Martingales (in discrete time)

Contents

1 Filtrations .................................................. 1
  1.1 Stochastic processes and filtrations ......................... 2
  1.2 Stopping times .......................................... 2

2 Martingales .................................................. 3
  2.1 Definition .............................................. 3
  2.2 Examples ............................................... 3
  2.3 Doob’s decomposition for submartingales ...................... 4
  2.4 A first stopping theorem ................................ 5

3 Convergence results .......................................... 5
  3.1 Almost sure convergence .................................. 5
  3.2 Convergence in $L^p$, $p > 1$ ................................ 6
  3.3 Convergence in $L^1$ .................................... 7

4 Law of Large Numbers and Central Limit Theorem ............. 7

5 Some extensions to the continuous time case .................. 8
  5.1 Filtrations, stopping times, processes ...................... 8
  5.2 Martingales ............................................ 10

1 Filtrations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space.

Notation: $\mathbb{N} = \{0, 1, \ldots\}$.

Definition 1 (Filtration). A non-decreasing sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of sub-$\sigma$-fields of $\mathcal{F}$ (i.e. $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ for all $n \in \mathbb{N}$) is called a filtration.

Let $\mathcal{F}_\infty = \sigma(\mathcal{F}_n, n \in \mathbb{N}) \subset \mathcal{F}$ the smallest $\sigma$-field such that $\mathcal{F}_n \subset \mathcal{F}_\infty$, for all $n \in \mathbb{N}$.
1.1 Stochastic processes and filtrations

A (real-valued) stochastic process is a sequence \((X_n)_{n \in \mathbb{N}}\) of (real-valued) random variables.

**Definition 2** (Natural filtration). Let \(\mathcal{F}^{(X)}_n = \sigma(X_0, \ldots, X_n)\), the filtration \((\mathcal{F}^{(X)}_n)_{n \in \mathbb{N}}\) is called the natural filtration associated with the stochastic process \((X_n)_{n \in \mathbb{N}}\).

**Definition 3** (Adapted process). A stochastic process \((X_n)_{n \in \mathbb{N}}\) is adapted to the filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\) if, for every \(n \in \mathbb{N}\), \(X_n\) is \(\mathcal{F}_n\)-measurable.

**Definition 4** (Predictable process). A stochastic process \((H_n)_{n \in \mathbb{N}}\) is predictable for the filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\) if \(H_0\) is constant, and for every \(n \in \mathbb{N}\), \(n \geq 1\), \(H_n\) is \(\mathcal{F}_{n-1}\)-measurable.

**Definition 5** (“Stochastic integral”). Let \((X_n)_{n \in \mathbb{N}}\) be an adapted process, and \((H_n)_{n \in \mathbb{N}}\) be a predictable process, with respect to a filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\).

Define \((H.X)_0 = 0\), and for \(n \in \mathbb{N}\), \(n \geq 1\),

\[(H.X)_n = \sum_{k=1}^{n} H_k \Delta X_k = \sum_{k=1}^{n} H_k (X_k - X_{k-1}),\]

where \(\Delta X_k = X_k - X_{k-1}\) denotes increments of \(X\).

1.2 Stopping times

**Definition 6** (Stopping time). Let \(T\) be a \(\mathbb{N} \cup \{\infty\}\)-valued random variable. It is a stopping time, for the filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\), if, for every \(n \in \mathbb{N}\), the event \(\{T \leq n\} \in \mathcal{F}_n\).

**Example 1.** Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration, and \((X_n)_{n \in \mathbb{N}}\) be an adapted process. Let \(A \subset \mathbb{R}\) be a Borel set. Then (with the convention \(\inf \emptyset = \infty\)),

\[T_A = \inf \{n \in \mathbb{N} \mid X_n \in A\}\]

is a stopping time for the filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\).

**Definition 7.** Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration and \(T\) a stopping time.

Define

\[\mathcal{F}_T = \{A \in \mathcal{F}_\infty \mid \forall n \in \mathbb{N}, A \cap \{T \leq n\} \in \mathcal{F}_n\}.\]

Then \(\mathcal{F}_T\) is a \(\sigma\)-field.

**Definition 8** (Stopped process). Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration, \((X_n)_{n \in \mathbb{N}}\) be an adapted process, and \(T\) be a stopping time.

For any \(n \in \mathbb{N}\), set \(X_n^T = X_{n \land T}\) (with the notation \(n \land m = \min(n, m)\)), and \(\mathcal{F}_n^T = \mathcal{F}_{n \land T}\) (note that \(n \land T\) is a stopping time).

The process \((X^T_n)_{n \in \mathbb{N}}\) is adapted with respect to the filtrations \((\mathcal{F}_n^T)_{n \in \mathbb{N}}\) and \((\mathcal{F}_n)_{n \in \mathbb{N}}\).

**Definition 9.** Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration, \((X_n)_{n \in \mathbb{N}}\) be an adapted process, and \(T\) be an almost surely finite stopping time: \(\mathbb{P}(T = \infty) = 0\).

Set \(X_T = \sum_{k \in \mathbb{N}} X_k 1_{T=k}\), then \(X_T\) is a \(\mathcal{F}_T\)-measurable random variable.

2
2 Martingales

2.1 Definition

Definition 10 (Martingale). Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration.

A stochastic process \((M_n)_{n \in \mathbb{N}}\) is a martingale if it is an adapted integrable process – for all \(n \in \mathbb{N}\), \(\mathbb{E}[|M_n|] < \infty\) – such that, for all \(n \in \mathbb{N}\),

\[
\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n.
\]

Definition 11 (Submartingale, Supermartingale). Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration.

- An adapted, integrable, stochastic process \((M_n)_{n \in \mathbb{N}}\) is a submartingale, if for all \(n \in \mathbb{N}\), \(\mathbb{E}[M_{n+1} | \mathcal{F}_n] \geq M_n\).

- An adapted, integrable, stochastic process \((M_n)_{n \in \mathbb{N}}\) is a supermartingale, if for all \(n \in \mathbb{N}\), \(\mathbb{E}[M_{n+1} | \mathcal{F}_n] \leq M_n\).

Proposition 1. Let \((M_n)_{n \in \mathbb{N}}\) be a martingale (resp. submartingale, resp. supermartingale). Then \(n \in \mathbb{N} \mapsto \mathbb{E}[M_n]\) is constant (resp. non-decreasing, resp. non-increasing).

Proposition 2. Let \((M_n)_{n \in \mathbb{N}}\) be a martingale, and \(\phi : \mathbb{R} \to \mathbb{R}\) be a convex function, such that for all \(n \in \mathbb{N}\), \(\mathbb{E}[|\phi(M_n)|] < \infty\).

Then \((\phi(M_n))_{n \in \mathbb{N}}\) is a submartingale.

Proposition 3. Let \((\mathcal{F}_n)_{n \in \mathbb{N}}\) be a filtration.

Let \((\Delta M_n)_{n \in \mathbb{N}, n \geq 1}\) be an adapted, integrable, stochastic process, such that

\[
\mathbb{E}[\Delta M_n | \mathcal{F}_{n-1}] = 0
\]

for all \(n \geq 1\). Such random variables are called martingale increments.

Define \(M_0 = 0\) and \(M_n = \sum_{k=1}^n \Delta M_k\) for all \(n \geq 1\). Then \((M_n)_{n \in \mathbb{N}}\) is a martingale with respect to \((\mathcal{F}_n)_{n \in \mathbb{N}}\).

2.2 Examples

Example 2. Let \((Z_n)_{n \in \mathbb{N}}\) be a sequence of i.i.d., integrable, random variables (independent and identically distributed).

Set \(S_0 = 0\), and \(S_n = \sum_{k=1}^n Z_k - n\mathbb{E}[Z_0]\). Then \((S_n)_{n \in \mathbb{N}}\) is a martingale (with respect to the natural filtration).

Example 3. Let \((S_n)_{n \in \mathbb{N}}\) be defined as in the previous example. Assume that \(\mathbb{E}[Z_0] = 0\), and that \(\mathbb{E}[^{\exp(\alpha Z_0)}] < \infty\), for some \(\alpha \in \mathbb{R}\).

Define \(M_0 = 1\), and, for \(n \geq 1\), \(M_n = \exp(\alpha S_n - n\Phi(\alpha))\), where \(\Phi(\alpha) = \log \mathbb{E}[\exp(\alpha Z_0)]\).

Then \((M_n)_{n \in \mathbb{N}}\) is a martingale.
Example 4. Let $X$ be an integrable random variable, and $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a filtration. Define $M_n = \mathbb{E}[X|\mathcal{F}_n]$. Then $(M_n)_{n \in \mathbb{N}}$ is a martingale.

Example 5. Let $(H_n)_{n \in \mathbb{N}}$ be an integrable, predictable process, with respect to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

- If $(M_n)_{n \in \mathbb{N}}$ is a martingale, then $H.M$ is a martingale.
- If $(M_n)_{n \in \mathbb{N}}$ is a submartingale, and if $H_n \geq 0$ almost surely for all $n \in \mathbb{N}$, then $H.M$ is a submartingale.

Example 6. Let $(X_n)_{n \in \mathbb{N}}$ denote a Markov chain on a state space $E$, with transition kernel $P$. Let $f : E \to \mathbb{R}$ such that $\mathbb{E}[f(X_0)] < \infty$.

Note that $\mathbb{E}[f(X_{n+1}) - Pf(X_n)|\mathcal{F}_n^{(X)}] = 0$ for all $n \in \mathbb{N}$. Define $M_n = \sum_{k=1}^{n} f(X_k) - Pf(X_{k-1})$, for $n \geq 1$, and $M_0 = 0$. Then $(M_n)_{n \in \mathbb{N}}$ is a martingale.

Observe that $M_n = f(X_n) - f(X_0) - \sum_{k=1}^{n-1} ((P - I)f)(X_k)$.

If $g = f - Pf$, define $S_n = \sum_{k=0}^{n} g(X_k)$. Observe that $S_n = M_n + f(X_0) - Pf(X_n)$, thus $(S_n - f(X_0) + Pf(X_0))_{n \in \mathbb{N}}$ is a martingale.

If $g$ is given first, $f$ is called (if it exists) a solution of the Poisson equation $g = f - Pf$.

2.3 Doob’s decomposition for submartingales

Theorem 1 (Doob’s decomposition). Let $(X_n)_{n \in \mathbb{N}}$ be a submartingale, with respect to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

There exists a unique decomposition

$$X_n = M_n + A_n$$

where

- $(M_n)_{n \in \mathbb{N}}$ is a martingale (with respect to the same filtration),
- $(A_n)_{n \in \mathbb{N}}$ is a predictable, integrable, and non-decreasing process, with $A_0 = 0$.

Formula: $\Delta A_n = \mathbb{E}[\Delta X_n|\mathcal{F}_{n-1}] \geq 0$.

Example 7. Let $(M_n)_{n \in \mathbb{N}}$ be a square-integrable martingale. The Doob’s decomposition of the submartingale defined by $X_n = M_n^2$ is obtained with

$$A_n = \langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}]$$.

Equivalently, $(M_n^2 - \langle M \rangle_n)_{n \in \mathbb{N}}$ is a martingale.

The process $(\langle M \rangle_n)_{n \in \mathbb{N}}$ is called the (predictable) quadratic variation of the martingale $(M_n)_{n \in \mathbb{N}}$.

In particular, $\mathbb{E}[M_n^2] = \mathbb{E}[M_0^2] + \sum_{k=1}^{n} \mathbb{E}[\Delta M_k^2]$, where $\Delta M_k = M_k - M_{k-1}$.

In fact, the martingale increments $(\Delta M_k)_{k \in \mathbb{N}^*}$ are orthogonal in $L^2$. 

4
2.4 A first stopping theorem

Theorem 2. Let \((M_n)_{n \in \mathbb{N}}\) be a martingale (resp. submartingale), and \(T\) be a stopping time, for some filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\).

Then the stopped process \((M_T^n)_{n \in \mathbb{N}}\) is a martingale (resp. submartingale) with respect to the filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\).

Indeed,

\[
M^n_T = M_0 + \sum_{k=1}^{\infty} 1_{k \leq n \wedge T}(M_k - M_{k-1}) = M_0 + \sum_{k=1}^{n} 1_{T \geq k}(M_k - M_{k-1}) = M_0 + (H.M)_n
\]

with \(H_n = 1_{T \geq n} = 1 - 1_{T \leq n-1} \geq 0\).

Theorem 3 (Stopping theorem, version 1). Let \((M_n)_{n \in \mathbb{N}}\) be a martingale, and \(T, S\) be two bounded stopping times \(- T \leq K \text{ almost surely, for some integer } K\) – for some filtration \((\mathcal{F}_n)_{n \in \mathbb{N}}\), such that almost surely \(S \leq T\).

Then \(E[M_T|\mathcal{F}_S] = M_S\).

In particular, \(E[M_T] = E[M_S] = E[M_0]\).

Indeed, \(M_T - M_S = (H.M)_K\), where \(H_n = 1_{T \geq n} - 1_{S \geq n} = 1_{S < n \leq T} \geq 0\).

In general, this identity is not true for unbounded stopping times. It holds true only with additional assumptions on the martingale.

Proposition 4. Let \((M_n)_{n \in \mathbb{N}}\) be a submartingale, and \(T, S\) be two bounded stopping times, such that \(S \leq T\) almost surely.

Then \(E[M_S] \leq E[M_T]\).

3 Convergence results

3.1 Almost sure convergence

Theorem 4. Let \((X_n)_{n \in \mathbb{N}}\) be a submartingale, such that

\[
\sup_{n \in \mathbb{N}} E[(X_n)^+] < \infty.
\]

Then there exists an integrable random variable \(X_\infty\), such that

\[X_n \to X_\infty, \text{ a.s.}\]

Note that \(E[|X_n|] = \frac{1}{2} E[(X_n)^+] - E[X_n] \leq \frac{1}{2} E[(X_n)^+] - E[X_0]\) by the submartingale property: integrability is a consequence of the almost sure convergence, thanks to Fatou's Lemma.
Corollary 5. If \((X_n)_{n \in \mathbb{N}}\) is a martingale, and is bounded in \(L^1\), then it converges almost surely. Moreover, \(X_\infty = \lim_{n \to \infty} X_n\) satisfies \(\mathbb{E}|X_\infty| < \infty\).

Indeed, \(\mathbb{E}[X_n^+] = \mathbb{E}[X_n^-] + \mathbb{E}[X_n] \leq \mathbb{E}[|X_n|] + \mathbb{E}[X_0]\).

Corollary 6. If \((X_n)_{n \in \mathbb{N}}\) is a nonnegative supermartingale, then it converges almost surely. Moreover, \(X_\infty = \lim_{n \to \infty} X_n\) satisfies \(\mathbb{E}|X_\infty| < \infty\), and \(X_n \geq \mathbb{E}[X_\infty | \mathcal{F}_n]\) for all \(n \in \mathbb{N}\).

Example 8. Even if \(X_\infty\) is integrable, in general the convergence does not hold in \(L^1\): it may happen that \(\mathbb{E}[X_\infty] \neq \mathbb{E}[X_0]\).

For instance, let \((Z_n)_{n \in \mathbb{N}}\) iid, with \(\mathbb{P}(Z = 1) = \mathbb{P}(Z = -1) = \frac{1}{2}\). Let \(a > 0\), and \(S_n = a + Z_1 + \ldots + Z_n\), and \(T = \inf\{n \in \mathbb{N}; S_n = 0\}\).

The stopped martingale \(X = (S_{n \wedge T})_{n \in \mathbb{N}}\) is nonnegative, thus converges almost surely. The limit is \(X_\infty = 0\) (\(S_{n \wedge T} \in \mathbb{Z}\) converges if and only if it is stationary). Thus \(\mathbb{E}[X_\infty] = 0 \neq \mathbb{E}[X_0] = 1\).

3.2 Convergence in \(L^p\), \(p > 1\)

Theorem 7 (Doob’s maximal inequality). Let \((X_n)_{n \in \mathbb{N}}\) be a submartingale. Set \(X_n^* = \sup_{0 \leq k \leq n} X_k\).

For any \(a \in (0, \infty)\) and \(n \in \mathbb{N}\),

\[a \mathbb{P}(X_n^* \geq a) \leq \mathbb{E}[X_n 1_{X_n^* \geq a}] \leq \mathbb{E}[X_n^*].\]

Indeed, \(A = \{X_n^* \geq a\} = \{T \leq n\}\), where \(T = \inf\{k \in \mathbb{N}; X_k \geq a\}\) is a stopping time. One has \(a \mathbb{P}(A) + \mathbb{E}[X_n 1_{A^c}] \leq \mathbb{E}[X_{n \wedge T}] \leq \mathbb{E}[X_n] = \mathbb{E}[X_n 1_A] + \mathbb{E}[X_n 1_{A^c}]\).

Theorem 8. Let \((X_n)_{n \in \mathbb{N}}\) be a nonnegative submartingale. Set \(X_n^* = \sup_{0 \leq k \leq n} X_k\).

For any \(p \in (1, \infty)\) and \(n \in \mathbb{N}\),

\[\|X_n^*\|_p \leq \frac{p}{p-1} \|X_n\|_p,\]

with \(\|X\|_p = (\mathbb{E}|X|^p)^{\frac{1}{p}}\).

Application: \(X_n = |M_n|\) where \((M_n)_{n \in \mathbb{N}}\) is a martingale.

Theorem 9. Let \((M_n)_{n \in \mathbb{N}}\) be a martingale, bounded in \(L^p\) for some \(p > 1\): \(\mathbb{E}[|X_n|^p] < \infty\).

Then \(M_n \to M_\infty\), almost surely and in \(L^p\).

Moreover, \(\mathbb{E}[|M_\infty|^p] = \lim_{n \to \infty} \mathbb{E}[|M_n|^p] = \sup_{n \in \mathbb{N}} \mathbb{E}[|M_n|^p]\), and \(M_\infty^* = \sup_{n \in \mathbb{N}} M_n\) satisfies \(\|M_\infty^*\|_p \leq \frac{p}{p-1} \|M_\infty\|_p\).
The particular case $p = 2$ can be treated as follows. One has $E[M_n^2] \leq \sup_{0 \leq k \leq n} M_k^2 \leq 4E[M_n^2] = 4E[(M)_n]$. The martingale is bounded in $L^2$ if and only if $E[(M)_\infty] < \infty$, with $(M)_\infty = \lim_{n \to \infty} (M)_n$. Since $E[(M_{n+p} - M_n)^2] = E[M_{n+p}^2] - E[M_n^2]$, showing that $(M_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2$ is straightforward. Using the maximal inequality and this $L^2$ convergence, one obtains that $E \sup_{k, \ell \geq n} |M_k - M_\ell|^2 \to 0$: thus almost surely $(M_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, hence almost sure convergence (without the need of the previous results).

### 3.3 Convergence in $L^1$

**Theorem 10.** Let $(M_n)_{n \in \mathbb{N}}$ be a martingale.

The following properties are equivalent:

1. $(M_n)_{n \in \mathbb{N}}$ converges almost surely and in $L^1$,
2. $(M_n)_{n \in \mathbb{N}}$ is uniformly integrable,
3. $(M_n)_{n \in \mathbb{N}}$ is closed: there exists $Z \in L^1$ such that $M_n = E[Z | F_n]$ for all $n \in \mathbb{N}$.

In addition, one may choose $Z = M_\infty$.

**Theorem 11** (Stopping theorem, version 2). Let $(M_n)_{n \in \mathbb{N}}$ be a uniformly integrable martingale, and $T, S$ be two stopping times, for some filtration $(F_n)_{n \in \mathbb{N}}$, such that almost surely $S \leq T$.

Then $M_T = E[M_\infty | F_T]$.


### 4 Law of Large Numbers and Central Limit Theorem

**Theorem 12** (Precised almost sure convergence). Let $(M_n)_{n \in \mathbb{N}}$ be a square-integrable martingale. Set $(M)_\infty = \lim_{n \to \infty} (M)_n$.

Then on the event $(\langle M \rangle_\infty < \infty)$, the sequence $(M_n)_{n \in \mathbb{N}}$ converges almost surely.

The result is already known in the case $E[(M)_\infty] < \infty$.

**Sketch of proof:**

- note that $(\langle M \rangle_\infty < \infty) = \bigcup_{b \in \mathbb{N}} \{T_b = \infty\}$, with stopping times $T_b = \{n \in \mathbb{N}; \langle M \rangle_{n+1} > b\}$.

- For any $b \in \mathbb{N}$, consider the martingale stopped at $T_b$. It is bounded in $L^2$: indeed
  \[ E[M_{n \wedge T_b}^2] = E[M_0^2] + E[(M)_{n \wedge T_b}] \leq E[M_0^2] + b. \]

- It thus converges almost surely. This means that, on the event $T_b = \infty$, $M_n = M_{n \wedge T_b}$ converges almost surely.
Theorem 13 (Law of Large Numbers). Let \((M_n)_{n\in\mathbb{N}}\) be a square-integrable martingale. Then, on the set \(\{\langle M \rangle_\infty = \infty\}\),
\[
\frac{M_n}{\langle M \rangle_n} \to 0 \quad n \to \infty
\]
almost surely.

Sketch of proof:

- On the set \(\{\langle M \rangle_\infty = \infty\}\), it is equivalent to prove \(\frac{M_n}{1 + \langle M \rangle_n} \to 0 \quad n \to \infty\).
- Let \(K_n = \sum_{k=1}^{n} \frac{\Delta M_k}{1 + \langle M \rangle_k}\). It is sufficient to prove that, on the event \(\{\langle M \rangle_\infty = \infty\}\), \(K_n\) converges almost surely: this follows from Kronecker’s lemma.
- Observe that \((K_n)_{n\in\mathbb{N}}\) is a martingale, with \(\langle K \rangle_n \leq \langle M \rangle_n\): it is a square-integrable martingale.
- According to the result above, it suffices to prove that \(\langle K \rangle_\infty < \infty\) almost surely.
- In fact, \(\langle K \rangle_\infty \leq 1\) almost surely. Precisely, set \(\beta_k = 1 + \langle M \rangle_k\) (non-decreasing sequence), and
\[
\langle K \rangle_n = \sum_{k=1}^{n} \frac{\Delta \beta_k}{\beta_k^2} \leq \sum_{k=1}^{n} \frac{\Delta \beta_k}{\beta_k \beta_k^{-1}} = \sum_{k=1}^{\infty} \frac{1}{\beta_k} - \frac{1}{\beta_k} = 1 - \frac{1}{1 + \langle M \rangle_n} \leq 1.
\]

Theorem 14 (Central Limit Theorem). Let \((M_n)_{n\in\mathbb{N}}\) be a square-integrable martingale.

Let \(s_n^2 = \mathbb{E}[\langle M \rangle_n]\).

Assume

- \(\frac{\langle M \rangle_n}{s_n} \to 1\) in probability,
- \(\frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2 1_{|\Delta M_k| \geq \epsilon s_n} | \mathcal{F}_k] \to 0\), for every \(\epsilon > 0\) (Lindeberg condition).

Then
\[
\frac{M_n}{s_n} \to \mathcal{N}(0,1),
\]
in distribution.

5 Some extensions to the continuous time case

5.1 Filtrations, stopping times, processes

Definition 12. A non-decreasing sequence \((\mathcal{F}_t)_{t\in\mathbb{R}^+}\) of sub-\(\sigma\)-fields of \(\mathcal{F}\) is called a filtration.
Definition 13. A stochastic process \((X_t)_{t \in \mathbb{R}^+}\) is adapted to the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) if, for every \(t \in \mathbb{R}^+\), \(X_t\) is \(\mathcal{F}_t\)-measurable.

The natural filtration associated with the stochastic process \((X_t)_{t \in \mathbb{R}^+}\) is given by \(\mathcal{F}_t = \sigma(X(s), 0 \leq s \leq t)\).

Let \(\mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \in \mathbb{R}^+)\).

Definition 14. Let \(T\) be a \([0, \infty)\)-valued random variable. It is a stopping time, for the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\), if, for every \(t \in \mathbb{R}^+\), the event \(\{T \leq t\} \in \mathcal{F}_t\).

The associated \(\sigma\)-field \(\mathcal{F}_T\) is defined by \(\mathcal{F}_T = \{A \in \mathcal{F}_\infty; \ \forall \ t \in \mathbb{R}^+, A \cap \{T \leq t\} \in \mathcal{F}_t\}\).

Proposition 5. Let \(T\) be a stopping time. For any \(n \in \mathbb{N}\), define random variables:

\[
T_n^+ = \left\lfloor \frac{2^n T}{2n} \right\rfloor + 1.
\]

For every \(n \in \mathbb{N}\), \(T_n^+\) is a stopping times.

In addition, almost surely, \(T \leq T_{n+1}^+ \leq T_n^+, \text{ and } T = \lim_{n \to \infty} T_n^+\).

Definition 15. A stochastic process \((X_t)_{t \in \mathbb{R}^+}\) is continuous, resp. càdlàg, if

\[
P\left(\{\omega \in \Omega; \ t \mapsto X_t(\omega) \in C(\mathbb{R}^+, \mathbb{R})\}\right) = 1,
\]

resp.

\[
P\left(\{\omega \in \Omega; \ t \mapsto X_t(\omega) \in D(\mathbb{R}^+, \mathbb{R})\}\right) = 1,
\]

with

- \(C(\mathbb{R}^+, \mathbb{R})\) the space of continuous real-valued functions,
- \(D(\mathbb{R}^+, \mathbb{R})\) the space of real-valued functions which are right-continuous, with left-limits at any point.

Example 9. Let \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) be a filtration, and \((X_t)_{t \in \mathbb{R}^+}\) be an adapted process. Let \(A \subset \mathbb{R}\) be a Borel set. Set

\[T_A = \inf \{n \in \mathbb{N} \ ; \ X_n \in A\} .\]

If \(A\) is closed and \(X\) is a continuous process, then \(T_A\) is a stopping time for the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\).

Proposition 6. Let \((\mathcal{F}_t)_{t \in \mathbb{R}^+}\) be a filtration, \((X_t)_{t \in \mathbb{R}^+}\) be an adapted càdlàg process, and \(T\) be a stopping time.

Then the stopped process \((X_{t \wedge T})_{t \in \mathbb{R}^+}\) is adapted.
5.2 Martingales

Definition 16. Let $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ be a filtration.

A stochastic process $(M_t)_{t \in \mathbb{R}^+}$ is a martingale if it is an adapted integrable process – for all $t \in \mathbb{R}^+$, $\mathbb{E}[|M_t|] < \infty$ – such that, for all $s, t \in \mathbb{R}^+$, with $s \leq t$,

$$\mathbb{E}[M_t | \mathcal{F}_s] = M_s.$$  

Examples of results for càdlàg martingales:

Theorem 15. Let $(X_t)_{t \in \mathbb{R}^+}$ be a càdlàg martingale, which is bounded in $L^1$: $\sup_{t \in \mathbb{R}^+} \mathbb{E}[|X_t|] < \infty$.

Then almost surely $X_t$ converges, when $t \to \infty$, to $X_\infty$. Moreover, $X_\infty \in (-\infty, \infty)$ almost surely.

Theorem 16. Let $(X_t)_{t \in \mathbb{R}^+}$ be a càdlàg martingale. Set $X_t^* = \sup_{s \in [0,t]} |X_s|$.

For any $a \in (0, \infty)$ and $t \in \mathbb{R}^+$,

$$a \mathbb{P}(X_t^* \geq a) \leq \mathbb{E}[|X_t|].$$  

Moreover, for every $p \in (1, \infty)$,

$$\|X_t^*\|_p \leq \frac{p}{p-1} \|X_t\|_p.$$  

Indeed, $X_t^* = \sup_{s \in \{t\} \cup [0,t] \cap \mathbb{Q}} |X_s|$.

Theorem 17. Let $(X_t)_{t \in \mathbb{R}^+}$ be a càdlàg martingale.

Let $T, S$ be two bounded stopping times, such that almost surely $S \leq T$.

Then $\mathbb{E}[M_T | \mathcal{F}_S] = M_S$.