying $\omega_q(0) = 0$, such that the following holds*:

- *For every $f \in \mathcal{D}'(\mathbb{R}^3)$ there exists at most one solution U in $\mathcal{M}_{3,q}$ satisfying $\|U\|_{\mathcal{M}_{3,q}} < \omega_q(\delta_q)$.*
- *For every tempered distribution f such that $\Delta^{-1}f \in \mathcal{M}_{3,q}$, and $\delta = \|\Delta^{-1}f\|_{\mathcal{M}_{3,q}} < \delta_q$, there exists a solution $U \in \mathcal{M}_{3,q}$ of* (1.1), *such that $\|U\|_{\mathcal{M}_{3,q}} \leq \omega_q(\delta)$.*

This result provides a satisfactory answer to Part 1 of the above program, but it seems difficult to make progress in Part 2 using such functional setting. For example, a very strong additional condition like $f \in \mathcal{S}_0(\mathbb{R}^3)$ (the space of functions in the Schwartz class with vanishing moments of all order), and f small, but only in the $\mathcal{M}_{3,q}$-norm (with $2 < q < 3$), seems to imply no interesting asymptotic properties for U (such as $U \in L^p$ with low p).

On the other hand the $\mathcal{M}_{3,q}$ spaces, as $q \uparrow 3$, become very close to $L^{3,\infty}$ as can be seen from relation (1.3). The purpose of our first theorem is to show that one can obtain propagation results according to Part 2 of our program, by strengthening a little the smallness assumption, and requiring that

$$\|\Delta^{-1}f\|_{L^{3,\infty}} < \varepsilon_1. \tag{2.1}$$

The continuous embedding of L^3 into the weak space $L^{3,\infty}$ implies that condition (2.1) will be fulfilled if, *i.e.*, $f \in \dot{H}_3^{-2}$ with small $\dot{H}_3^{-2}$-norm. Moreover, the continuous embedding

$$\dot{H}^{-\frac{3}{2}} \subset \dot{H}_3^{-2}$$

shows that the case of forces $f \in \dot{H}^{-\frac{3}{2}}$ with small $\dot{H}^{-\frac{3}{2}}$-norm is also encompassed by (2.1).

We now state our first theorem.

Theorem 2.2. *There exists an absolute constant $\varepsilon_1 > 0$ with the following properties*:

- *If $f \in \mathcal{S}'(\mathbb{R}^3)$ is such that $\Delta^{-1}f \in L^{3,\infty}$ and satisfying condition* (2.1), *then there exists a solution $U \in L^{3,\infty}$ of* (1.1) *such that*

$$\|U\|_{L^{3,\infty}} \leq 2\|\Delta^{-1}\mathbb{P}f\|_{L^{3,\infty}}. \tag{2.2}$$

  (*The uniqueness holds in the more general setting of Theorem* 2.1).
- *Let $\frac{3}{2} < p < \infty$. If U is the above solution then we have more precisely*

$$U \in L^{3,\infty} \cap L^p \quad \textit{if and only if} \quad \mathbb{P}f \in \dot{H}_p^{-2}.$$

In this case (and if $p \neq 3$) $U \in L^q$ for all q such that $3 < q \leq p$ (or $p \leq q < 3$). Moreover, U belongs to $L^{3,\infty} \cap L^\infty$ (respectively, $U \in L^{3,\infty} \cap \mathrm{BMO}$) if and only if $\Delta^{-1}\mathbb{P}f \in L^\infty$ (respectively, $\Delta^{-1}\mathbb{P}f \in \mathrm{BMO}$).

Remark 2.3. Important examples of solutions that can be obtained through this theorem are those corresponding to external forces $f = (f_1, f_2, f_3)$ with components of the with components of the form $\varepsilon\delta$, where δ is the Dirac mass at the origin. Notice that $f \notin \dot{H}^{-3/2}$. However, assumption (2.1) is fulfilled, because $\Delta^{-1}f(x) = \frac{\varepsilon}{|x|}(c_1, c_2, c_3)$.

In fact, due to the invariance under rotations of (1.1), in this case one one can always fix a coordinate system in a way such that $f = (\varepsilon\delta, 0, 0)$. The solutions that one obtains in this way are well-known: they are the axi-symmetric solutions (around the x_1 axis) discovered by Landau sixty years ago, with ordinary differential equations methods. These are solutions that are singular at the origin—in fact the components of the velocity field are homogeneous functions of degree -1 — and smooth outside zero. They can also be seen as self-similar stationary solutions of the non-stationary Navier–Stokes equations.

We refer to [8] for an explicit expressions and other interesting properties about these solutions and to [25] (see also [27]) for related uniqueness results.

Remark 2.4. The particular case $p = 2$ is physically relevant since it corresponds to finite energy solutions. The conclusion $U \in L^2$ was obtained by Bjorland and Schonbek [3], under a technical smallness assumption non invariant under scaling. Part (2) of Theorem 2.2 improves their result. Indeed the same conclusion can be reached under the more general conditions (2.1) and $f \in \dot{H}^{-2}$. In particular, it follows that $f \in \dot{H}^{-\frac{3}{2}} \cap \dot{H}^{-2}$ with f small in $\dot{H}^{-\frac{3}{2}}$ would be enough to get $U \in L^2$. This fact was pointed out to the first and the last author by an anonymous referee of their paper [3].

Roughly speaking, for $f \in \dot{H}^{-\frac{3}{2}}$, the additional requirement $f \in \dot{H}^{-2}$ (which turns out to be also necessary for obtaining $U \in L^2$, up to a modification of f with an additive potential force, which in any case would change only the pressure of the flow), is formally equivalent to the additional vanishing condition $\hat{f}(\xi) = o(|\xi|^{\frac{1}{2}})$ as $|\xi| \to 0$.

Remark 2.5. The first conclusion of Theorem 2.2 bears some relations with the work of Kozono and Yamazaki [19] and Yamazaki [29], where they also obtained existence results of (possibly non-stationary) solutions in Lorentz-spaces and in unbounded domains. However, the assumptions in [19, 29] on the external force reads $f = \operatorname{div} F$, with F small in $L^{\frac{3}{2},\infty}$. Their condition is more stringent than our condition (2.1) because it involves more regularity (one more derivative, or more precisely, one less anti-derivative) on f.

The first part of Theorem 2.2 is also related to the work of Cannone and Karch [9]. There, the authors constructed non-stationary solutions of Navier–Stokes in the whole space in $L^\infty_t(L^{3,\infty})$ with initial data small in $L^{3,\infty}$ and external force such that

$$\sup_{t>0}\left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}f(s)\,ds \right\|_{L^{3,\infty}}$$

is small. With some modifications of their proofs it would be possible to deduce the first conclusion of our theorem from their result, by considering time-independent external forces (in this case the above condition boils down to (2.1)). We prefer however to give a self-contained proof directly in the stationary case, because this allows us to obtain necessary and sufficient conditions. Moreover, none of these papers addressed the construction of solution in L^p with $p < 3$.

We recall a well known fixed point Lemma for bilinear forms that will be needed in the sequel. The proof can be found in [7].

Lemma 2.6. *Let X be a Banach space and* $B : X \times X \to X$ *a bilinear map. Let* $\|\cdot\|_X$ *denote the norm in X. If for all* $x_1, x_2 \in X$ *one has*

$$\|B(x_1, x_2)\|_X \leq \eta \|x_1\|_X \|x_2\|_X.$$

Then for all $y \in X$ *satisfying* $4\eta\|y\|_X < 1$, *the equation*

$$x = y + B(x, x),$$

has a solution $x \in X$ *satisfying and uniquely defined by the condition*

$$\|x\|_X \leq 2\|y\|_X.$$

Remark 2.7. The proof of this lemma also shows that $x = \lim_{k\to\infty} x_k$ where the approximate solutions x_k are defined by $x_0 = y$ and $x_k = y + B(x_{k-1}, x_{k-1})$. Moreover $\|x_k\|_X \leq 2\|y\|_X$ for all k.

Proof of Theorem 2.2. We use a method of mixed bilinear estimates, inspired from [17]. Let us set

$$U_0 \equiv -\Delta^{-1}\mathbb{P}f, \quad B(U, V) \equiv \Delta^{-1}\mathbb{P}\nabla \cdot (U \otimes V).$$

Then the system (1.1) can be rewritten as

$$U = U_0 + B(U, U) \tag{2.3}$$

and the solutions of this equations are indeed weak solutions of (1.1). This equation can be solved applying the standard fixed point method as described in Lemma 2.6 in space $L^{3,\infty}$. We have the estimate

$$\|B(U, V)\|_{L^{3,\infty}} \leq C_1 \|U\|_{L^{3,\infty}} \|V\|_{L^{3,\infty}}, \tag{2.4}$$

for some $C_1 > 0$ independent on U and V. Note that an estimate similar to (2.4) has been proved e.g. by Meyer in [21] in the case of the non-stationary Navier–Stokes equations (the bilinear operator B is slightly different in that case).

To prove (2.4), we only have to observe that the symbol $\hat{m}(\xi)$ of the pseudo-differential operator $\Delta^{-1}\mathbb{P}\,\mathrm{div}$ is a homogeneous function of degree -1, such that $\hat{m}(\xi) \in C^\infty(\mathbb{R}^3\backslash\{0\})$. Thus, the corresponding kernel m is a homogeneous function of degree -2, smooth outside the origin (more precisely $m = (m_{j,h,k})_{j,h,k=1,2,3}$ and $m_{j,h,k}$ are homogeneous functions of degree -2).

In particular,

$$B(U, V) = m(D)(U \otimes V) \quad \text{with } m \in L^{\frac{3}{2},\infty}.$$

Thence,

$$\|m(D)v\|_{L^{p_2,q_1}} \le C(p_1, q_1)\|v\|_{L^{p_1,q_1}}, \quad \frac{1}{p_2} = \frac{1}{p_1} + \frac{2}{3} - 1, \quad \begin{cases} 1 < p_1 < 3, \\ 1 \le q_1 \le \infty, \end{cases} \tag{2.5}$$

by the Young inequality stated in Proposition 1.2. Applying this to $v = U \otimes V$ and using Proposition 1.1 to deduce that for $U, V \in L^{3,\infty}$ one has that $v \in L^{\frac{3}{2},\infty}$, we get estimate (2.4) with $C_1 = C(\frac{3}{2}, \infty)$. Hence by Lemma 2.6 it follows that, provided that $4\|U_0\|C_1 < 1$, there exists a solution of (2.3) satisfying (2.2).

To prove Part 2, we make use of approximate solutions of $\Phi(U) = U_0 + B(U, U)$. That is we choose a sequence satisfying $U_k = U_0 + B(U_{k-1}, U_{k-1})$ and use Remark 2.7 to state that $U_k \to U$ in $L^{3,\infty}$ as $k \to \infty$. We show now that

$$\|B(U_k, U_k)\|_{L^p} \le C(p)\|U_k\|_{L^{3,\infty}}\|U_k\|_{L^p}, \quad \frac{3}{2} < p < \infty, \tag{2.6}$$

valid for some positive function $p \mapsto C(p)$, continuous on $(\frac{3}{2}, \infty)$. To obtain this estimate we use the Hölder inequality given in Proposition 1.1 to deduce that $\|U_k \otimes U_k\|_{L^{\frac{3p}{3+p},p}} \le C_2(p)\|U_k\|_{L^p}\|U_k\|_{L^{3,\infty}}$. Relation (2.5) for $p_1 = \frac{3p}{3+p}$ and $q_1 = p$ completes the proof of (2.6).

By Part 1, applied to the approximations U_k, we know that $\|U_k\|_{L^{3,\infty}} \le 2\|U_0\|_{L^{3,\infty}}$. Choose $\|U_0\|_{L^{3,\infty}} \le c_0\varepsilon_1$, then we get from (2.6), for $\frac{3}{2} < p < \infty$

$$\|U_{k+1}\|_{L^p} \le \|U_0\|_{L^p} + 2c_0C(p)\varepsilon_1\|U_k\|_{L^p}. \tag{2.7}$$

If $\mathbb{P}f \in \dot{H}_p^{-2}$, then $U_0 \in L^p$ and so, by induction, $\|U_k\|_{L^p} < \infty$ for all k. Provided $2c_0C(p)\varepsilon_1 < 1$, iterating inequality (2.7) implies that U_k is uniformly bounded in L^p with respect to k, and hence $U \in L^p$.

However, $C(p)$ blows up as $p \to \frac{3}{2}$ or $p \to \infty$, and we want to have a smallness assumption independent of p. To circumvent this difficulty, we replace, if necessary, the constant ε_1 of Part 1 of the theorem with a smaller absolute constant (still denoted ε_1), in a such way that $2c_0\varepsilon_1 < 1/\sup_{2\le p\le 7} C(p)$. Then the above argument yields the conclusion of the "if part" of the theorem in the case $2 \le p \le 7$. To prove the "only if" part one simply uses estimate (2.6) with $U_k = U$ together with (2.3) to get $U_0 = -\Delta^{-1}\mathbb{P}f \in L^p$, hence $\mathbb{P}f \in \dot{H}_p^{-2}$.

Let us now consider the case $\mathbb{P}f \in \dot{H}_p^{-2}$, $\frac{3}{2} < p < 2$. Then $U_0 \in L^p \cap L^{3,\infty}$ and by interpolation $U_0 \in L^2$, so using the case $2 \le p \le 7$ we get that $U \in L^2$. On the other hand, according to Proposition 1.2 the space $L^{\frac{3}{2},\infty}$ is stable under convolution with L^1-functions so

$$B(U, U) = m(D)(U \otimes U) \in L^{\frac{3}{2},\infty}.$$

But from estimate (2.4) we know that $B(U, U) \in L^{3,\infty}$. By interpolation, $B(U, U) \in L^p$. Combining this with equality (2.3) yields $U \in L^p$. Conversely, suppose that $U \in L^p$. Since we already know that the solution $U \in L^{3,\infty}$, we deduce by interpolation that $U \in L^2$. The argument above shows that $B(U, U) \in L^p$. Hence by (2.4) it follows that $U_0 \in L^p$, and this, in turn is equivalent to $\mathbb{P}f \in \dot{H}_p^{-2}$.

We now consider the case $U_0 \in L^p$ with $7 < p \leq \infty$ (this is equivalent to $\mathbb{P}f \in \dot{H}_p^{-2}$ if $p < \infty$). Since $U_0 \in L^{3,\infty}$, we have by interpolation that $U_0 \in L^4 \cap L^7$. From the previous case, we infer that $U \in L^4 \cap L^7$. By interpolation, we also have that $U \in L^{6,2}$. Using Proposition 1.1 this implies that $U \otimes U \in L^{3,1}$, so from Proposition 1.2 and recalling $m \in L^{\frac{3}{2},\infty}$ we get that

$$B(U, U) = m(D)(U \otimes U) \in L^\infty.$$

But we also know that $B(U, U) \in L^{3,\infty}$, so by interpolation $B(U, U) \in L^p$. From (2.3) we conclude that $U \in L^p$. The same argument also shows that $U \in L^p$ implies $U_0 \in L^p$.

Finally, the BMO case follows in the same way. Indeed, the argument above shows that if U_0 or U belong to BMO, then $B(U, U) \in L^\infty$. But $L^\infty \subset$ BMO, so $B(U, U) \in$ BMO. From relation (2.3) we see that $U \in$ BMO iff $U_0 \in$ BMO. This completes the proof of Theorem 2.2. □

Remark 2.8. With the same proof, one can show the following equivalent condition for the stationary solution U constructed in Theorem 2.2 to belong to $L^{p,r}$. If $p \in (\frac{3}{2}, \infty)$ and $r \in [1, \infty]$ then $U \in L^{p,r}$ if and only if $\Delta^{-1}\mathbb{P}f \in L^{p,r}$.

3. Pointwise Behavior in $\mathbb{R}^3$ and Asymptotic Profiles

In the previous section we dealt with forces such that $\Delta^{-1}f \in L^{3,\infty}$. Since the typical example of a function in $L^{3,\infty}$ is $|x|^{-1}$, it is natural to ask which supplementary properties are satisfied when $|\Delta^{-1}f(x)| \leq \varepsilon|x|^{-1}$. below provides a rather complete answer.

In particular, we will obtain exact asymptotic profiles in the far field for decaying solutions of (1.1). Starting with the work of Finn (see [13] and the references therein), a lot is known about the spatial asymptotics of stationary solutions in unbounded domains. The case of the whole space that we treat in this section is of course simpler than the case of exterior domains or aperture domains considered e.g. in [14]. Nevertheless, focusing on this case allow us to put weaker (and more natural) smallness assumptions on the force, thus providing a more transparent presentation of the problem.

We observe here that, despite the unboundedness of $\mathbb{P}$ in the $\dot{E}_\theta$ spaces, it is fairly easy to ensure e.g. that $\Delta^{-1}\mathbb{P}f \in \dot{E}_1$. Indeed, one has for example that

$$\|\Delta^{-1}\mathbb{P}f\|_{\dot{E}_1} \leq C(\|f\|_{\dot{E}_3} + \|f\|_{L^1}). \tag{3.1}$$

Notice that all the norms in inequality (3.1) are invariant under scaling. The above inequality can be proved with a simple size estimate (using that $\Delta^{-1}\mathbb{P}$ is a convolution operator with a kernel $\tilde{m}$ satisfying $|\tilde{m}(x)| \leq C|x|^{-1}$). The same conclusion $\Delta^{-1}\mathbb{P}f \in \dot{E}_1$ can be obtained also via the Fourier transform (using classical results in [24]), assuming, e.g., $f = \nabla \cdot F$ where F is a two dimensional tensor with homogeneous components of degree -2, smooth outside the origin.

Let us recall the imbedding $\dot{E}_1 \hookrightarrow L^{3,\infty}$, thus a smallness assumption in the space $\dot{E}_1$ implies a smallness assumption in $L^{3,\infty}$.

The spirit of Theorem 3.1 below is close to a previous work of the second author (see [6]) in which similar conclusions are shown for the time-dependent Navier–Stokes equation in the whole space.

Theorem 3.1. *There exists an absolute constant* $\varepsilon_2 > 0$ (*with* ε_2 a priori *smaller than the constant* ε_1 *of Theorem* 2.2) *such that*:

- *If* $f \in \mathcal{S}'(\mathbb{R}^3)$ *is such that* $\Delta^{-1}\mathbb{P}f \in \dot{E}_1$ *and* $\|\Delta^{-1}\mathbb{P}f\|_{\dot{E}_1} < \varepsilon_2$, *then the solution* $U \in L^{3,\infty}$ *obtained in Theorem* 2.2 *satisfies*

$$\|U\|_{\dot{E}_1} \leq 2\|\Delta^{-1}\mathbb{P}f\|_{\dot{E}_1}.$$

- *Let* $0 \leq \theta \leq 2$. *Under the additional assumption* $\Delta^{-1}\mathbb{P}f \in \dot{E}_\theta$, *we have also* $U \in \dot{E}_\theta$.
- *In particular, if* $\Delta^{-1}\mathbb{P}f \in \dot{E}_0 \cap \dot{E}_2$, *with small* $\dot{E}_1$-*norm, then* U *satisfies the pointwise estimate*

$$|U(x)| \leq C(1+|x|)^{-2}.$$

In this case the solution U *has the following profile as* $|x| \to \infty$:

$$U(x) = -\Delta^{-1}\mathbb{P}f(x) + m(x) : \left(\int U \otimes U\right) + O\big(|x|^{-3}\log(|x|)\big), \tag{3.2}$$

where $m = (m_{j,h,k})$ *is the kernel of* $\Delta^{-1}\mathbb{P}\,div$ *and* $m_{j,h,k}(x)$ *are homogeneous functions of degree* -2, C^∞ *outside zero. Furthermore,*

$$m(x) : \left(\int U \otimes U\right) \equiv 0 \quad \text{if and only if} \quad \exists c \in \mathbb{R} \text{ s.t. } \int U_h U_k = c\delta_{h,k}, \tag{3.3}$$

for $h, k = 1, 2, 3$, *where* $\delta_{h,k} = 0$ *or* 1 *if* $h \neq k$ *or* $h = k$.

Remark 3.2. Let us be more explicit with our notation: by definition, for $j = 1, 2, 3$,

$$\left[m(x) : \int (U \otimes U)\right]_j = \sum_{h,k=1}^{3} m_{j,h,k}(x)\left(\int U_h(y)U_k(y)dy\right).$$

Moreover $m_{j,h,k}(x) = \partial_h M_{j,k}(x)$, where $M_{j,k}$ is the tensor appearing in the fundamental solution of the Stokes equation. The computation of M goes back to Lorentz (1896). See [14, Vol. I, p. 190] for the explicit formula.

Remark 3.3. For example, it follows from this theorem that, if $f \in \mathcal{S}(\mathbb{R}^3)$ is such that $0 \notin \operatorname{supp}\hat{f}$ and f satisfies the previous smallness assumption, then

$$U(x) \simeq m(x) : \left(\int U \otimes U\right), \quad \text{as } |x| \to \infty$$

provided that the right-hand side does not vanish. Indeed, we have in this case $\Delta^{-1}\mathbb{P}f \in \mathcal{S}(\mathbb{R}^3)$. In particular $|U(x)| \leq C(1+|x|)^{-2}$. But the improved estimate $U(x) = o(|x|^{-2})$ as $|x| \to \infty$ holds if and only if the flow satisfies the orthogonality relations (3.3). Of course, generically it is not the case. This implies the optimality of the restriction $\theta \leq 2$ in Theorem 3.1 as well as the optimality of the restriction $p > \frac{3}{2}$ appearing in Theorem 2.2. It is possible to relax the condition that $0 \notin \operatorname{supp}\hat{f}$ assuming, instead that $|\hat{f}(\xi)| \leq C|\xi|^k$ for a sufficiently large $k > 0$. As noticed in [3], this is essentially an oscillatory condition on f, describing the large time behavior of the solution of the Cauchy problem for the heat equation.

Remark 3.4. Examples of (exceptional) stationary flows satisfying the orthogonality relations (3.3), and such that $U(x) = O\big(|x|^{-3}\log(|x|)\big)$, are easily constructed by taking f satisfying the assumptions of the previous remark and additional suitable symmetries. An axi-symmetry condition would not be enough: one rather needs here polyhedral-type symmetries. The suitable symmetries to be imposed on f can be classified exactly as done in [4], in the case of the non-stationary Navier–Stokes equations. For example the two conditions $Rf(x) = f(Rx)$ and $Sf(x) = f(Sx)$ where R, S are the orthogonal transformations in $\mathbb{R}^3$ $R : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$ and $S : (x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3)$ are sufficient. See [4] for explicit examples of this type of vector fields.

On the other hand, explicit examples of solutions $U = U_f$ which *do not* satisfy the orthogonality relations can be obtained simply by taking $f = \eta f_0$ with $\eta > 0$ sufficiently small and $f_0 \in \dot{H}^{-2}$ satisfying the conditions of Part 2 of Theorem 2.2 with $p = 2$ (this implies that $U_{f_0} \in L^2$). If, in addition, *there is no* $c \in \mathbb{R}$ such that

$$\int (\Delta^{-1}\mathbb{P}f_0)_h(\Delta^{-1}\mathbb{P}f_0)_k = c\delta_{h,k},$$

then U_f cannot satisfy the orthogonality relations, provided $\eta > 0$ is small enough. The proof of this claim relies on an argument that has been used in [5] in the setting of the non-stationary Navier–Stokes equations. These observations lead us to the following theorem, containing the announced non-existence result of generic solutions in L^p, $p \leq \frac{3}{2}$.

Theorem 3.5. *Let $f_0 = (f_1, f_2, f_3)$ be a divergence-free vector field such that $\hat{f} \in C_0^\infty(\mathbb{R}^3)$ and $0 \notin supp(\hat{f})$. Assume also that the matrix*

$$\left(\int \frac{(\hat{f}_0)_j\overline{(\hat{f}_0)_k}}{|\xi|^4}\,d\xi\right)_{j,k}$$

is not a scalar multiple of the identity. Then there exists $\eta_0 > 0$ such that the solution of (1.1) *with $f = \eta f_0$ and $0 < \eta \leq \eta_0$ satisfies,*

$$c\left(\frac{x}{|x|}\right)|x|^{-2} \leq |U(x)| \leq C|x|^{-2}, \quad |x| \gg 1, \tag{3.4}$$

where $C > 0$ is independent on x and $c(\frac{x}{|x|}) > 0$ on a set of positive surface measure on the unit sphere. In particular, $U \notin L^p(\mathbb{R}^3)$ for all $1 \leq p \leq \frac{3}{2}$.

Proof of Theorem 3.1. We already have, by Theorem 2.2, a solution in $L^{3,\infty}$. To see that such solution belongs more precisely, to $\dot{E}_1$ we only have to prove the estimate

$$\|B(U, V)\|_{\dot{E}_1} \leq C\|U\|_{\dot{E}_1}\|V\|_{\dot{E}_1}, \tag{3.5}$$

for some $C > 0$ independent on U and V. Indeed, an application of Lemma 2.6 shows the existence and the uniqueness of the solution U in $\dot{E}_1$. This solution also belongs to $L^{3,\infty}$ since $\dot{E}_1 \subset L^{3,\infty}$. Of course, the re-application of the fixed point argument requires that we replace the constant $\varepsilon_1 > 0$ of Theorem 2.2 by a smaller one. Relation (3.5) is a particular case of the following lemma:

Lemma 3.6. *Let θ_1, θ_2 be two real numbers such that $1 < \theta_1 + \theta_2 < 3$. There exists a constant C such that*

$$\|B(U, V)\|_{\dot{E}_{\theta_1+\theta_2-1}} \leq C\|U\|_{\dot{E}_{\theta_1}} \|V\|_{\dot{E}_{\theta_2}}.$$

Moreover

$$\|B(U, U)\|_{\dot{E}_2} \leq C(\|U\|^2_{\dot{E}_{\frac{3}{2}}} + \|U\|^2_{L^2}).$$

Proof. Recall that $B(U, V) = m * (U \otimes V)$ with m homogeneous of degree -2. Since $|m(x)| \leq C|x|^{-2}$ we can bound

$$B(U, V) = \int m(x-y) : (U \otimes V)(y)dy \leq C\|U\|_{\dot{E}_{\theta_1}} \|V\|_{\dot{E}_{\theta_2}} \int \frac{1}{|x-y|^2|y|^{\theta_1+\theta_2}} dy.$$

It is easy to show that the last integral is a function of $|x|$ homogeneous of order $1 - \theta_1 - \theta_2$, so it can be bounded by $C|x|^{1-\theta_1-\theta_2}$.

To show the second part, we decompose

$$B(U, V) = \left(\int_{|y| \leq \frac{|x|}{2}} + \int_{\frac{|x|}{2} \leq |y|}\right) m(x-y) : (U \otimes V)(y)dy = I_1 + I_2.$$

We have

$$|I_2| \leq C\|U\|^2_{\dot{E}_{\frac{3}{2}}} \int_{\frac{|x|}{2} \leq |y|} \frac{1}{|x-y|^2|y|^3} dy \leq \frac{C}{|x|^2} \|U\|^2_{\dot{E}_{\frac{3}{2}}}$$

where we used the same scaling argument as above to deduce the last inequality. Next, we write for I_1

$$|I_1| \leq C \int_{|y| \leq \frac{|x|}{2}} \frac{1}{|x-y|^2} |U(y)|^2 dy \leq \frac{C}{|x|^2} \int_{|y| \leq \frac{|x|}{2}} |U(y)|^2 dy \leq \frac{C}{|x|^2} \|U\|^2_{L^2}. \qquad \square$$

Let us now prove Part 2 of Theorem 3.1. We have the additional information $\Delta^{-1}\mathbb{P}f \in \dot{E}_\theta$. We argue as in the proof of Theorem 2.2. That is we define $\Phi(U) = U_0 + B(U, U)$ and we choose a sequence satisfying $U_k = \Phi(U_{k-1})$. From Lemma 3.6 we have the estimate

$$\|B(U_k, U_k)\|_{\dot{E}_\theta} \leq C_\theta \|U_k\|_{\dot{E}_1} \|U_k\|_{\dot{E}_\theta}, \quad 0 < \theta < 2,$$

for some positive function $\theta \mapsto C_\theta$, continuous on $(0, 2)$. As in Theorem 2.2 part 2 it follows that the sequence of approximate solutions U_k remains bounded in $\dot{E}_\theta$, provided that $\Delta^{-1}\mathbb{P}f \in \dot{E}_\theta$, for some $\theta \in (0, 2)$, and

$$2C_\theta \|\Delta^{-1}\mathbb{P}f\|_{\dot{E}_1} < 1.$$

The continuity of C_θ allows to obtain the conclusion of the theorem (with a smallness assumption independent on θ), at least for e.g. $\theta \in \left[\frac{1}{2}, \frac{7}{4}\right]$. We had to exclude a neighborhood of $\theta = 0$ and of $\theta = 2$, where C_θ blows-up.

In the case $\frac{7}{4} < \theta \leq 2$, we know that $\Delta^{-1}\mathbb{P}f \in \dot{E}_1 \cap \dot{E}_\theta \subset \dot{E}_1 \cap \dot{E}_{\frac{7}{4}}$. So, from the previous case we deduce that the solution U satisfies $U \in \dot{E}_1 \cap \dot{E}_{\frac{7}{4}} \subset L^2 \cap \dot{E}_{\frac{3}{2}}$. Using again Lemma 3.6 we infer that $B(U, U) \in \dot{E}_2$. But we also know that $B(U, U) \in \dot{E}_1$ so $B(U, U) \in \dot{E}_\theta$. The conclusion in the case $\frac{7}{4} < \theta \leq 2$ now follows from equation (2.3).

It remains to consider the case $0 < \theta < \frac{1}{2}$ (the case $\theta = 0$ is contained in Theorem 2.2, since $\dot{E}_0 = L^\infty$). As above, we show that $U \in \dot{E}_{\frac{1}{2}} \cap \dot{E}_1$ so $U \in \dot{E}_{\frac{\theta+1}{2}}$. From Lemma 3.6 we get that $B(U, U) \in \dot{E}_\theta$ so $U \in \dot{E}_\theta$. The proof of Part 2 of Theorem 3.1 is now completed.

Let us prove Part 3. We will show using decay properties of m and a Taylor expansion that for any solution such that $|U(x)| \leq C(1 + |x|)^{-2}$, we have

$$\Delta^{-1}\mathbb{P}\nabla \cdot (U \otimes U)(x) = m(x) : \int U \otimes U + O\big(|x|^{-3}\log(|x|)\big), \quad \text{as } |x| \to \infty. \tag{3.6}$$

But,

$$\begin{aligned}
\Delta^{-1}\mathbb{P}\nabla \cdot (U \otimes U)(x) &= \int m(x - y) : U \otimes U(y)dy \\
&= m(x) : \int U \otimes U - m(x) : \int_{|y| \geq |x|/2} U \otimes U \\
&\quad + \int_{|y| \leq |x|/2} [m(x - y) - m(x)] : U \otimes U(y)dy \\
&\quad + \int_{|x-y| \leq |x|/2} m(x - y) : U \otimes U(y)dy \\
&\quad + \int_{|y| \geq |x|/2, |x-y| \geq |x|/2} m(x - y) : U \otimes U(y)dy.
\end{aligned}$$

The only properties on the kernel m that we will use are $|m(x)| \leq C|x|^{-2}$ and $|\nabla m(x)| \leq C|x|^{-3}$. We need to show that all the terms on the RHS of the last inequality (excepted the first one) are bounded by $C|x|^{-3}\log|x|$ for large $|x|$. This follow easily since $U \in L^2 \cap \dot{E}_2$. For large $|x|$, the second, the fourth and the last term on the right-hand side are in fact bounded by $C|x|^{-3}$. The third term is bounded by $C|x|^{-3}\log|x|$, for large $|x|$, as it can be checked applying the Taylor formula to m. This implies both the asymptotic profiles (3.6) and (3.2).

To conclude, it remains to show that the homogeneous functions

$$\sum_{h,k} m_{j,h,k}(x) \int U_h U_k, \quad j = 1, 2, 3,$$

vanish identically if and only if the matrix $\int U \otimes U$ is a scalar multiple of the identity. We reproduce a computation similar to that in [6, 22]: taking the Fourier transform, the above vanishing condition is proved to be equivalent to

$$\sum_{h,k} \hat{m}_{j,h,k}(\xi) \int U_h U_k = \sum_h \frac{\mathrm{i}\xi_h}{|\xi|^2} \int U_j U_h - \sum_{h,k} \frac{\mathrm{i}\xi_j\xi_h\xi_k}{|\xi|^4} \int U_h U_k = 0, \quad \text{for a.e. } \xi \in \mathbb{R}^3.$$

The conclusion is now obvious. □

We end this section with the proof of Theorem 3.5.

Proof of Theorem 3.5. We start by choosing η_0 sufficiently small such that

$$\eta_0 \|\Delta^{-1}\mathbb{P} f_0\|_{L^{3,\infty}} \leq \varepsilon_1,$$

where ε_1 is the smallness constant from Theorem 2.2. According to Theorem 2.2, for $0 < \eta \leq \eta_0$ there exists a unique solution $U \in L^{3,\infty} \cap L^2$ of (1.1) with $f = \eta f_0$ such that $\|U\|_{L^{3,\infty}} \leq 2\eta \|\Delta^{-1}\mathbb{P} f_0\|_{L^{3,\infty}}$. It suffices to show that the orthogonality relations (3.3) does not hold true for U.

Let $W_0 = -\Delta^{-1}\mathbb{P} f_0$ and $U_0 = \eta W_0$. The hypothesis implies that the matrix $\int W_0 \otimes W_0$ is not a scalar multiple of the identity. This means that there exists $j \neq k$ such that either $\int W_0^j W_0^k \neq 0$ or $\int |W_0^j|^2 \neq \int |W_0^k|^2$, where W_0^j denotes the jth component of W_0. We will suppose that $\int W_0^j W_0^k \neq 0$, the other case being entirely similar.

We have

$$\begin{aligned} \left| \int U^j U^k - \int U_0^j U_0^k \right| &= \left| \int (U^j - U_0^j) U^k + \int U_0^j (U^k - U_0^k) \right| \\ &\leq \|U - U_0\|_{L^2} (\|U\|_{L^2} + \|U_0\|_{L^2}). \end{aligned} \tag{3.7}$$

From (2.3) and (2.6) with $p = 2$ and U_k replaced by U we deduce that

$$\|U - U_0\|_{L^2} = \|B(U, U)\|_{L^2} \leq C(2)\|U\|_{L^2}\|U\|_{L^{3,\infty}} \leq 2C(2)\eta\|W_0\|_{L^{3,\infty}}\|U\|_{L^2}. \tag{3.8}$$

Therefore

$$\|U\|_{L^2} \leq \|U_0\|_{L^2} + \|U - U_0\|_{L^2} \leq \eta\|W_0\|_{L^2} + 2C(2)\eta_0\|W_0\|_{L^{3,\infty}}\|U\|_{L^2}.$$

If we further strengthen the smallness assumption on η_0 by

$$\eta_0 \leq \frac{1}{4C(2)\|W_0\|_{L^{3,\infty}}}$$

we get that

$$\|U\|_{L^2} \leq 2\eta\|W_0\|_{L^2}.$$

Relation (3.8) combined with the previous estimate implies that

$$\|U - U_0\|_{L^2} \leq 4C(2)\eta^2\|W_0\|_{L^{3,\infty}}\|W_0\|_{L^2}.$$

Using the two previous bounds in (3.7) implies that

$$\left| \int U^j U^k - \eta^2 \int W_0^j W_0^k \right| \leq 12C(2)\eta^3\|W_0\|_{L^{3,\infty}}\|W_0\|_{L^2}^2.$$

Finally

$$\begin{aligned} \left| \int U^j U^k \right| &\geq \eta^2 \left| \int W_0^j W_0^k \right| - \left| \int U^j U^k - \eta^2 \int W_0^j W_0^k \right| \\ &\geq \eta^2 \left| \int W_0^j W_0^k \right| - 12C(2)\eta^3\|W_0\|_{L^{3,\infty}}\|W_0\|_{L^2}^2 > 0 \end{aligned}$$

if we further assume that

$$\eta_0 \le \frac{\left|\int W_0^j W_0^k\right|}{24C(2)\|W_0\|_{L^{3,\infty}}\|W_0\|_{L^2}^2}. \qquad \square$$

4. Stability of the Stationary Solutions

Consider now a mild formulation of the Navier–Stokes equations with time independent forcing function f satisfying, as usual, to a smallness condition as in (2.1),

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}f\,ds - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)ds. \tag{4.1}$$

The two main goals of this section are the following. First we want to establish conditions on u_0 to ensure that the above system has a solution $u \in L^\infty(\mathbb{R}_+, L^{p,\infty})$. Next, we want to find the largest possible class of solutions u to (4.1) for which we can say that $u(t)$ converges to the steady solution U given by Theorem 2.2 corresponding to the same force f. This class will be general enough to include non-stationary solutions in $L^{3,\infty}$ with large initial data. We will show in particular that *a priori* global solutions, verifying a mild regularity condition but initially large in $L^{3,\infty}$, become small in $L^{3,\infty}$ after some time. Only the singularity at infinity of the initial velocity needs to be small in some sense which is made rigorous in (4.17). For example, we allow an initial velocity u_0 bounded by $C/|x|$ everywhere and bounded by $\varepsilon/|x|$ for large x, with C arbitrary and ε small.

We recall that *a priori* large non-stationary solutions in $\dot{B}_{p,q}^{-1+\frac{3}{p}}$ and VMO^{-1} of the Navier–Stokes equations without forcing are known to converge to zero in these spaces (see [1, 15]). However, in our case, convergence to zero will not necessarily hold true for $\|u(t)-U\|_{L^{3,\infty}}$, due to the fact that the smooth function in $\mathscr{S}(\mathbb{R}^3)$ are not dense in $L^{3,\infty}$. Thus, only weaker convergence results should be expected.

Theorem 4.3 collects our results on the stability of small solutions u, extending, for flows in $\mathbb{R}^3$ with time independent forcing term, those of [2, 9, 19, 29] to the case $\frac{3}{2} < p < 3$, and providing some additional information also for $p > 3$. Theorem 4.7 contains the convergence result of large solutions u to small stationary solutions U. Its proof relies on some energy estimates inspired by [1, 15] and on the results on the stability of small solutions prepared in Theorem 4.3.

To begin we first recall a lemma which will be useful for estimating the integral terms on the right-hand side of (4.1) in $L^{p,\infty}$ spaces. We notice that for the case $p=3$, $q=3/2$ the Lemma below was obtained in several papers, among them the first seems to be in Yamazaki's paper [29]. Variants of this lemma can also be found in [21], in a slightly less general form, and in [20].

Lemma 4.1. *Given any $p \in (\frac{3}{2}, \infty)$ let $q = \frac{3p}{p+3}$. For $0 \le \sigma < t$, the operator*

$$\widetilde{L}_\sigma(\phi)(t) = \int_\sigma^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\phi(s)ds$$

satisfies

$$\|\widetilde{L}_\sigma(\phi)(t)\|_{L^{p,\infty}} \le C(p)\sup_{0<s<t}\|\phi(s)\|_{L^{q,\infty}} \tag{4.2}$$

where $C(p)$ denotes a constant independent of σ.

Proof. Let $F(t)$ be the kernel of the operator $e^{t\Delta}\mathbb{P}\,\mathrm{div}$. First recall the rescaling relation

$$F(x,t) = t^{-2}F(x/\sqrt{t}, 1)$$

and that $F(\cdot, 1) \in L^1 \cap L^\infty$.

We consider separately the following two pieces

$$A_1 = \int_{t-\lambda^*}^{t} F(t-s) * \phi(s)ds \quad \text{and} \quad A_2 = \int_{\sigma}^{t-\lambda^*} F(t-s) * \phi(s)ds.$$

The idea of the estimate is to find, given any fixed λ, a λ^* so that $|A_2| < \lambda/2$. With this choice of λ^* we can estimate the Lebesgue measure of the set $\{x : \widetilde{L}_\sigma(\phi)| > \lambda\}$ in terms of A_1 only. In that direction we establish two preliminary estimates. The first is a an application of Young's inequality stated in Proposition 1.2:

$$\|A_2\|_{L^\infty} \le C\int_{\sigma}^{t-\lambda^*} \|F(t-s)\|_{L^{\alpha,1}} \|\phi\|_{L^{q,\infty}} ds.$$

Here, $\alpha = \frac{3p}{2p-3}$. The estimate $\|F(t-s)\|_{L^{\alpha,1}} \le C(t-s)^{-1-\frac{3}{2p}}$ (that follows from the rescaling properties of F) implies

$$\|A_2\|_{L^\infty} \le C(p)(\lambda^*)^{-\frac{3}{2p}} \|\phi\|_{X_q^{\sigma,t}}. \tag{4.3}$$

Here we have introduced the notation $X_q^{\sigma,t} = L^\infty((\sigma,t), L^{q,\infty})$. Similarly, $\|F(t-s)\|_{L^1} \le (t-s)^{-\frac{1}{2}}$ and

$$\|A_1\|_{L^{q,\infty}} \le \int_{t-\lambda^*}^{t} \|F(t-s)\|_{L^1} \|\phi\|_{L^{q,\infty}} ds \le (\lambda^*)^{\frac{1}{2}} \|\phi\|_{X_q^{\sigma,t}}. \tag{4.4}$$

We proceed with the bound for $\|\widetilde{L}_\sigma(\phi)\|_{L^{p,\infty}}$. Using the definition of the norm and the triangle inequality,

$$\|\widetilde{L}_\sigma(\phi)(t)\|_{L^{p,\infty}} \le \sup_{\lambda>0} \lambda\,\mathrm{mes}\{x : |A_1| + |A_2| > \lambda\}^{\frac{1}{p}}.$$

For each $\lambda > 0$ we may choose λ^* such that the right-hand side of (4.3) is equal to $\lambda/2$. With this choice of λ^*,

$$\lambda\,\mathrm{mes}\{x : |A_1| + |A_2| > \lambda\}^{\frac{1}{p}} \le \lambda\,\mathrm{mes}\{x : |A_1| > \lambda/2\}^{\frac{1}{p}}.$$

Also, using (4.4):

$$\lambda\,\mathrm{mes}\{x : |A_1| > \lambda/2\}^{\frac{1}{p}} \le \lambda^{1-\frac{q}{p}} \|A_1\|_{L^{q,\infty}}^{\frac{q}{p}} \le C\|\phi\|_{X_q^{\sigma,t}}.$$

Taking the supremum over all $\lambda > 0$ establishes (4.2). □

The following lemma concerns the large time behavior in $L^{3,\infty}$ of solutions of the heat equation. It will provide a better understanding of the statements of our two next theorems.

Lemma 4.2. *Let $f \in L^{3,\infty}$.*

- *Let $\varepsilon > 0$ be arbitrary. Then f can be decomposed as $f = f_1 + f_2$ with $f_1 \in L^2$ and $\|f_2\|_{L^{3,\infty}} < \varepsilon$ if and only if $\limsup_{R\to 0} R\, mes\{|f| > R\}^{\frac{1}{3}} < \varepsilon$.*
- *If $\lim_{R\to 0} R\, mes\{|f| > R\}^{\frac{1}{3}} = 0$ then $e^{t\Delta} f \to 0$ in $L^{3,\infty}$ as $t \to \infty$.*
- *There exists some $g \in L^{3,\infty}$ such that $e^{t\Delta} g \to 0$ in $L^{3,\infty}$ as $t \to \infty$ and such that $\limsup_{R\to 0} R\, mes\{|g| > R\}^{\frac{1}{3}} \neq 0$.*

Proof. Assume first that $f = f_1 + f_2$ with $f_1 \in L^2$ and $\|f_2\|_{L^{3,\infty}} \leq \varepsilon$. We estimate

$$\text{mes}\{|f_1| > R\} \leq \frac{1}{R^2} \int_{\mathbb{R}^3} |f_1|^2$$

so that $\limsup_{R\to 0} R\,\text{mes}\{|f_1| > R\}^{\frac{1}{3}} = 0$. We also have that

$$\limsup_{R\to 0} R\,\text{mes}\{|f_2| > R\}^{\frac{1}{3}} \leq \sup_{R>0} R\,\text{mes}\{|f_2| > R\}^{\frac{1}{3}} = \|f_2\|_{L^{3,\infty}} < \varepsilon.$$

Let $\delta \in (0,1)$. Since $\{|f| > R\} \subset \{|f_1| > \delta R\} \cup \{|f_2| > (1-\delta)R\}$ we have that

$$\begin{aligned}
&\limsup_{R\to 0} R\,\text{mes}\{|f| > R\}^{\frac{1}{3}} \\
&\quad \leq \limsup_{R\to 0} R\big(\text{mes}\{|f_1| > \delta R\} + \text{mes}\{|f_2| > (1-\delta)R\}\big)^{\frac{1}{3}} \\
&\quad \leq \limsup_{R\to 0} R\,\text{mes}\{|f_1| > \delta R\}^{\frac{1}{3}} + \limsup_{R\to 0} R\,\text{mes}\{|f_2| > (1-\delta)R\}^{\frac{1}{3}} \\
&\quad = \frac{1}{\delta}\limsup_{R\to 0} R\,\text{mes}\{|f_1| > R\}^{\frac{1}{3}} + \frac{1}{1-\delta}\limsup_{R\to 0} R\,\text{mes}\{|f_2| > R\}^{\frac{1}{3}} \\
&\quad \leq \frac{1}{1-\delta}\|f_2\|_{L^{3,\infty}}.
\end{aligned}$$

Letting $\delta \to 0$ implies that $\limsup_{R\to 0} R\,\text{mes}\{|f| > R\}^{\frac{1}{3}} \leq \|f_2\|_{L^{3,\infty}} < \varepsilon$.

Conversely, assume that $\limsup_{R\to 0} R\,\text{mes}\{|f| > R\}^{\frac{1}{3}} < \varepsilon$. There exists R_ε such that

$$\sup_{0<R<R_\varepsilon} R\,\text{mes}\{|f| > R\}^{\frac{1}{3}} < \varepsilon.$$

We set $f_1 = f\chi_{\{|f|>R_\varepsilon\}}$ and $f_2 = f\chi_{\{|f|\leq R_\epsilon\}}$ where χ denotes the characteristic function. Clearly $|f_2| \leq R_\epsilon$ and $|f_2| \leq |f|$ so that

$$\|f_2\|_{L^{3,\infty}} = \sup_{0<R<R_\varepsilon} R\,\text{mes}\{|f_2| > R\}^{\frac{1}{3}} \leq \sup_{0<R<R_\varepsilon} R\,\text{mes}\{|f| > R\}^{\frac{1}{3}} < \varepsilon.$$

It remains to show that $f_1 \in L^2(\mathbb{R}^3)$. Let $N_\varepsilon \in \mathbb{Z}$ be such that $R_\varepsilon > 2^{N_\varepsilon}$. Then

$$\{|f| > R_\varepsilon\} \subset \bigcup_{n\geq N_\varepsilon} \{2^n < |f| \leq 2^{n+1}\}$$

so

$$\int_{\mathbb{R}^3} |f_1|^2 = \int_{\{|f|>R_\varepsilon\}} |f|^2 \le \sum_{n=N_\epsilon}^{\infty} \int_{\{2^n<|f|\le 2^{n+1}\}} |f|^2 \le \sum_{n=N_\epsilon}^{\infty} 4^{n+1} \operatorname{mes}\{2^n < |f|\}$$
$$\le \sum_{n=N_\epsilon}^{\infty} \frac{4}{2^n} \|f\|_{L^{3,\infty}}^3 < \infty.$$

This shows the first part of the lemma.

Assume now that $\lim_{R\to 0} R \operatorname{mes}\{|f| > R\}^{\frac{1}{3}} = 0$ and let $\varepsilon > 0$ be arbitrary. Using the first part we decompose $f = f_1 + f_2$ with $f_1 \in L^2$ and $\|f_2\|_{L^{3,\infty}} < \varepsilon$. The standard decay estimates for the heat equation implies that $\|e^{t\Delta} f_1\|_{L^{3,\infty}} < Ct^{-\frac{1}{4}} \|f_1\|_{L^2} \to 0$ as $t \to \infty$. Moreover, $\|e^{t\Delta} f_2\|_{L^{3,\infty}} \le \|f_2\|_{L^{3,\infty}} < \varepsilon$. We infer that

$$\limsup_{t\to\infty} \|e^{t\Delta} f\|_{L^{3,\infty}} \le \limsup_{t\to\infty} (\|e^{t\Delta} f_1\|_{L^{3,\infty}} + \|e^{t\Delta} f_2\|_{L^{3,\infty}}) \le \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\limsup_{t\to\infty} \|e^{t\Delta} f\|_{L^{3,\infty}} = 0$, as required.

To prove the third part of the lemma, we choose

$$g(x) = \frac{e^{i|x|^2}}{\langle x\rangle}, \quad \langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

It is a straightforward calculation to check that

$$\limsup_{R\to 0} R \operatorname{mes}\{|g| > R\}^{\frac{1}{3}} = \left(\frac{4\pi}{3}\right)^{\frac{1}{3}}.$$

On the other hand, we will show that $e^{\frac{1}{4}\Delta} g \in L^2$ which by the decay estimates for the heat equation implies that $\|e^{t\Delta} g\|_{L^{3,\infty}} < C(t - \frac{1}{4})^{-\frac{1}{4}} \|e^{\frac{1}{4}\Delta} g\|_{L^2} \to 0$ as $t \to \infty$.

Since the kernel of the operator $e^{\frac{1}{4}\Delta}$ is $\pi^{-\frac{3}{2}} e^{-|x|^2}$ one has that

$$\begin{aligned}
e^{\frac{1}{4}\Delta} g(x) &= \pi^{-\frac{3}{2}} \int_{\mathbb{R}^3} \frac{e^{i|x-y|^2}}{\langle x-y\rangle} e^{-|y|^2}\, dy \\
&= \pi^{-\frac{3}{2}} e^{i|x|^2} \int_{\mathbb{R}^3} e^{-2ix\cdot y} \frac{e^{(i-1)|y|^2}}{\langle x-y\rangle} dy \\
&= \pi^{-\frac{3}{2}} \frac{e^{i|x|^2}}{\langle x\rangle^2} \int_{\mathbb{R}^3} \left(1 - \frac{1}{4}\Delta_y\right) e^{-2ix\cdot y} \frac{e^{(i-1)|y|^2}}{\langle x-y\rangle} dy \\
&= \pi^{-\frac{3}{2}} \frac{e^{i|x|^2}}{\langle x\rangle^2} \int_{\mathbb{R}^3} e^{-2ix\cdot y} \left(1 - \frac{1}{4}\Delta_y\right) \left[\frac{e^{(i-1)|y|^2}}{\langle x-y\rangle}\right] dy.
\end{aligned}$$

The integral in the last term is bounded uniformly with respect to x. Indeed, all derivatives of $e^{(i-1)|y|^2}$ are integrable and all derivatives of $\frac{1}{\langle x-y\rangle}$ are uniformly bounded in x and y. We deduce that $|e^{\frac{1}{4}\Delta} g(x)| \le C\langle x\rangle^{-2}$ which implies that $e^{\frac{1}{4}\Delta} g \in L^2$. This completes the proof of the lemma. □

We state now our stability result for small solutions.

Theorem 4.3. *There exists an absolute constant $\varepsilon_3 > 0$ with the following properties:*

- *If $f, u_0 \in \mathcal{S}'(\mathbb{R}^3)$ are such that*

$$\|\Delta^{-1}\mathbb{P}f\|_{L^{3,\infty}} + \|u_0\|_{L^{3,\infty}} < \varepsilon_3 \tag{4.5}$$

then there is a unique solution $u \in L^\infty(\mathbb{R}_+, L^{3,\infty})$ of (4.1), weakly continuous with respect to $t \in [0, \infty)$, satisfying

$$\sup_{s>0} \|u(s)\|_{L^{3,\infty}} \leq 2\|u_0\|_{L^{3,\infty}} + 4\|\Delta^{-1}\mathbb{P}f\|_{L^{3,\infty}}. \tag{4.6}$$

- *Let $p \in (\frac{3}{2}, \infty)$ and suppose in addition to (4.5) that $u_0 \in L^{p,\infty}$. If u is the above solution then,*

$$u \in L^\infty(\mathbb{R}_+, L^{p,\infty}) \quad \text{if and only if} \quad \Delta^{-1}\mathbb{P}f \in L^{p,\infty}.$$

- *Let $p \in (\frac{3}{2}, \infty)$, and $q > \min\{3, p\}$. Suppose in addition to (4.5) that $u_0 \in L^{p,\infty}$ and $\Delta^{-1}\mathbb{P}f \in L^{p,\infty}$. Let also $U \in L^{3,\infty} \cap L^{p,\infty}$ be the unique stationary solution given by Theorem 2.2 (see also Remark 2.8). (We assume here that $\varepsilon_3 \leq \varepsilon_1$, the constant introduced in Theorem 2.2).*

  (i) *There is a function $\varepsilon(q) > 0$ such that if $\varepsilon_3 < \varepsilon(q)$ then, for some constant $C > 0$,*

$$\|u(t) - U\|_{L^q} \leq Ct^{-\frac{3}{2}(\frac{1}{\min(3,p)} - \frac{1}{q})}, \quad \forall \min\{p, 3\} < q < \infty. \tag{4.7}$$

  In particular, $u(t) - U \to 0$ in L^q as $t \to \infty$ for all $q > \min\{3, p\}$.

  (ii) *If $\frac{3}{2} < p \leq 3$, then $u(t) \rightharpoonup U$ weakly in $L^{p,\infty}$ as $t \to \infty$. Moreover, $u(t) \to U$ strongly in $L^{p,\infty}$ if and only if $e^{t\Delta}(u_0 - U) \to 0$ in $L^{p,\infty}$.*

  (iii) *If $\frac{3}{2} < p < 3$, then the conclusion of the previous item can be strengthened as follows:*

$$\|u(t) - U - e^{t\Delta}(u_0 - U)\|_{L^q} \leq Ct^{\frac{1}{2} + \frac{3}{2q} - \frac{3}{p}} \tag{4.8}$$

  for all $\frac{3p}{6-p} \leq q \leq p$ and for some constant $C > 0$ independent of t.

  In particular, $u(t) - U \to 0$ in L^q if and only if $e^{t\Delta}(u_0 - U) \to 0$ in L^q as $t \to \infty$, for all $\frac{3p}{6-p} < q \leq p$.

Notice that in (4.7) neither $u(t)$ nor U belong in general to L^q. Similarly, the terms appearing in the left-hand side of (4.8) in general do not belong, separately, to L^q. In other words, the difference $u(t) - U$ is better behaved than the solutions themselves.

Remark 4.4. In the particular case $p = q = 2$, the preceding theorem contains an interesting variant of the stability result for finite-energy solutions obtained in [3] with a different method. Indeed, consider a stationary solution $U \in L^2 \cap L^{3,\infty}$ and a perturbation $w_0 \in L^2 \cap L^{3,\infty}$. According to conclusion (iii), the solution u of the non-stationary Navier–Stokes equations starting from $u_0 = U - w_0$ satisfies, under the above smallness assumptions, $u(t) \to U$ in L^2 as $t \to \infty$ (we use here that $e^{t\Delta}w_0 \to 0$

in L^2). Explicit convergence rates can be given, e.g., if the perturbation belongs to additional function spaces. For instance, when $w_0 \in L^{\frac{3}{2},\infty} \cap L^{3\infty}$, then

$$\|u(t) - U\|_2 \leq Ct^{-1/4}, \quad \text{as } t \to \infty.$$

Remark 4.5. Let us present some further immediate consequences of this theorem. If the perturbation satisfies $w_0 \in L^3$, then $e^{t\Delta}w_0 \to 0$ in L^3 and so in $L^{3,\infty}$ as $t \to \infty$. This in turn implies, by (ii),

$$u(t) \to U \quad \text{in } L^{3,\infty} \text{ as } t \to \infty.$$

More generally, according to the second part of Lemma 4.2, such a conclusion remains valid when $\lim_{R\to 0} R\,\mathrm{mes}\{|w_0| > R\}^{\frac{1}{3}} = 0$. However, notice that neither $w_0 \in L^{3,\infty}$ is sufficient nor $\lim_{R\to 0} R\,\mathrm{mes}\{|w_0| > R\}^{\frac{1}{3}} = 0$ is necessary to ensure this result.

In the same way, in the case $\frac{3}{2} < p < 3$, the condition $\lim_{R\to 0} R\,\mathrm{mes}\{|w_0| > R\}^{\frac{1}{p}} = 0$ implies that $u(t) \to U \in L^{p,\infty}$ as $t \to \infty$. But in this case the stronger condition $w_0 \in L^p$ would imply also, by (iii), the stronger conclusion $u(t) \to U$ in L^p.

Remark 4.6. The proof of Theorem 4.3 below will show that Equation (4.8) holds true for the wider range $\max(1, \frac{p}{2}) \leq q < \frac{3p}{3-p}$. We did not state the full range for q because the most interesting case is $q \leq p$ and also because it would require showing that in the case $p < 3$, the statement (i) is true with a constant $\varepsilon(q)$ independent of q. This additional fact is easy to prove with a recursive argument, but since it is not really necessary we prefer to skip it.

Proof of Theorem 4.3. We estimate the forcing term in (4.1) by integrating the heat kernel in time then relying on a fixed point argument making use of Lemma 4.1. The relation

$$\int_0^t e^{(t-s)\Delta} ds = e^{t\Delta}\Delta^{-1} - \Delta^{-1} \tag{4.9}$$

that follows since both operators have the same symbols, gives

$$\left\| \int_0^t e^{(t-s)\Delta}\mathbb{P}f\, ds \right\|_{L^{3,\infty}} \leq 2\|\Delta^{-1}\mathbb{P}f\|_{L^{3,\infty}}. \tag{4.10}$$

We used above that $e^{t\Delta}$ is a convolution operator with a function of norm L^1 equal to 1. Given (4.10), the first part of this theorem follows from the work of Cannone and Karch [9]. But the proof takes only a few lines, so we give it for the sake of the completeness.

Using again that the kernel of $e^{t\Delta}$ is of L^1 norm equal to 1 we deduce that $\|e^{t\Delta}u_0\|_{L^{3,\infty}} \leq \|u_0\|_{L^{3,\infty}}$. Therefore, if we denote $\tilde{u}_0 = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\mathbb{P}f\, ds$ one has that

$$\|\tilde{u}_0\|_{L^\infty(\mathbb{R}_+;L^{3,\infty})} \leq \|u_0\|_{L^{3,\infty}} + 2\|\Delta^{-1}\mathbb{P}f\|_{L^{3,\infty}}.$$

To apply the fixed point argument we introduce the notation $\widetilde{B}(u, v) = \widetilde{L}_0(u \otimes v)$ and rewrite (4.1) as

$$u = \tilde{u}_0 - \widetilde{B}(u, u). \tag{4.11}$$

The bound (4.2), with $p = 3$, and hence $q = \frac{3}{2}$ combined with the Hölder inequality from Proposition 1.1 yields

$$\|\widetilde{B}(u, v)(t)\|_{L^{3,\infty}} \leq C\left(\sup_{s>0} C\|u(s)\|_{L^{3,\infty}}\right)\left(\sup_{s>0} \|v(s)\|_{L^{3,\infty}}\right).$$

We apply this estimate combined with the fixed point argument given in Lemma 2.6 to the operator $\widetilde{\Phi}(u) = \tilde{u}_0 - \widetilde{B}(u, u)$ in the space $L^\infty(\mathbb{R}_+, L^{3,\infty})$. This approach yields the existence of a unique solution $u \in L^\infty(\mathbb{R}_+, L^{3,\infty})$ provided $4C\|\tilde{u}_0\|_{L^\infty(\mathbb{R}_+;L^{3,\infty})} < 1$. Lemma 2.6 also insures that the solution satisfies $\|u\|_{L^\infty(\mathbb{R}_+;L^{3,\infty})} \leq 2\|\tilde{u}_0\|_{L^\infty(\mathbb{R}_+;L^{3,\infty})}$, establishing part 1 of the theorem.

To prove the second part of the theorem we establish first the cases $p \in [2, 7]$ with a fixed point argument then treat the other cases with an interpolation argument. First, combine (4.2) with the Hölder inequality to establish

$$\|\widetilde{B}(u, u)(t)\|_{L^{p,\infty}} \leq C(p)\left(\sup_{s>0} \|u(s)\|_{L^{p,\infty}}\right)\left(\sup_{s>0} \|u(s)\|_{L^{3,\infty}}\right). \tag{4.12}$$

Let $\widetilde{C}$ be the maximum value of the constant in the above equation for $p \in [2, 7]$, we require $8\varepsilon_3 < 1/\widetilde{C}$. Considering again the sequence of approximate solutions (u_i) constructed in the usual way, and making use of (4.6) we see

$$\|u_{i+1}(t)\|_{L^{p,\infty}} \leq \sup_{s>0} \|\tilde{u}_0\|_{L^{p,\infty}} + \frac{1}{2} \sup_{s>0} \|u_i(s)\|_{L^{p,\infty}}.$$

From this estimate the "if" statement in the second claim follows for $p \in [2, 7]$.

If $p \in (\frac{3}{2}, 2)$, through interpolation we find that for all $r \in (2, 3)$ we have that $\tilde{u}_0 \in L^\infty(\mathbb{R}_+; L^{r,\infty})$ and therefore $u \in L^\infty(\mathbb{R}_+; L^{r,\infty})$. Appealing to (4.2) and again combining it with the Hölder inequality we see

$$\|\widetilde{B}(u, u)(t)\|_{L^{p,\infty}} \leq C \sup_{t>0} \|u(s)\|^2_{L^{r,\infty}} \tag{4.13}$$

where $r = \frac{6p}{p+3} \in (2, 3)$, hence the right-hand side is bounded. Combining this estimate with (4.11) is enough to prove the "if" statement in the case $p \in \left(\frac{3}{2}, 2\right)$. If $p \in (7, \infty)$ we again interpolate to get $u \in L^\infty(\mathbb{R}_+; L^{r,\infty})$ for all $r \in (3, 6)$. Choosing again $r = \frac{6p}{p+3} \in (3, 6)$ in (4.13) finishes the "if" statement in the second claim. To establish the "only if" part of the claim combine (4.12) with (4.11) and notice that the right-hand side of (4.9) tends to $-\Delta^{-1}$ as $t \to \infty$. The weak continuity $u(t) \to u(t')$ for $t \to t'$ and $t' \in [0, \infty)$ (the continuity is actually in the strong topology of $L^{3,\infty}$ for $t' \in (0, \infty)$) is proved as in [21].

It remains to prove the third part of the theorem, the stability results for stationary solutions. We begin with Claim (i). Let $q > \min\{3, p\}$. It is worth noticing that for $q > 3$, a stability result in the $L^{q,\infty}$-norm, as well as a decay estimate of the form $\|u(t) - v(t)\|_{L^q} \leq Ct^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{q})}$ was stated in [9, Proposition 4.3]. However, it seems that the argument briefly sketched in [9] cannot be directly applied to the case where the second solution $v(t)$ is stationary, because a non-obvious generalization of Lemma 4.1 would be needed. Therefore, we provide a detailed proof of estimate (4.7).

Let $w = U - u$ and $w^0 = U - u_0$. Then this difference w satisfies the mild PDE

$$w(t) = e^{t\Delta}w^0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes w + w \otimes U)(s)ds. \tag{4.14}$$

Moreover, our smallness assumptions on u_0 and f and the usual fixed point Lemma 2.6 imply that w can be obtained as the limit in $L^\infty(\mathbb{R}^+, L^{3,\infty})$ of the approximating sequence (w_k), defined by

$$w_{k+1} = e^{t\Delta}w^0 - \widetilde{B}(u, w_k) - \widetilde{B}(w_k, U),$$

where the recursive relation starts with $w_0(x, t) = e^{t\Delta}w^0$. Moreover, this sequence (w_k) is bounded in $L^\infty(\mathbb{R}^+, L^{p,\infty})$.

By the semigroup property (recall that $F(x, t)$ denotes the kernel of $e^{t\Delta}\mathbb{P}\,\mathrm{div}$):

$$\widetilde{B}(u, v)(t) = e^{t\Delta/2}\widetilde{B}(u, v)(t/2) + \int_{t/2}^t F(t-s) * (u \otimes v)(s)ds.$$

We deduce

$$\begin{aligned} w_{k+1}(t) = e^{t\Delta}w^0 - e^{t\Delta/2}\widetilde{B}(u, w_k)(t/2) - e^{t\Delta/2}\widetilde{B}(w_k, U)(t/2) \\ - \int_{t/2}^t F(t-s) * (u \otimes w_k)(s)ds - \int_{t/2}^t F(t-s) * (w_k \otimes U)(s)ds. \end{aligned}$$

Now let $r = \min(3, p)$ and denote

$$M = \max\left\{ \|w^0\|_{L^{r,\infty}}, \|U\|_{L^{r,\infty}}, \sup_{s>0} \|u(s)\|_{L^{r,\infty}} \right\}.$$

By Lemma 4.1 and using that the sequence w_k is bounded in $L^\infty(\mathbb{R}^+, L^{3,\infty})$,

$$\|\widetilde{B}(u, w_k)(t/2)\|_{L^{r,\infty}} + \|\widetilde{B}(w_k, U)(t/2)\|_{L^{r,\infty}} \le C_r M.$$

A heat kernel estimate now implies, for all $q > r$ and for some constant $C'_r > 0$ independent of q,

$$\begin{aligned} \|w_{k+1}(t)\|_{L^{q,\infty}} \le C'_r M t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} + \left\| \int_{t/2}^t F(t-s) * (u \otimes w_k)(s)ds \right\|_{L^{q,\infty}} \\ + \left\| \int_{t/2}^t F(t-s) * (w_k \otimes U)(s)ds \right\|_{L^{q,\infty}}. \end{aligned}$$

From Lemma 4.1 with Hölder's inequality we have

$$\begin{aligned} \left\| \int_{t/2}^t F(t-s) * (u \otimes w_k)(s)ds \right\|_{L^{q,\infty}} + \left\| \int_{t/2}^t F(t-s) * (w_k \otimes U)(s)ds \right\|_{L^{q,\infty}} \\ \le C''_q \varepsilon_3 \sup_{s\in[t/2,t]} \|w_k(s)\|_{L^{q,\infty}}. \end{aligned}$$

Let

$$W_k(t) \equiv \sup_{\tau\in[t,\infty)} \|w_k(\tau)\|_{L^{q,\infty}}.$$

then

$$W_{k+1}(t) \le C'_r M t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} + C''_q \varepsilon_3 W_k(t/2).$$

Iterating this inequality implies

$$\begin{aligned} W_k(t) &\le C'_r M \sum_{n=0}^{k-1} \Big(C''_q\, \varepsilon_3\, 2^{\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}\Big)^n t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} + \big(C''_q \varepsilon_3\big)^k W_0(t/2^k) \\ &\le 2C'_r M t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} + C(r,q)\Big(C''_q \varepsilon_3 2^{\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}\Big)^k t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}, \end{aligned}$$

provided

$$C''_q \varepsilon_3 2^{\frac{3}{2}(\frac{1}{r}-\frac{1}{q})} < \frac{1}{2}.$$

A slightly more stringent smallness condition and independent on $r > \frac{3}{2}$ is, e.g.,

$$\varepsilon_3 < \varepsilon(q) := \frac{1}{4C''_q}. \tag{4.15}$$

Now assuming (4.15) and letting $k \to \infty$ we get,

$$\|w(t)\|_{L^{q,\infty}} \le 2C'_r M t^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}, \quad \text{for } r = \min(3, p) \text{ and } q > r. \tag{4.16}$$

Writing the above estimate for $q-\eta$ and $q+\eta$, for some $\eta > 0$ small enough and interpolating the L^q-space between $L^{q-\eta,\infty}$ and $L^{q+\eta,\infty}$ shows that the above estimate remains valid with $\|w(t)\|_{L^q}$ on the left-hand side. This establishes the stability result (4.7).

We now prove Claim (ii). The weak convergence $u(t) \rightharpoonup U$ in $L^{p,\infty}$ for $\frac{3}{2} < p \le 3$ is obvious since the solution $u(t)$ is bounded in $L^{p,\infty}$ and goes to U in the sense of distributions (even in L^q, $q > 3$, as implied by the previous part of the proof).

On the other hand, the proof of the necessary and sufficient condition for the strong convergence result in the $L^{3,\infty}$-norm is given in [9, Theorem 2.2] and in [9, Corollary 4.1], hence we will skip it. The necessary and sufficient condition for the strong convergence result in the $L^{p,\infty}$-norm, with $\frac{3}{2} < p < 3$ is a direct consequence of Claim (iii) which we now prove.

Let now $\frac{3}{2} < p < 3$ and $\frac{3p}{6-p} \le q \le p$. Given these restrictions, there exists some $q_1 \in [p, 4]$ such that the following relations hold true:

$$\frac{1}{q} - \frac{1}{p} \le \frac{1}{q_1} \le \frac{1}{p}, \quad \frac{1}{q_1} < \min\Big(1 - \frac{1}{p}, \frac{1}{3} + \frac{1}{q} - \frac{1}{p}\Big).$$

We go back to the equation for w given in (4.14). We estimate $\|w(t) - w_0(t)\|_{L^q}$ using Propositions 1.1 and 1.2, the bound $\|w(t)\|_{L^{q_1,\infty}} \le Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q_1}\right)}$ (consequence of (4.7) with $q = q_1$) and the fact that $U \in L^{p,\infty}$ and $u \in L^\infty((0,\infty), L^{p,\infty})$. We get

$$\begin{aligned}\|w(t) - w_0(t)\|_{L^q} &\le C\int_0^t (t-s)^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p}-\frac{1}{q_1})}\|(u\otimes w + w\otimes U)(s)\|_{L^{\frac{q_1 p}{p+q_1},\infty}}ds\\ &\le C\int_0^t (t-s)^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p}-\frac{1}{q_1})}\|w(s)\|_{L^{q_1,\infty}}\big(\|u(s)\|_{L^{p,\infty}} + \|U\|_{L^{p,\infty}}\big)ds\\ &\le C\int_0^t (t-s)^{-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p}-\frac{1}{q_1})}s^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q_1})}ds\\ &\le Ct^{\frac{1}{2}+\frac{3}{2q}-\frac{3}{p}}.\end{aligned}$$

The theorem is now completely proved.

We finally show our stability result for large solutions.

Theorem 4.7. *There exists an absolute constant $\varepsilon_4 > 0$ with the following property. Let $u \in L^\infty_{loc}([0,\infty); L^{3,\infty}) \cap L^4_{loc}([0,\infty); L^4)$ be a global solution of the evolutionary Navier–Stokes equations with a constant in time forcing f such that $\Delta^{-1}\mathbb{P}f \in L^{3,\infty} \cap L^4$ and*

$$A(u_0, f) \equiv \limsup_{R\to 0} R\, mes\{|u_0| > R\}^{\frac{1}{3}} + \|\Delta^{-1}\not\supseteq f\|_{L^{3,\infty}} < \varepsilon_4. \tag{4.17}$$

Let $U \in L^{3,\infty} \cap L^4$ be the unique stationary solution constructed in Theorem 2.2. Then we have that

- $\limsup_{t\to\infty} \|u(t)\|_{L^{3,\infty}} \le 22A(u_0, f)$;
- *$u(t) \rightharpoonup U$ weakly in $L^{3,\infty}$ as $t \to \infty$;*
- *$u(t) \to U$ in $L^{3,\infty}$ as $t \to \infty$ if and only if $e^{t\Delta}(u_0 - U) \to 0$ strongly in $L^{3,\infty}$ as $t \to \infty$.*

Proof. The idea of the proof is the same as in [15] where it was proved that any global solution of the Navier–Stokes equations without external force goes to 0 in the Besov spaces $\dot{B}^{-1+\frac{3}{p}}_{p,q}$ when the time becomes large (see also [1] for the case of VMO^{-1}). It consists in decomposing the initial velocity in a small part plus a square integrable part. The small part remains small by the small data theory and the square-integrable part will become small at some point by using some energy estimates.

Here we use Lemma 4.2 to decompose $u_0 = v_0 + w_0$ where $v_0 \in L^2 \cap L^{3,\infty}$ and $\|w_0\|_{L^{3,\infty}} < 2A(u_0, f)$. Assuming that $3\varepsilon_4 < \varepsilon_3$ where ε_3 is the constant from Theorem 4.3, we can apply that theorem to construct a global solution w of the Navier–Stokes equations with forcing term f, initial velocity w_0 and such that

$$\|w(t)\|_{L^{3,\infty}} \le 8A(u_0, f) \quad \text{for all } t \ge 0.$$

Moreover, according to relation (4.7) the solution w satisfies the following decay estimate $\sup_{t>0} t^{\frac{1}{8}}\|w(t) - U\|_{L^4} < \infty$. Since $U \in L^4$ we infer that $w \in L^4_{loc}([0,\infty); L^4)$.

The difference $v = u - w$ verifies the following PDE:

$$\partial_t v - \Delta v + u\cdot\nabla v + v\cdot\nabla w + \nabla p' = 0 \tag{4.18}$$

whose integral form reads

$$v(t) = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v + v\otimes w)(s)ds. \tag{4.19}$$

We show first that $v\in C^0([0,\infty);L^2)$. The first term on the right-hand side above clearly belongs to this space. We show that so does the second term. The kernel $F(t)$ of the operator $e^{t\Delta}\mathbb{P}\,\mathrm{div}$ is of the form $F(x,t) = t^{-2}F(\frac{x}{\sqrt{t}},1)$ with $F(\cdot,1)\in L^1\cap L^\infty\subset L^{p,q}$ for all $1<p<\infty$ and $1\leq q\leq\infty$. In particular, $\|F(t)\|_{L^{\frac{6}{5},2}}\leq t^{-\frac{3}{4}}$ so that $F\in L^1_{loc}([0,\infty);L^{\frac{6}{5},2})$. By the Hölder inequality we also have that $u\otimes v + v\otimes w\in L^\infty_{loc}([0,\infty);L^{\frac{3}{2},\infty})$. Since the last term in (4.19) is the space-time convolution of F with $u\otimes v+v\otimes w$, we infer that it belongs to $C^0([0,\infty);L^2)$.

For $0<\delta<1$, let J_δ be a smoothing operator that multiplies in the frequency space by a cut-off function bounded by 1 which is a smoothed out version of the characteristic function of the annulus $\{\delta<|\xi|<\frac{1}{\delta}\}$. We also introduce an approximation of the identity φ_η in time.

Given the additional regularity found for v above, we remark that we can multiply the equation of v expressed in (4.18) by $\varphi_\eta*\varphi_\eta*J_\delta^2 v$ and integrate in space and time from t_0 to t, with $t_0>0$, to obtain that

$$\begin{aligned}\|\varphi_\eta * J_\delta v(t)\|_{L^2}^2 &+ 2\int_{t_0}^t \|\nabla\varphi_\eta * J_\delta v(s)\|_{L^2}^2\,ds\\ &= \|\varphi_\eta * J_\delta v(t_0)\|_{L^2}^2 + 2\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla(\varphi_\eta*\varphi_\eta*J_\delta^2 v)\cdot v\\ &\quad + 2\int_0^t\int_{\mathbb{R}^3} v\cdot\nabla(\varphi_\eta*\varphi_\eta*J_\delta^2 v)\cdot w.\end{aligned} \tag{4.20}$$

We let now $\eta\to 0$. Given the time continuity of v with values in L^2, we have that $\varphi_\eta * J_\delta v(t)\to J_\delta v(t)$ and $\varphi_\eta * J_\delta v(t_0)\to J_\delta v(t_0)$ in L^2 as $\eta\to 0$. The other terms in (4.20) pass easily to the limit $\eta\to 0$. Therefore, taking first the limit $\eta\to 0$ in (4.20), and second $t_0\to 0$ and using again that $v\in C^0[0,\infty);L^2)$ we get that

$$\begin{aligned}\|J_\delta v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla J_\delta v(s)\|_{L^2}^2\,ds = \|J_\delta v_0\|_{L^2}^2 &+ 2\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla J_\delta^2 v\cdot v\\ &+ 2\int_0^t\int_{\mathbb{R}^3} v\cdot\nabla J_\delta^2 v\cdot w.\end{aligned} \tag{4.21}$$

We bound the last two terms on the right-hand side as follows

$$\begin{aligned}2\int_0^t\int_{\mathbb{R}^3} u\cdot\nabla J_\delta^2 v\cdot v + 2\int_0^t\int_{\mathbb{R}^3} v\cdot\nabla J_\delta^2 v\cdot w &\leq 2\int_0^t \|\nabla J_\delta^2 v\|_{L^2}\|v\|_{L^4}(\|u\|_{L^4}+\|w\|_{L^4})\\ &\leq \frac{1}{2}\int_0^t \|\nabla J_\delta v\|_{L^2}^2 + \int_0^t \|v\|_{L^4}^2(\|u\|_{L^4}^2+\|w\|_{L^4}^2).\end{aligned}$$

Plugging this in (4.21) yields

$$\|J_\delta v(t)\|_{L^2}^2 + \int_0^t \|\nabla J_\delta v(s)\|_{L^2}^2\,ds \leq \|J_\delta v_0\|_{L^2}^2 + \int_0^t \|v\|_{L^4}^2(\|u\|_{L^4}^2+\|w\|_{L^4}^2).$$

Since $u, v, w \in L^4_{loc}([0,\infty); L^4)$, the right-hand side above is uniformly bounded with respect to δ. Letting $\delta \to 0$ implies thanks to the Beppo-Levi theorem that $\int_0^t \|\nabla v(s)\|_{L^2}^2\, ds < \infty$, that is $v \in L^2_{loc}([0,\infty); H^1)$.

We go back to (4.21) and estimate

$$\begin{aligned} 2\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla J_\delta^2 v \cdot v &= 2\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla J_\delta^2 v \cdot (1 - J_\delta^2) v \\ &\le 2\|u\|_{L^4(0,t;L^4)} \|\nabla v\|_{L^2(0,t;L^2)} \|(1 - J_\delta^2)v\|_{L^4(0,t;L^4)} \\ &\to 0 \quad \text{as } \delta \to 0. \end{aligned} \tag{4.22}$$

We observe now that $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^{6,2}(\mathbb{R}^3)$. This imbedding follows from the Young inequality for Lorentz spaces after noticing that $(-\Delta)^{-\frac{1}{2}}$ is a convolution operator with a function bounded by $\frac{C}{|x|^2}$ which therefore belongs to $L^{\frac{3}{2},\infty}$. We use this fact together with the Hölder inequality to bound the last term in (4.21) as follows

$$2\int_0^t \int_{\mathbb{R}^3} v \cdot \nabla J_\delta^2 v \cdot w \le C\int_0^t \|v\|_{L^{6,2}} \|\nabla J_\delta^2 v\|_{L^2} \|w\|_{L^{3,\infty}} \le CA(u_0, f) \int_0^t \|\nabla v\|_{L^2}^2. \tag{4.23}$$

Using (4.22) and (4.23) in (4.21), letting $\delta \to 0$ and using the Beppo-Levi theorem we infer that

$$\|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s)\|_{L^2}^2\, ds \le \|v_0\|_{L^2}^2 + CA(u_0, f)\int_0^t \|\nabla v\|_{L^2}^2.$$

If we further assume that $C\varepsilon_4 \le 1$, then $CA(u_0, f) \le 1$ so the relation above implies that $v \in L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$. By interpolation and from the imbedding $\dot{H}^{\frac{1}{2}} \subset L^{3,\infty}$ we infer that $v \in L^4(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}) \subset L^4(\mathbb{R}_+; L^{3,\infty})$. So there exists a time $T = T(\varepsilon_4)$ such that $\|v(T)\|_{L^{3,\infty}} < A(u_0, f)$. Since we also have that $\|w(T)\|_{L^{3,\infty}} < 8A(u_0, f)$ we infer that $\|u(T)\|_{L^{3,\infty}} < 9A(u_0, f)$. Assuming further that $10\varepsilon_4 < \varepsilon_3$, Theorem 4.3 allows to construct a small solution starting from time T, a solution whose $L^{3,\infty}$ norm will be bounded by $22A(u_0, f)$. We will prove below a uniqueness result stating that u must be equal to this small solution starting from time T. Once this is proved, the first part of the theorem follows. Moreover, using again that our solution u becomes small after the time T, the second and the third part of the theorem are consequences of Theorem 4.3. Except that the equivalent condition for $u(t)$ to converge strongly to U in $L^{3,\infty}$ is that $e^{t\triangle}(u(T) - U) \to 0$ strongly in $L^{3,\infty}$ as $t \to \infty$. To finish the proof it therefore suffices to show that

$$e^{t\triangle}(u(T) - U) \overset{t\to\infty}{\longrightarrow} 0 \quad \text{in } L^{3,\infty} \iff e^{t\triangle}(u_0 - U) \overset{t\to\infty}{\longrightarrow} 0 \quad \text{in } L^{3,\infty}.$$

This is a consequence of the following sequence of equivalence relations:

$$\begin{aligned} e^{t\triangle}(u(T) - U) \overset{t\to\infty}{\longrightarrow} 0 \quad \text{in } L^{3,\infty} &\iff e^{t\triangle}(w(T) - U) \overset{t\to\infty}{\longrightarrow} 0 \quad \text{in } L^{3,\infty} \\ &\iff w(t) \overset{t\to\infty}{\longrightarrow} U \quad \text{in } L^{3,\infty} \\ &\iff e^{t\triangle}(w_0 - U) \overset{t\to\infty}{\longrightarrow} 0 \quad \text{in } L^{3,\infty} \\ &\iff e^{t\triangle}(u_0 - U) \overset{t\to\infty}{\longrightarrow} 0 \quad \text{in } L^{3,\infty}. \end{aligned}$$

We used above that $v(T), v_0 \in L^2$ and the decay estimates for the heat equation to deduce the first and fourth lines of the relation above, and Theorem 4.3 twice for w, starting from time $t = 0$ and from time $t = T$ to deduce the second and third lines. This completes the proof provided that we prove the announced uniqueness result.

Let $\bar{u}$ be the small solution starting from time T with initial velocity $u(T)$ constructed in Theorem 4.3 and set $\bar{v} = \bar{u} - w$. As above we have that $\bar{v} \in C^0([T; \infty); L^2) \cap L^2([T; \infty); \dot{H}^1)$. Then $v - \bar{v}$ solves the following equation:

$$\partial_t(v - \bar{v}) - \Delta(v - \bar{v}) + u \cdot \nabla(v - \bar{v}) + (v - \bar{v}) \cdot \nabla\bar{u} = -\nabla p_1.$$

As in the previous argument, one can prove that this relation can be multiplied by $v - \bar{v}$ and integrated from T to t to get that, for all $t \geq T$,

$$\begin{aligned}
\|(v - \bar{v})(t)\|_{L^2}^2 + 2\int_T^t \|\nabla(v - \bar{v})\|_{L^2}^2 &= \int_T^t \int (v - \bar{v}) \cdot \nabla(v - \bar{v}) \cdot \bar{u} \\
&\leq C\int_T^t \|\nabla(v - \bar{v})\|_{L^2}^2 \|\bar{u}\|_{L^{3,\infty}} \\
&\leq CA(u_0, f)\int_T^t \|\nabla(v - \bar{v})\|_{L^2}^2 \\
&\leq \int_T^t \|\nabla(v - \bar{v})\|_{L^2}^2
\end{aligned}$$

provided that $A(u_0, f)$ is sufficiently small. We infer that $v(t) = \bar{v}(t)$, that is $u(t) = \bar{u}(t)$ for all $t \geq T$. This completes the proof of the theorem. □

Remark 4.8. We also have stability in $L^{p,\infty}$ for large solutions. More precisely, suppose that in addition to the hypothesis of Theorem 4.7 we assume that $u_0 \in L^{p,\infty}$ with $p \in (\frac{3}{2}, 3)$. Then $u \in L^\infty(\mathbb{R}_+; L^{p,\infty})$, $u(t) \rightharpoonup U$ weakly in $L^{p,\infty}$ and $u(t) - e^{t\Delta}(u_0 - U) \to U$ in $L^{p,\infty}$ as $t \to \infty$. This follows easily after applying Theorem 4.3 starting from the time T when the solution becomes small. One only needs to show the following two facts:

- if $u_0 \in L^{p,\infty}$ and $\Delta^{-1}\mathbb{P}f \in L^{p,\infty}$ then $u \in L^\infty(0, T; L^{p,\infty})$;
- $e^{(t-T)\Delta}(u(T) - U) - u^{t\Delta}(u_0 - U) \to 0$ strongly in $L^{p,\infty}$ as $t \to \infty$.

To prove the first assertion, we observe that, with the notation from the proof of Theorem 4.3 (namely the notation used in relation (4.11)) one has that $\tilde{u}_0 \in L^\infty(\mathbb{R}_+; L^{p,\infty})$. Moreover, by the Hölder inequality and using the standard decay estimates for the heat equation we can bound

$$\begin{aligned}
\|\widetilde{B}(u, u)(t)\|_{L^p} &\leq \int_0^t \|e^{(t-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u)(s)\|_{L^p}\, ds \\
&\leq C\int_0^t (t - s)^{-\frac{3}{2}+\frac{3}{2p}} \|u(s)\|_{L^{3,\infty}}^2 \\
&\leq Ct^{-\frac{1}{2}+\frac{3}{2p}} \sup_{0<s<t} \|u(s)\|_{L^{3,\infty}}^2.
\end{aligned} \tag{4.24}$$

We infer that $\widetilde{B}(u, u) \in L^\infty(0, T; L^p) \subset L^\infty(0, T; L^{p,\infty})$, so by (4.11) we also have that $u \in L^\infty(0, T; L^{p,\infty})$. To show the second assertion, we observe that it is sufficient to prove that

$$u(T) - U - e^{t\Delta}(u_0 - U) \in L^{q,\infty}$$

for some $q < p$. But $u - U$ verifies the PDE

$$\partial_t(u - U) - \Delta(u - U) + u \cdot \nabla u - U \cdot \nabla U = -\nabla p_2$$

whose mild formulation implies that

$$u(T) - U - e^{t\Delta}(u_0 - U) = -\int_0^T e^{(T-s)\Delta}\mathbb{P}\nabla \cdot (u \otimes u - U \otimes U)(s)ds.$$

The same estimate as in (4.24) shows now that the right-hand side belongs to $L^{q,\infty}$ for any $\frac{3}{2} < q < 3$, in particular for some $q < p$.

Moreover, if $u_0 \in L^p$, then the previous argument shows that $u \in L^\infty(0, T; L^p)$. From Theorem 4.3 applied starting from time T we infer that $u \in L^\infty(\mathbb{R}_+; L^p)$ and $u(t) \to U$ in L^p as $t \to \infty$.

Remark 4.9. We observe that the condition imposed on the initial velocity by the hypothesis of Theorem 4.7 does not imply that u_0 is close in $L^{3,\infty}$ to the smooth functions in $\mathcal{S}(\mathbb{R}^3)$. Indeed, that would require to have that the quantity $\limsup_{R\to\infty} R\,\mathrm{mes}\{|u_0| > R\}^{\frac{1}{3}}$ is small too. This condition is not necessary in Theorem 4.7.

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