Surface approximation, discrete varifolds, and regularized first variation

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Introduction

Shape visualization and processing are fundamental tasks in many fields from mechanical engineering to physics, biology, chemical engineering, medicine, astronomy, etc., and are the subject of very active research in image processing and computer graphics. For there is a huge variety of applications and a large variety of capture systems, there are many discrete models for representing a shape: point clouds, meshes, pixel/voxel representations, splines, level sets representations, etc. All these models carry very different informations, and it is hopeless to look after equivalence relations between them. It makes sense however to try to look at all these models in a common framework, possibly weaker, which would provide a common formalism for studying a large variety of both discrete and continuous shapes.

Such a common framework has been proposed in [Bue14, Bue15]: the class of varifolds. Varifolds have been introduced by Almgren in 1965 [Alm65] to study the existence of critical points of the area functional. They have several nice properties in a variational context: compactness, mass continuity, criteria of rectifiability, a notion of multiplicity, and a weak notion of curvature called the first variation [All72, Sim83]. In addition, the varifold structure is flexible enough to describe not only classical continuous objects as curves, surfaces, rectifiable sets, etc., but also "discrete" objects like meshes, point clouds, volumetric representations. etc.

Previous contributions in the literature also involve tools from geometric measure theory to define a convenient notion of curvature or to study more generally discrete surface approximation. For instance, a unified notion of *curvature measure* valid both for surfaces and their discrete approximations, and based on *normal cycles*, is introduced in [CSM06]. First defined for surfaces and triangulations [Mor08], it has been recently extended in [CCLT09] to more general discretizations like point clouds. The accuracy of the approximation of the surface is measured in terms of Hausdorff distance while the error between the curvature measure of the surface and the curvature measure of the approximation is controlled in terms of the *Bounded Lipschitz distance* which is similar to the Wasserstein distance. Shape comparison is another application of geometric measure theory: Charon and Trouvé[CT13] endow triangulated surfaces with a varifold structure, and define a distance between varifolds, both computable from a numerical point of view and adapted to shape matching.

In the first section of this paper, we follow [Bue15] and introduce *discrete volumetric varifolds* and *point cloud varifolds* to represent both volumetric surface models and point clouds. We study convergence issues, and in particular, in Theorem 2.6 , we estimate in terms of the Bounded Lipschitz distance the quality of the approximation of rectifiable varifolds by either discrete volumetric varifolds or point cloud varifolds.

We already mentioned that any varifold can be endowed with a notion of generalized curvature, namely the first variation. However, the convergence of a sequence of varifolds $(V_i)_i$ does not imply the convergence of the associated first variations, unless the latter are uniformly bounded, that is, if

$$\sup_i \|\delta V_i\| < +\infty \,.$$

This condition is not always satisfied neither by weakly–$*$ converging sequences of discrete volumetric varifolds, nor by point cloud varifolds, hence the following question arises:

Question 1. *What conditions on a weakly–$*$ converging sequence of varifolds (possibly not rectifiable) ensure that the limit varifold has bounded first variation? that the first variations converge?*

We start from the following observation: even when a varifold does not have bounded first variation, the first variation is by definition a linear form on C_c^1, which can thus always be tested on a C^1 or Lipschitz function, so that it is possible to regularize the first variation δV of a varifold V with a Lipschitz or even more regular kernel ρ_ε. This regularization $\delta V * \rho_\varepsilon$ has an explicit expression (given in Proposition 3.1). Such regularization allows to give an answer to Question 1 (see Theorem 3.5). Moreover, it provides a notion of approximate curvature for a varifold V (depending on the choice of a regularizing kernel ρ_ε):

$$H_\varepsilon = -\frac{\delta V * \rho_\varepsilon}{\|V\| * \rho_\varepsilon} \,, \tag{1}$$

which converges $\|V\|$–almost everywhere to the classical mean curvature as soon as the varifold V has bounded first variation (see Proposition 3.6).

It remains to study the behavior of the approximate mean curvature (1) on sequences of discrete varifolds. We thus need to make a connection between the scale parameter ε of the approximation and the scale of the discrete objects (the size δ_i of the mesh $\mathcal{K}_i$, assuming that we are considering a sequence of discrete volumetric varifolds V_i associated to a varifold V and to a sequence of meshes $(\mathcal{K}_i)_i$, see Definition 9).

Question 2. *Are there conditions on the infinitesimal sequences $(\varepsilon_i)_i$ and $(\delta_i)_i$ ensuring that the approximate mean curvatures of discrete volumetric varifolds $(V_i)_i$ associated with a rectifiable varifold V and with the sequence of meshes $(\mathcal{K}_i)_i$ of size $\delta_i \to 0$, converge to the mean curvature of V?*

The answer is yes, under some additional assumptions on the regularity of V and ρ, as well as on the relation between the parameters ε_i and δ_i, see Proposition 2.4 and Theorem 3.10.

Then, in order to understand better the approximate first variation provided by the regularization $\delta V * \rho_\varepsilon$, we address the question of representing $\delta V * \rho_\varepsilon$ as the first variation of some conveniently chosen varifold:

Question 3. *Given a d–varifold V, is the regularization $\delta V * \rho_\varepsilon$ of the first variation δV, the first variation $\delta\left(\widehat{V}_\varepsilon\right)$ of some varifold $\widehat{V}_\varepsilon$? If this is the case, is $\widehat{V}_\varepsilon$ the regularization (in a sense to be defined) of V?*

The construction of $\widehat{V}_\varepsilon$ can be done explicitly (Theorem 4.2). Indeed for every $\psi \in C_c^0(\Omega \times G_{d,n})$,

$$\left\langle \widehat{V}_\varepsilon, \psi \right\rangle = \langle V, (y, S) \mapsto \psi(\cdot, S) * \rho_\varepsilon(y) \rangle \,.$$

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We observe that the mass $\|\widehat{V}_\varepsilon\| = \|V\| * \rho_\varepsilon$ is the convolution of $\|V\|$ and in Proposition 4.4, we point out that the tangential part $\widehat{\nu}_x^\varepsilon$ of $\widehat{V}_\varepsilon = \|\widehat{V}_\varepsilon\| \otimes \widehat{\nu}_x^\varepsilon$ is generally neither a Dirac mass nor a combination of Dirac masses.

The final section of the paper is devoted to numerical experiments in order to illustrate on 2D subsampled parametric test shapes and on 3D point clouds our notion of approximate mean curvature. In particular, we study the behavior with respect to the choice of a regularizing parameter, or the ratio between the number of points and the scale parameter ε.

Notations

From now on, we fix d, $n \in \mathbb{N}$ with $1 \le d < n$ and an open set $\Omega \subset \mathbb{R}^n$. We adopt the following notations.

- $A \triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference.

- $B_r(x) = \{y \mid |y - x| < r\}$ is the open ball in $\mathbb{R}^n$ of center x and radius r.

- $B^\delta = \bigcup_{x \in B} B_\delta(x) = \{y \in \mathbb{R}^n \mid d(y, B) \le \delta\}$.

- $\omega_d = \mathcal{L}^d(B_1(0))$ is the d–volume of the unit ball in $\mathbb{R}^d$.

- Being A, B two open sets then $A \subset\subset \Omega$ means that A is relatively compact in B.

- Let $A \subset \Omega$ then $A^c = \Omega \setminus A$ denotes the complementary of A in Ω.

- $G_{d,n}$ is the Grassmannian of all d-dimensional vector subspaces of $\mathbb{R}^n$ equipped with the metric
$$d(T, P) = \|\Pi_T - \Pi_P\|$$
with $\Pi_T \in M_n(\mathbb{R})$ being the matrix of the orthogonal projection onto T and $\|\cdot\|$ a norm on $M_n(\mathbb{R})$. Measures on $\Omega \times G_{d,n}$ are considered with respect to the Borel algebra on $\Omega \times G_{d,n}$.

- Given a continuous $\mathbb{R}^m$–valued function f defined in Ω, its support spt f is the closure in Ω of $\{y \in \Omega \mid f(y) \ne 0\}$.

- $C_c^k(\Omega)$ is the space of real continuous compactly supported functions of class C^k ($k \in \mathbb{N}$) in Ω.

- $C_o^0(\Omega)$ is the closure of $C_c^0(\Omega)$ for the sup norm.

- $\text{Lip}_k(\Omega)$ is the space of Lipschitz functions in Ω with Lipschitz constant $\le k$.

- We denote by $|\mu|$ the total variation of a measure μ.

- $\mathcal{M}_{loc}(\Omega)^m$ is the space of $\mathbb{R}^m$–valued Radon measures and $\mathcal{M}(\Omega)^m$ is the space of R^m–valued finite Radon measures.

- $\mathcal{L}^n$ is the n–dimensional Lebesgue measure.

- $\mathcal{H}^d$ is the d–dimensional Hausdorff measure.

1 Generalities on varifolds

We recall here a few facts about varifolds, see [Sim83] for a more complete survey of varifolds theory.

Definition 1 (Rectifiable d–varifold). *Given an open set $\Omega \subset \mathbb{R}^n$, let M be a countably d–rectifiable set and θ be a non negative function with $\theta > 0$ $\mathcal{H}^d$–almost everywhere in M. A rectifiable d–varifold $V = v(M, \theta)$ in Ω is a positive Radon measure on $\Omega \times G_{d,n}$ of the form $V = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ i.e.*

$$\int_{\Omega \times G_{d,n}} \varphi(x, T) \, dV(x, T) = \int_M \varphi(x, T_x M) \, \theta(x) \, d\mathcal{H}^d(x) \quad \forall \varphi \in \mathrm{C}_c(\Omega \times G_{d,n}, \mathbb{R})$$

where $T_x M$ is the approximative tangent space at x which exists $\mathcal{H}^d$–almost everywhere in M. The function θ is called the multiplicity *of the rectifiable varifold.*

Let us turn to the general notion of varifold:

Definition 2 (General d–varifold). *Let $\Omega \subset \mathbb{R}^n$ be an open set. A d–varifold in Ω is a positive Radon measure on $\Omega \times G_{d,n}$.*

Remark 1.1. As $\Omega \times G_{d,n}$ is locally compact, Riesz Theorem allows to identify Radon measures on $\Omega \times G_{d,n}$ and continuous linear forms on $\mathrm{C}_c(\Omega \times G_{d,n})$ (we used this fact in the definition of rectifiable d–varifolds) and the convergence in the sense of varifolds is then the weak–$*$ convergence.

Definition 3 (Convergence of varifolds). *A sequence of d–varifolds $(V_i)_i$ weakly–$*$ converges to a d–varifold V in Ω if, for all $\varphi \in \mathrm{C}_c(\Omega \times G_{d,n})$,*

$$\langle V_i, \varphi \rangle = \int_{\Omega \times G_{d,n}} \varphi(x, P) \, dV_i(x, P) \xrightarrow[i \to \infty]{} \langle V, \varphi \rangle = \int_{\Omega \times G_{d,n}} \varphi(x, P) \, dV(x, P) \, .$$

Definition 4 (Mass). *The mass of a general varifold V is the positive Radon measure defined by $\|V\|(B) = V(\pi^{-1}(B))$ for every $B \subset \Omega$ Borel, with*

$$\begin{cases} \pi : & \Omega \times G_{d,n} & \to & \Omega \\ & (x, S) & \mapsto & x \end{cases}$$

In particular, the mass of a d–rectifiable varifold $V = v(M, \theta)$ is the measure $\|V\| = \theta \mathcal{H}^d_{|M}$.

The set of d–varifolds is endowed with a notion of generalized curvature called *first variation*.

Definition 5 (First variation of a varifold). *The first variation of a d–varifold in $\Omega \subset \mathbb{R}^n$ is the linear functional*

$$\delta V : \begin{array}{ccc} \mathrm{C}_c^1(\Omega, \mathbb{R}^n) & \to & \mathbb{R} \\ X & \mapsto & \int_{\Omega \times G_{d,n}} \mathrm{div}_P X(x) \, dV(x, P) \end{array}$$

It is a distribution of order 1.

For $P \in G$ and $X = (X_1, \ldots, X_n) \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$, the operator div_P is defined as

$$\mathrm{div}_P(x) = \sum_{j=1}^n \langle \nabla^P X_j(x), e_j \rangle = \sum_{j=1}^n \langle \Pi_P(\nabla X_j(x)), e_j \rangle \text{ whith } (e_1, \ldots, e_n) \text{ the canonical basis of } \mathbb{R}^n.$$

The linear functional δV is generally not continuous with respect to the C_c^0 topology. When it is true, we say that the varifold has *locally bounded first variation*:

Definition 6 (Locally bounded first variation). *We say that a d–varifold on Ω has locally bounded first variation when the linear form δV is continuous that is to say, for every compact set $K \subset \Omega$ there is a constant c_K such that for every $X \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$ with $\mathrm{spt}\, X \subset K$,*

$$|\delta V(X)| \leq c_K \sup_K |X| \,.$$

If a d–varifold V has locally bounded first variation, the linear form δV can be extended into a continuous linear form on $\mathrm{C}_c(\Omega, \mathbb{R}^n)$, and by Riesz Theorem, there exists a Radon measure on Ω (still denoted δV) such that

$$\delta V(X) = \int_\Omega X \cdot \delta V \quad \text{for every } X \in \mathrm{C}_c(\Omega, \mathbb{R}^n)$$

Thanks to Radon-Nikodym Theorem, we can derive δV with respect to $\|V\|$ and there exist a function $H \in \left(\mathrm{L}^1_{loc}(\Omega, \|V\|)\right)^n$ and a measure δV_s singular to $\|V\|$ such that

$$\delta V = -H\|V\| + \delta V_s \,.$$

The function H is called the *generalized mean curvature vector*. Thanks to the divergence theorem, it coincides with the classical notion of mean curvature if $V = v(M, 1)$ with M a C^2 submanifold. We now define and study varifolds structures on discrete objects.

We now recall what is the *Bounded Lipschitz distance* between two Radon measures. It is also called the *flat metric* and it can be seen as a modified 1–Wasserstein distance which allows the comparison of measures with different masses. In contrast, the 1–Wasserstein distance between two measures with different masses is infinite.

Definition 7 (Bounded Lipschitz distance / Flat distance). *Let μ and ν be two Radon measures on a locally compact metric space (X, d), the quantity*

$$\Delta^{1,1}(\mu, \nu) = \sup \left\{ \left| \int_X \varphi \, d\mu - \int_X \varphi \, d\nu \right| \, : \, \varphi \in \mathrm{Lip}(X),\ \|\varphi\|_\infty \leq 1 \ and\ \mathrm{lip}(\varphi) \leq 1 \right\}$$

defines a distance in the space of Radon measure in X, called the Bounded Lipschitz distance.

Remark 1.2. As for Wasserstein distances (see [Vil09]), the Bounded Lipschitz distance has a dual formulation (see [PR14]).

We shall also use the following localized notion of $\Delta^{1,1}$ convergence, in the case of varifolds:

Definition 8. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let μ and ν be two d–varifolds in Ω. For any open ball $B \subset \Omega$ we define*

$$\Delta^{1,1}_B(\mu, \nu) = \sup \left\{ \left| \int_{\Omega \times G_{d,n}} \varphi \, d\mu - \int_{\Omega \times G_{d,n}} \varphi \, d\nu \right| \, : \, \begin{array}{c} \varphi \in \mathrm{Lip}(\Omega \times G_{d,n}),\ \|\varphi\|_\infty \leq 1,\ \mathrm{lip}(\varphi) \leq 1 \\ and\ \mathrm{spt}\,\varphi \subset B \times G_{d,n} \end{array} \right\} \,.$$

2 Discrete varifolds

Most surface representations can be endowed with a varifold structure (triangulations, point clouds, pixelizations). Here we focus on non-structured discretizations for a couple of reasons. First, non-structured discretizations are more flexible as they do not require or impose a-priori constraints on the geometry or topology of the surfaces: one may think for instance to the Plateau problem, that is, to the minimization of the area of surfaces spanning some common boundary, but without any a-priori assumption on the topology of the competing surfaces. Second, more structured discretizations can be treated with more specific tools from discrete geometry [BSSZ08], while for less structured ones (like pixelizations and point clouds) the frameworks proposed in the existing literature are quite limited and mainly focused on the reconstruction of regular or triangulated surfaces (see for instance [HDD$^+$92]). For instance, in this second case it is more difficult to design robust approximation schemes or curvature estimators.

We consider two main classes of such discretizations: volumetric varifolds (which were introduced in [Bue15]) and point cloud varifolds.

2.1 Volumetric varifolds and point cloud varifolds

A mesh of an open set Ω is a countable and locally finite partition

$$\mathcal{K} = \bigsqcup_{K \in \mathcal{K}} K$$

of Ω. For the moment, no other assumptions on the shape of the cells or on the geometry of the mesh are needed except that *the size of the mesh*

$$\delta = \sup_{K \in \mathcal{K}} \operatorname{diam} K$$

is finite.

Let us introduce the notion of *discrete volumetric varifold* (see [Bue15]). Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mathcal{K}$ be a mesh of Ω. Roughly speaking, given for instance a d–rectifiable set $M \subset \mathbb{R}^n$ (a curve, a surface...) we can define for any cell $K \in \mathcal{K}$, a mass m_K (the length of the piece of curve in the cell, the area of the piece of surface in the cell...) and a mean tangent plane P_K as

$$m_K = \mathcal{H}^d(M \cap K) \text{ and } P_K \in \operatorname*{arg\,min}_{S \in G_{d,n}} \int_{M \cap K} |T_x M - S| \, d\mathcal{H}^d(x) \,,$$

and similarly, given a rectifiable d–varifold V, defining

$$m_K = \|V\|(K) \text{ and } P_K \in \operatorname*{arg\,min}_{S \in G_{d,n}} \int_{K \times G_{d,n}} |P - S| \, dV(x, P) \,,$$

gives what we call a volumetric approximation of V. This yields the following definition of discrete volumetric varifold:

Definition 9. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider a mesh $\mathcal{K}$ of Ω and a set of pairs $\{(m_K, P_K)\}_{K \in \mathcal{K}} \subset \mathbb{R}_+ \times G_{d,n}$. We can associate this set of pairs with the d–varifold*

$$V_\mathcal{K} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \,, \quad \text{where } |K| = \mathcal{L}^n(K) \,.$$

This d–varifold is not rectifiable since its support is n–rectifiable but not d–rectifiable. We will refer to the set of d–varifolds of this special form as discrete volumetric varifolds.
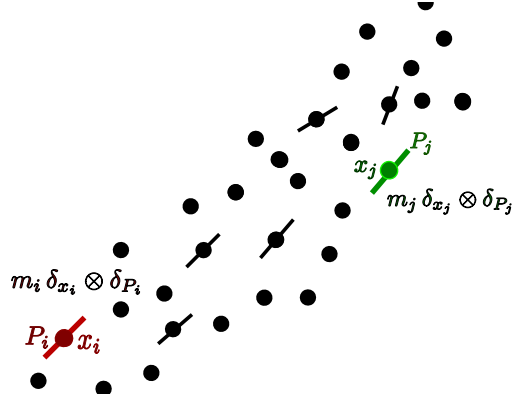
We now define a varifold structure on point clouds.

Definition 10 (Point cloud varifolds)**.** *Let $\{x_i\}_{i=1\ldots N} \subset \mathbb{R}^n$ be a point cloud, weighted by the masses $\{m_i\}_{i=1\ldots N}$ and provided with directions $\{P_i\}_{i=1\ldots N} \subset G_{d,n}$. We can thus associate the set of triplets $\{(\{x_i, m_i, P_i\} : i = 1, \ldots, N\}$ with a d–varifold on $\mathbb{R}^n \times G_{d,n}$:*

$$V = \sum_{i=1}^{N} m_i\, \delta_{x_i} \otimes \delta_{P_i} \,,$$

so that for $\varphi \in \mathrm{C}_c^0(\Omega \times G_{d,n})$,

$$\int \varphi\, dV = \sum_{i=1}^{N} \varphi(x_i, P_i) \,.$$

2.2 A one-to-one correspondence between volumetric varifolds and point cloud varifolds

First of all, let us point out the following fact:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider a mesh $\mathcal{K}$ of Ω of size $\eta = \sup\limits_{K \in \mathcal{K}} \operatorname{diam} K$ and a family $\{x_K, m_K, P_K\}_{K \in \mathcal{K}} \subset \mathbb{R}^n \times \mathbb{R}_+ \times G_{d,n}$ such that $x_K \in K$, for all $K \in \mathcal{K}$. Define the volumetric varifold $V_{\mathcal{K}}^{vol}$ and the point cloud varifold $V_{\mathcal{K}}^{pt}$ as*

$$V_{\mathcal{K}}^{vol} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}_{|K}^n \otimes \delta_{P_K} \ and \ \sum_{K \in \mathcal{K}} m_K \delta_{x_K} \otimes \delta_{P_K} \,.$$

Then, for any $B \subset \Omega$, setting $B^\eta = \{x \in \Omega : d(x, B) \le \eta\}$, we obtain $\mu_\eta := \|V_{\mathcal{K}}^{vol}\|(B^\eta) = \|V_{\mathcal{K}}^{pt}\|(B^\eta)$ and $\mu = \|V_{\mathcal{K}}^{vol}\|(\Omega) = \|V_{\mathcal{K}}^{pt}\|(\Omega)$, and moreover

$$\Delta_B^{1,1}(V_{\mathcal{K}}^{vol}, V_{\mathcal{K}}^{pt}) \le \eta\mu_\eta \ and \ thus \ \Delta^{1,1}(V_{\mathcal{K}}^{vol}, V_{\mathcal{K}}^{pt}) \le \eta\mu$$

Proof. Let $\varphi \in \mathrm{Lip}_1(\mathbb{R}^n \times G_{d,n})$ such that $\mathrm{spt}\,\varphi \subset B \subset \Omega$, then

$$\left| \int_{\Omega \times G_{d,n}} \varphi \, dV_{\mathcal{K}}^{vol} - \int_{\Omega \times G_{d,n}} \varphi \, dV_{\mathcal{K}}^{pt} \right| = \left| \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \int_K \varphi(x, P_K) \mathcal{L}^n(x) - \sum_{K \in \mathcal{K}} m_K \varphi(x_K, P_K) \right|$$

$$\leq \sum_{\substack{K \in \mathcal{K} \\ B \cap K \neq \emptyset}} m_K \fint_K |\varphi(x, P_K) - \varphi(x_K, P_K)| \, d\mathcal{L}^n(x)$$

$$\leq \sum_{\substack{K \in \mathcal{K} \\ B \cap K \neq \emptyset}} \fint_K \underbrace{\mathrm{lip}(\varphi)}_{=1} |x - x_K| \, d\mathcal{L}^n(x)$$

$$\leq \eta \sum_{\substack{K \in \mathcal{K} \\ B \cap K \neq \emptyset}} m_K$$

$$= \eta \, \|V_{\mathcal{K}}^{vol}\| \left(\bigcup_{B \cap K \neq \emptyset} K \right) = \eta \, \|V_{\mathcal{K}}^{pt}\| \left(\bigcup_{B \cap K \neq \emptyset} K \right).$$

And we conclude since $\displaystyle\bigcup_{B \cap K \neq \emptyset} K \subset \{x \in \Omega \, : \, d(x, B) \leq \eta\}$. $\qquad\square$

Remark 2.2. Proposition 2.1 allows to switch between discrete volumetric and point cloud varifolds, up to some quantified error in terms of bounded Lipschitz distance. Indeed, given a discrete volumetric varifold

$$V_{\mathcal{K}}^{vol} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K}$$

one can choose a finite set of points $x_K \in K$, one for each $K \in \mathcal{K}$, and define the point cloud varifold

$$V_{\mathcal{K}}^{pt} = \sum_{K \in \mathcal{K}} m_K \delta_{x_K} \otimes \delta_{P_K} \,,$$

so that by Proposition 2.1 one has

$$\Delta^{1,1}(V_{\mathcal{K}}^{vol}, V_{\mathcal{K}}^{pt}) \leq \eta \|V_{\mathcal{K}}^{vol}\|(\Omega) \,.$$

Conversely, given a point cloud varifold

$$V_N^{pt} = \sum_{i=1}^N m_i \delta_{x_i} \otimes \delta_{P_i} \,,$$

one can consider a mesh $\mathcal{K}$ with mesh-size η smaller than the minimum distance between x_i and x_j for $i, j = 1, \ldots, N$ with $i < j$. Then one defines $m_K = m_i$ and $P_K = P_i$ if and only if $x_i \in K$, while $m_K = 0$ and $P_K = $ any d-plane, if and only if $x_i \notin K$ for all $i = 1, \ldots, N$ (note that the previous assumption on the mesh size implies that at most one point x_i belongs to the same cell K). Finally, one defines the discrete volumetric varifold

$$V_{\mathcal{K}}^{vol} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \,.$$

It is the immediate to check that $\|V_{\mathcal{K}}^{vol}\|(\Omega) = \sum_j m_j = \|V_N\|(\Omega) < +\infty$ and, thanks to Proposition 2.1, to conclude that

$$\Delta^{1,1}(V_N, V_{\mathcal{K}}^{vol}) \leq \eta \|V_N\|(\Omega) \,.$$

2.3 Approximation of rectifiable varifolds by discrete varifolds

In this section, we prove that the family of discrete volumetric varifolds and the family of point cloud varifolds approximate well the space of rectifiable varifolds in the sense of weak−∗ convergence. Moreover, we give a way of quantifying this approximation in terms of the mesh size and the mean oscillation of tangent planes.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V a d−varifold in Ω. Let let $(\mathcal{K}_i)_{i\in\mathbb{N}}$ be a sequence of meshes of Ω, and set*

$$\delta_i = \sup_{K \in \mathcal{K}_i} \operatorname{diam}(K) \quad \forall\, i \in \mathbb{N} \,.$$

Then, there exist a sequence of point cloud varifolds $(V_i^{pt})_i$ and a sequence of volumetric varifolds $(V_i^{vol})_i$ such that for any $B \subset \Omega$,

$$\max\left(\Delta_B^{1,1}(V, V_i^{pt}),\ \Delta_B^{1,1}(V, V_i^{vol})\right) \leq \delta_i \|V\|(B^{\delta_i}) + \sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(B^{\delta_i} \cap K) \times G_{d,n}} \|P - S\|\, dV(x, S) \,.$$

$$(2)$$

Proof. Let us explain the construction for a fixed i. We define the volumetric varifold V_i^{vol} and the point cloud varifold V_i^{pt} as

$$V_i^{vol} = \sum_{K \in \mathcal{K}_i} \frac{m_K^i}{|K|} \mathcal{L}^n \otimes \delta_{P_K^i} \quad \text{and} \quad V_i^{pt} = \sum_{K \in \mathcal{K}_i} m_K^i \delta_{x_K^i} \otimes \delta_{P_K^i} \,,$$

with

$$m_K^i = \|V\|(K), \quad x_K^i \in K \quad \text{and} \quad P_K^i \in \arg\min_{P \in G_{d,n}} \int_{K \times G_{d,n}} \|P - S\|\, dV(x, S) \,.$$

Then, for any open ball $B \subset \Omega$ and $\varphi \in \operatorname{Lip}_1(\mathbb{R}^n \times G_{d,n})$ with $\operatorname{spt} \varphi \subset B \times G_{d,n}$, we set

$$\Delta_i^{vol}(\varphi) = \int_{\Omega \times G_{d,n}} \varphi\, dV_i^{vol} - \int_{\Omega \times G_{d,n}} \varphi\, dV$$

and obtain

$$\left|\Delta_i^{vol}(\varphi)\right| = \left| \sum_{K \in \mathcal{K}_i} \int_{K \cap B} \varphi(x, P_K^i) \frac{\|V\|(K)}{|K|}\, d\mathcal{L}^n(x) - \sum_{K \in \mathcal{K}_i} \int_{(K \cap B) \times G_{d,n}} \varphi(y, T)\, dV(y, T) \right|$$

$$\leq \sum_{\substack{K \in \mathcal{K}_i \\ K \cap B \neq \emptyset}} \fint_{x \in K} \int_{(y,T) \in K \times G_{d,n}} \underbrace{\left|\varphi(x, P_K^i) - \varphi(y, T)\right|}_{\leq (|x-y| + \|P_K^i - T\|)}\, dV(y, T)\, d\mathcal{L}^n(x)$$

$$\leq \delta_i \sum_{\substack{K \in \mathcal{K}_i \\ K \cap B \neq \emptyset}} \|V\|(K) + \sum_{K \in \mathcal{K}_i} \int_{K \times G_{d,n}} \left\|P_K^i - T\right\|\, dV(y, T)$$

$$\leq \delta_i \|V\|(B^{\delta_i}) + \sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(B^{\delta_i} \cap K) \times G_{d,n}} \|P - T\|\, dV(y, T) \,.$$

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By a similar computation we derive the analogous estimate for V_i^{pt}: setting

$$\Delta_i^{pt}(\varphi) = \int_{\Omega \times G_{d,n}} \varphi \, dV_i^{vol} - \int_{\Omega \times G_{d,n}} \varphi \, dV$$

we find

$$\left| \Delta_i^{pt}(\varphi) \right| = \left| \sum_{K \in \mathcal{K}_i} \varphi(x_K^i, P_K^i) \|V\|(K) \, d\mathcal{L}^n(x) - \sum_{K \in \mathcal{K}_i} \int_{(K \cap B) \times G_{d,n}} \varphi(y,T) \, dV(y,T) \right|$$

$$\leq \sum_{\substack{K \in \mathcal{K}_i \\ K \cap B \neq \emptyset}} \fint_{x \in K} \int_{(y,T) \in K \times G_{d,n}} \underbrace{\left| \varphi(x, P_K^i) - \varphi(y,T) \right|}_{\leq \left(|x-y| + \| P_K^i - T \| \right)} \, dV(y,T) \, d\mathcal{L}^n(x)$$

$$\leq \delta_i \|V\|(B^{\delta_i}) + \sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(B^{\delta_i} \cap K) \times G_{d,n}} \|P - T\| \, dV(y,T) \,,$$

which concludes the proof. $\qquad\square$

We now study, for a given varifold V, the convergence of the term

$$\sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(B^{\delta_i} \cap K) \times G_{d,n}} \|P - T\| \, dV(y,T) \,.$$

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(\mathcal{K}_i)_{i \in \mathbb{N}}$ be a sequence of meshes of Ω. Set $\delta_i = \sup_{K \in \mathcal{K}_i} \mathrm{diam}(K)$ and assume that $\delta_i \to 0$ as $i \to \infty$. Let $V = v(M, \theta)$ be a rectifiable d–varifold in Ω with mass $\|V\|(\Omega) < +\infty$. Then it holds*

$$\sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(\Omega \cap K) \times G_{d,n}} \|P - T\| \, dV(y,T) \xrightarrow[i \to \infty]{} 0 \,. \tag{3}$$

Moreover, if there exist constants $C, C', \beta, \gamma > 0$ and a decomposition $\mathcal{K}_i = \mathcal{K}_i^{reg} \sqcup \mathcal{K}_i^{sing}$ for all i, such that

$$\|T_x M - T_y M\| \leq C |x - y|^\beta, \ \forall x, y \in K, \ \forall K \in \mathcal{K}_i^{reg} \tag{4}$$

and

$$\|V\| \left(\mathcal{K}_i^{sing} \right) \leq C' \delta_i^\gamma \,, \tag{5}$$

then one can find $C'' > 0$ such that

$$\sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(B^{\delta_i} \cap K) \times G_{d,n}} \|P - T\| \, dV(y,T) \leq C'' \delta_i^{\min(\beta, \gamma)} \|V\|(B^{\delta_i}) \tag{6}$$

for all $B \subset \Omega$ and all $i \in \mathbb{N}$.

Proof. Step 1. For all i, there exists $A^i : \Omega \to \mathcal{M}_n(\mathbb{R})$ constant in each cell $K \in \mathcal{K}_i$ and such that

$$\int_{\Omega \times G_{d,n}} \left\| A^i(y) - T \right\| \, dV(y,T) = \int_{y \in \Omega} \left\| A^i(y) - T_y M \right\| \, d\|V\|(y) \xrightarrow[i \to +\infty]{} 0 \,.$$

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Indeed, let $\varepsilon > 0$, as $x \mapsto T_x M \in \mathrm{L}^1(\Omega, \mathcal{M}^n(\mathbb{R}), \|V\|)$ then, there exists $A : \Omega \to \mathcal{M}^n(\mathbb{R}) \in \mathrm{Lip}(\Omega)$ such that

$$\int_{y \in \Omega} \|A(y) - T_y M\| \, d\|V\|(y) < \varepsilon \,.$$

For all i and $K \in \mathcal{K}_i$, define for $x \in K$,

$$A^i(x) = A^i_K = \frac{1}{\|V\|(K)} \int_K A(y) \, d\|V\|(y) \,.$$

Then

$$\int_{y \in \Omega} \|A^i(y) - T_y M\| \, d\|V\|(y) \le \int_{y \in \Omega} \|A^i(y) - A(y)\| \, d\|V\|(y) + \int_{y \in \Omega} \|A(y) - T_y M\| \, d\|V\|(y)$$

$$\le \varepsilon + \sum_{K \in \mathcal{K}_i} \int_{y \in K} \left\| \frac{1}{\|V\|(K)} \int_K A(u) \, d\|V\|(u) - A(y) \right\| \, d\|V\|(y)$$

$$\le \varepsilon + \sum_{K \in \mathcal{K}_i} \frac{1}{\|V\|(K)} \int_{y \in K} \int_{u \in K} \|A(u) - A(y)\| \, d\|V\|(u) \, d\|V\|(y)$$

$$\le \varepsilon + \delta_i \mathrm{lip}(A) \|V\|(\Omega) \le 2\varepsilon \text{ for } i \text{ large enough.}$$

Step 2. For all i, there exists $T^i : \Omega \to G_{d,n}$ constant in each cell $K \in \mathcal{K}_i$ such that

$$\int_{\Omega \times G_{d,n}} \|T^i(y) - T\| \, dV(y, T) = \int_{y \in \Omega} \|T^i(y) - T_y M\| \, d\|V\|(y) \xrightarrow[i \to +\infty]{} 0 \,.$$

Indeed, let $\varepsilon > 0$, thanks to Step 1, fix i and $A^i : \Omega \to \mathcal{M}_n(\mathbb{R})$ such that

$$\sum_{K \in \mathcal{K}_i} \int_K \|A^i(y) - T_y M\| \, d\|V\|(y) < \varepsilon \,,$$

so that $\int_K \|A^i(y) - T_y M\| \, d\|V\|(y) = \varepsilon^i_K$ with $\sum_{K \in \mathcal{K}_i} \varepsilon^i_K < \varepsilon$. In particular, for all $K \in \mathcal{K}_i$, there exists $y_K \in K$ such that

$$\|A^i(y_K) - T_{y_K} M\| \le \frac{\varepsilon^i_K}{\|V\|(K)} \,.$$

Define $T^i : \Omega \to G_{d,n}$, constant in each cell, by $T^i(y) = T_{y_K} M$ for $K \in \mathcal{K}_i$ and $y \in K$, and then,

$$\int_{\Omega \times G_{d,n}} \|T^i(y) - T\| \, dV(y, T) = \sum_{K \in \mathcal{K}_i} \int_K \|T_{y_K} M - T_y M\| \, d\|V\|(y) \tag{7}$$

$$\le \sum_{K \in \mathcal{K}_i} \int_K \|T_{y_K} M - \underbrace{A^i(y)}_{=A^i(y_K)}\| \, d\|V\|(y) + \int_{\Omega \times G_{d,n}} \|A^i(y) - T\| \, dV(y, T)$$

$$\le \sum_{K \in \mathcal{K}_i} \int_K \frac{\varepsilon^i_K}{\|V\|(K)} d\|V\|(y) + \varepsilon$$

$$\le 2\varepsilon \,.$$

Step 3. $\displaystyle\sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(\Omega \cap K) \times G_{d,n}} \|P - T\| \, dV(y, T) \xrightarrow[i \to \infty]{} 0.$

Indeed, thanks to Step 2, let $T^i : \Omega \to G_{d,n}$ constant in each cell $K \in \mathcal{K}_i$: for all $y \in K$, $T^i(y) = T_K^i$, and such that $\displaystyle\int_{\Omega \times G_{d,n}} \|T^i(y) - T\| \, dV(y, T) \xrightarrow[i \to +\infty]{} 0.$ We have,

$$\sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(\Omega \cap K) \times G_{d,n}} \|P - T\| \, dV(y, T) \leq \sum_{K \in \mathcal{K}_i} \int_{K \times G_{d,n}} \|T_K^i - T\| \, dV(y, T)$$

$$= \int_{\Omega \times G_{d,n}} \|T^i(y) - T\| \, dV(y, T)$$

$$\xrightarrow[i \to +\infty]{} 0 \, .$$

Step 4. Assume now that (4) and (5) hold, then define $T_K^i = T_{y_K} M$ for each cell $K \in \mathcal{K}_i$ and for some $y_K \in K$. Then

$$\sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(B^{\delta_i} \cap K) \times G_{d,n}} \|P - T\| \, dV(y, T) \leq \sum_{K \in \mathcal{K}_i} \int_{K \cap B^{\delta_i}} \|T_{y_K} M - T_y M\| \, d\|V\|(y)$$

$$\leq \sum_{K \in \mathcal{K}_i^{reg}} \int_{K \cap B^{\delta_i}} \|T_{y_K} M - T_y M\| \, d\|V\|(y)$$

$$+ \sum_{K \in \mathcal{K}_i^{reg}} \int_{K \cap B^{\delta_i}} \|T_{y_K} M - T_y M\| \, d\|V\|(y)$$

$$\leq \sum_{K \in \mathcal{K}_i} \int_{K \cap B^{\delta_i}} C|y_K - y|^\beta \, dV(y, T) + 2\|V\|(\mathcal{K}^{sing} \cap B^{\delta_i})$$

$$\leq C\delta_i^\beta \|V\|(B^{\delta_i}) + 2C'\delta_i^\gamma$$

$$\leq C'' \delta_i^{\min(\beta, \gamma)} \|V\|(B^{\delta_i}) \, .$$

$\square$

Remark 2.5. A remarkable case where assumptions (4) and (5) are both satisfied is when the varifold V is associated with the interface set of a quasiminimal N-cluster in $\mathbb{R}^2$ and $\mathbb{R}^3$ (see [Tay76, Dav10]). Such clusters arise for instance as minimizers of the total length/area of their set of interfaces, under volume constraints or up to additional bulk energy terms. By showing (6) we provide a quantitative estimate on the local oscillation of tangent planes, which will be used in the proof of Theorem 2.6 below.

We can now show that rectifiable varifolds can be approximated by discrete varifolds (either point cloud varifolds or volumetric varifolds).

Theorem 2.6. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set, let* $(\mathcal{K}_i)_{i \in \mathbb{N}}$ *be a sequence of meshes of* Ω, *and set*

$$\delta_i = \sup_{K \in \mathcal{K}_i} \operatorname{diam}(K) \quad \forall \, i \in \mathbb{N} \, .$$

Let $V = v(M, \theta)$ *is a rectifiable* d–*varifold in* Ω *with* $\|V\|(\Omega) < \infty$. *Then there exist a sequence of point cloud varifolds* $(V_i^{pt})_i$ *and a sequence of volumetric varifolds* $(V_i^{vol})_i$ *such that*

- $V_i^{pt} \xrightarrow[i\to\infty]{*} V$ and $V_i^{vol} \xrightarrow[i\to\infty]{*} V$.

- $\Delta^{1,1}(V_i^{pt}, V) \xrightarrow[i\to\infty]{} 0$ and $\Delta^{1,1}(V_i^{vol}, V) \xrightarrow[i\to\infty]{} 0$

If moreover (4) *and* (5) *hold, then there exists $C > 0$ such that for all $B \subset \Omega$,*

$$\max\left(\Delta_B^{1,1}(V_i^{pt}, V),\ \Delta_B^{1,1}(V_i^{vol}, V)\right) \leq C\|V\|(B^{\delta_i})\ \delta_i^{\min(\beta,\gamma)}. \tag{8}$$

Proof. Let $(V_i^{pt})_i$ and $(V_i^{vol})_i$ be the two sequences of discrete varifolds constructed in Proposition 2.3. In the following we write V_i instead of either V_i^{pt} or V_i^{vol}, for more simplicity. By (2) and (3) applied with $B = \Omega$, we have that

$$\Delta^{1,1}(V_i, V) \leq \delta_i\|V\|(\Omega) + \sum_{K \in \mathcal{K}_i} \min_{P \in G_{d,n}} \int_{(\Omega \cap K) \times G_{d,n}} \|P - S\|\, dV(x, S) \xrightarrow[i\to\infty]{} 0.$$

This implies that for any $\varphi \in \mathrm{Lip}(\Omega \times G_{d,n})$,

$$\langle V_i, \varphi \rangle \xrightarrow[i\to+\infty]{} \langle V, \varphi \rangle. \tag{9}$$

If $\varphi \in \mathrm{Lip}_k(\Omega \times G_{d,n})$, consider $\frac{1}{k}\varphi \in \mathrm{Lip}_1(\Omega \times G_{d,n})$ and use the linearity of (9). It remains to check the case $\varphi \in \mathrm{C}_c^0(\Omega \times G_{d,n})$ to have the weak–$*$ convergence. Let $\varphi \in \mathrm{C}_c^0(\Omega \times G_{d,n})$ and $\varepsilon > 0$. We can extend φ into $\overline{\varphi} \in \mathrm{C}_c^0(\Omega \times \mathcal{M}_n(\mathbb{R}))$ by Tietze-Urysohn theorem since $G_{d,n}$ is closed. Then, by density of $\mathrm{Lip}(\Omega \times \mathcal{M}_n(\mathbb{R}))$ in $\mathrm{C}_c^0(\Omega \times \mathcal{M}_n(\mathbb{R}))$ with respect to the uniform topology, there exists $\overline{\psi} \in \mathrm{Lip}(\Omega \times \mathcal{M}_n(\mathbb{R}))$ such that $\left\|\overline{\varphi} - \overline{\psi}\right\|_\infty < \varepsilon$. Let now $\psi \in \mathrm{Lip}(\Omega \times G_{d,n})$ be the restriction of $\overline{\psi}$ to $\Omega \times G_{d,n}$, then,

$$\begin{aligned}
|\langle V, \varphi \rangle - \langle V_i, \varphi \rangle| &\leq |\langle V, \varphi \rangle - \langle V, \psi \rangle| + |\langle V, \psi \rangle - \langle V_i, \psi \rangle| + |\langle V_i, \psi \rangle - \langle V_i, \varphi \rangle| \\
&\leq \|V\|(\Omega)\|\varphi - \psi\|_\infty + |\langle V, \psi \rangle - \langle V_i, \psi \rangle| + \|V_i\|(\Omega)\|\varphi - \psi\|_\infty.
\end{aligned}$$

As $\|V_i\|(\Omega) = \|V\|(\Omega)$ for all i by definition of V_i and $|\langle V, \psi \rangle - \langle V_i, \psi \rangle| \xrightarrow[i\to+\infty]{} 0$ by (9), there exists i large enough such that

$$|\langle V, \varphi \rangle - \langle V_i, \varphi \rangle| \leq (2\|V\|(\Omega) + 1)\ \varepsilon,$$

which concludes the general case.

Finally, the local estimate (8) is a consequence of Propositions 2.3 and 2.4 (more precisely it directly follows from the estimates (2) and (6)). $\qquad\square$

3 Regularized first variation and quantitative conditions of rectifiability for sequences of varifolds

Given a sequence of approximating d–varifolds $(V_i)_i$ weakly–$*$ converging to some d–varifold, a sufficient condition for V to have locally bounded first variation, i.e. for δV to be a Radon measure, is

$$\sup_i \|\delta V_i\| < +\infty. \tag{10}$$

However, point clouds do not have bounded first variation and, as shown in Example 6 in [Bue15], the typical sequences of discrete volumetric varifolds that one would consider as approximations of a smooth varifold have non-uniformly bounded first variations. Therefore, it is not natural to require that the first variation δV_i of V_i is a (uniformly bounded) Radon measure. Nevertheless, it is a distribution of order 1, hence a uniform control of a suitable sequence of regularizations of δV_i will be enough to ensure that the limit varifold V has a locally bounded first variation. Moreover, these regularizations will provide a "scale-dependent" notion of first variation particularly suitable for discrete varifolds.

3.1 Regularized first variation

We fix a non negative function $\rho \in C^1(\mathbb{R}^n)$ such that

$$\int \rho = 1 \text{ and } \operatorname{spt} \rho \subset B_1(0) , \tag{11}$$

and for any $\varepsilon > 0$ we let $\rho_\varepsilon(x) = \dfrac{1}{\varepsilon^n} \rho\left(\dfrac{x}{\varepsilon}\right)$. The ε-regularized first variation of a varifold is introduced in the following proposition.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω with finite mass $\|V\|(\Omega)$. Then, its first variation δV can be naturally extended as a linear continuous functional on $\mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$. Moreover, $\delta V * \rho_\varepsilon \in \mathrm{L}^1(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$ one has*

$$\delta V * \rho_\varepsilon(x) = \int_{B_\varepsilon(x) \times G_{d,n}} \nabla^S \rho_\varepsilon(y-x) \, dV(y,S) = \frac{1}{\varepsilon^{n+1}} \int_{B_\varepsilon(x) \times G_{d,n}} \nabla^S \rho\left(\frac{y-x}{\varepsilon}\right) dV(y,S). \tag{12}$$

Proof. First of all, notice that $(x,S) \mapsto \operatorname{div}_S X(x)$ is continuous and bounded, while V is a finite Radon measure, thus for any $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ one can define

$$\delta V(X) = \int_\Omega \operatorname{div}_S X(x) \, dV(x,S)$$

and obtain

$$\delta V(X) \le \|V\|(\Omega) \, \|X\|_{\mathrm{C}^1} ,$$

which means that the linear extension is continuous with respect to the C^1-norm. By definition, for any $X \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$ we have by definition

$$\langle \delta V * \rho_\varepsilon, X \rangle = \langle \delta V, X * \rho_\varepsilon \rangle = \langle V, (y,S) \mapsto \operatorname{div}_S(X * \rho_\varepsilon)(y) \rangle .$$

For every $y \in \mathbb{R}^n$ we find $\operatorname{div}_S(X * \rho_\varepsilon)(y) = X * \nabla^S \rho_\varepsilon(y) := \sum_{i=1}^n X_i * \partial_i^S \rho_\varepsilon(y)$, thus by Fubini-Tonelli's theorem we get

$$\langle \delta V * \rho_\varepsilon, X \rangle = \int_{\Omega \times G_{d,n}} (X * \nabla^S \rho_\varepsilon)(y) \, dV(y,S)$$

$$= \int_{\Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} X(x) \nabla^S \rho_\varepsilon(y-x) \, d\mathcal{L}^n(x) \, dV(y,S)$$

$$= \int_{x \in \mathbb{R}^n} X(x) \left(\int_{B_\varepsilon(x) \times G_{d,n}} \nabla^S \rho_\varepsilon(y-x) \, dV(y,S) \right) d\mathcal{L}^n(x) ,$$

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which proves (12). The fact that $\delta V * \rho_\varepsilon \in \mathrm{L}^1(\mathbb{R}^n)$ is an immediate consequence of the Lipschitz property of the kernel ρ_ε. $\qquad\square$

Remark 3.2. Notice that $\delta V * \rho_\varepsilon$ is well-defined on $\mathbb{R}^n$ even when δV is not bounded.

Remark 3.3. If we take a Lipschitz kernel that is not of class C^1, formula (12) still holds, but only for $\mathcal{L}^n$-almost all $x \in \mathbb{R}^n$. This may be a problem, since we rather need the formula to be valid for $\|V\|$-almost all $x \in \Omega$. However, in some cases it is possible to prove (12) even though the kernel is no more than Lipschitz. Indeed, when the varifold V is d-rectifiable one can "intrinsically" define the tangential gradient of a Lipschitz function $\|V\|$-almost everywhere: in this case, formula (12) turns out to be valid. Let us better explain this point with a simple example. Let us consider the *tent kernel* $T : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$T(z) = \max\left(\lambda_n^{-1}(1 - |z|),\ 0\right), \tag{13}$$

where $\lambda_n = \int_{|z| \le 1}(1 - |z|)\, d\mathcal{L}^n(z)$. We take a d-rectifiable varifold $V = v(M, \theta)$ with bounded mass and first variation, then according to Proposition 3.2 in [LM09] the first variation measure $\delta V(B_r(x))$ of a ball can be expressed in terms of integrated conormals on the boundary. More precisely, for $\|V\|$-almost all $x \in M$ and for almost every $r > 0$ it holds

$$\delta V(B_r(x)) = -\int_{\partial B_r(x) \cap M} \eta(y)\theta(y)\, d\mathcal{H}^{d-1}(y)\,,$$

where $\eta(y) = \dfrac{\Pi_{T_y M}(y - x)}{|\Pi_{T_y M}(y - x)|}$ is the outward conormal vector. If we average this relation by integrating in $r \in (0, \varepsilon)$, we obtain via coarea formula

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{r=0}^{\varepsilon} \delta V(B_r(x))\, dr &= -\frac{1}{\varepsilon}\int_{B_\varepsilon(x)\cap M} \frac{\Pi_{T_y M}(y - x)}{|y - x|}\theta(y)\, d\mathcal{H}^d(y) \\
&= -\frac{1}{\varepsilon}\int_{B_\varepsilon(x)\times G_{d,n}} \frac{\Pi_S(y - x)}{|y - x|}\, dV(y, S)\,.
\end{aligned}
\tag{14}
$$

At the same time, (12) formally becomes

$$\delta V * T_\varepsilon(x) = -\frac{1}{\lambda_n \varepsilon^{n+1}}\int_{B_\varepsilon(x)\times G_{d,n}} \frac{\Pi_S(y - x)}{|y - x|}\, dV(y, S)\,, \tag{15}$$

hence we see that (14) and (15) are the same formula, up to a scaling factor due to normalization of the tent function.

In the case of the tent function, it is also possible to define the ε-regularized first variation of a d-varifold even when the varifold is not rectifiable. The idea is to notice that the tent function can be uniformly approximated by smooth functions, such that the gradients uniformly converge in any compact subset of $B_1(0) \setminus \{0\}$. Therefore, formula (12) can be recovered for the limit tent kernel, unless $\|V\|(\{x\}) > 0$. In order to fix the problem occurring when single points are charged by the mass of the varifold, one can observe that all radial regularizations of the tent kernel have a null gradient at the origin, hence the formula will become valid as soon as we understand that, formally, the tent function has a null gradient at the origin (see in particular Example 3.9).

In the next two results we exploit some basic properties of the ε-regularized first variation.

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d-varifold in Ω with $\|V\|(\Omega) < +\infty$. Then for any $X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$,*

$$|\langle \delta V * \rho_\varepsilon, X \rangle - \langle \delta V, X \rangle| \leq \|V\|(\Omega \cap (\mathrm{spt}\, X + B_\varepsilon(0))) \, \|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \xrightarrow[\varepsilon \to 0]{} 0 \,.$$

Moreover, if V has bounded first variation then

$$\delta V * \rho_\varepsilon \xrightarrow[\varepsilon \to 0]{*} \delta V \,.$$

Proof. Let $X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$. As $\langle \delta V * \rho_\varepsilon, X \rangle = \langle \delta V, \rho_\varepsilon * X \rangle$ then

$$|\langle \delta V * \rho_\varepsilon, X \rangle - \langle \delta V, X \rangle| = |\langle \delta V, \rho_\varepsilon * X - X \rangle| \leq \|V\|(\Omega) \, \|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \,.$$

And $\|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \xrightarrow[\varepsilon \to 0]{} 0$ leads to the conclusion. If moreover V has bounded first variation, then for all $X \in \mathrm{C}_c^0(\mathbb{R}^n, \mathbb{R}^n)$,

$$|\langle \delta V * \rho_\varepsilon, X \rangle - \langle \delta V, X \rangle| \leq \|\delta V\| \, \|\rho_\varepsilon * X - X\|_\infty \xrightarrow[\varepsilon \to 0]{} 0 \,.$$

$\square$

In the next theorem (technically, a partial generalization of Allard's compactness theorem for rectifiable varifolds) we show that, given an infinitesimal sequence $(\varepsilon_i)_i$ of positive numbers and a sequence of d-varifolds $(V_i)_i$ with uniformly bounded total masses, such that $\delta V_i * \rho_{\varepsilon_i}$ satisfies a uniform boundedness assumption, there exists a subsequence of V_i that weakly-$*$ converges to a limit varifold V with bounded first variation. If additionally the masses $\|V\|_i$ of the varifolds in the sequence satisfy a uniform lower density bound, then V is rectifiable. Notice that the sequence V_i is required to be neither of bounded first variation, nor rectifiable.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $(V_i)_i$ be a sequence of d-varifolds. Assume that there exists a positive, decreasing and infinitesimal sequence $(\varepsilon_i)_i$, such that*

$$M := \sup_i \{ \|V_i\|(\Omega) + \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1} \} < +\infty \,. \tag{16}$$

Then there exists a subsequence $(V_{\varphi(i)})_i$ weakly–$*$ converging in Ω to a d–varifold V, V has bounded first variation and $\|V\|(\Omega) + |\delta V|(\Omega) \leq M$. Moreover, if we further assume the existence of $\theta_0, r_0 > 0$ such that, for any $0 < r < r_0$ and for $\|V_i\|$-almost every $x \in \Omega$,

$$\|V_i\|(B_r(x)) \geq \theta_0 r^d \,, \tag{17}$$

then the limit varifold V obtained above is d-rectifiable.

Proof. Since M is finite, there exists a subsequence $(V_{\varphi(i)})_i$ weakly–$*$ converging in Ω to a d–varifold V. By Proposition 3.4, for any $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$ we obtain

$$\left| \langle \delta V_{\varphi(i)} * \rho_{\varepsilon_{\varphi(i)}}, X \rangle - \langle \delta V, X \rangle \right| \leq \left| \langle \delta V_{\varphi(i)} * \rho_{\varepsilon_{\varphi(i)}}, X \rangle - \langle \delta V_{\varphi(i)}, X \rangle \right| + \left| \langle \delta V_{\varphi(i)}, X \rangle - \langle \delta V, X \rangle \right|$$

$$\leq \underbrace{\|V_i\|(\Omega)}_{\leq C < +\infty} \left\| X * \rho_{\varepsilon_{\varphi(i)}} - X \right\|_{\mathrm{C}^1} + \left| \langle \delta V_{\varphi(i)}, X \rangle - \langle \delta V, X \rangle \right|$$

$$\xrightarrow[i \to \infty]{} 0 \,.$$

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Consequently, for any $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$ one has $|\langle \delta V, X \rangle| \leq \sup_i \| \delta V_i * \rho_{\varepsilon_i} \|_{\mathrm{L}^1} \| X \|_\infty$. We conclude that δV extends into a continuous linear form in $\mathrm{C}_c^0(\Omega, \mathbb{R}^n)$ whose norm is bounded by $\sup_i \| \delta V_i * \rho_{\varepsilon_i} \|_{\mathrm{L}^1}$, thus $\| V \|(\Omega) + |\delta V|(\Omega) \leq M$.

Assuming the additional hypothesis (17), we can pass to the limit and prove the same inequality for V. By Theorem 5.5(1) in [All72] we obtain the last part of the claim. □

3.2 Approximate mean curvature

In the previous section we have considered the ε-regularization of the first variation of a varifold. Here we introduce some ε-approximations of the generalized mean curvature of a rectifiable varifold $V = v(M, \theta)$ with bounded first variation (i.e., of the absolutely continuous part of the first variation with respect to the mass) and study their properties with special emphasis on quantified error estimates. To this end, it is worth recalling that the first variation measure δV can be decomposed as the sum of an absolutely continuous part $-H \| V \|$ and of a singular part δV_s, and that the Borel function H appearing in this decomposition is the so-called *generalized mean curvature* of V.

In the next proposition we consider a first type of approximation, based on convolution of both first variation and mass by means of the same family of kernels $\{\rho_\varepsilon\}_{\varepsilon > 0}$. This will be the privileged choice here, even though more general choices could be made (see Remark 3.8).

Proposition 3.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M, \theta)$ be a rectifiable d–varifold with bounded first variation $\delta V = -H \| V \| + \delta V_s$. Assume that ρ is a radial kernel and define*

$$H_\varepsilon(x) = -\frac{\delta V * \rho_\varepsilon(x)}{\| V \| * \rho_\varepsilon(x)} .$$

Then, for $\| V \|$–almost any $x \in \Omega$ it holds

$$H_\varepsilon(x) \xrightarrow[\varepsilon \to 0]{} H(x) .$$

Moreover, if the singular part $\delta_s V$ is null and the generalized mean curvature H is L-Lipschitz on M, then for $\| V \|$-almost all $x \in M$ we have

$$|H_\varepsilon(x) - H(x)| \leq L\varepsilon . \tag{18}$$

Proof. For $x \in \Omega$ we find

$$
\begin{aligned}
|H_\varepsilon(x) - H(x)| &= \frac{1}{\| V \| * \rho_\varepsilon(x)} \left| (H \| V \| - \delta V_s) * \rho_\varepsilon(x) - H(x) \left(\| V \| * \rho_\varepsilon(x) \right) \right| \\
&\leq \frac{1}{\| V \| * \rho_\varepsilon(x)} \left| (H \| V \|) * \rho_\varepsilon(x) - H(x) \left(\| V \| * \rho_\varepsilon(x) \right) \right| + \frac{|\delta V_s * \rho_\varepsilon(x)|}{\| V \| * \rho_\varepsilon(x)} \\
&\leq \frac{1}{\| V \| * \rho_\varepsilon(x)} \int_{y \in \mathbb{R}^n} |H(x) - H(y)| \, \rho_\varepsilon(x - y) \, d\| V \|(y) + \frac{|\delta V_s| * \rho_\varepsilon(x)}{\| V \| * \rho_\varepsilon(x)}
\end{aligned}
$$

For $\| V \|$-almost every x, by definition of the approximate tangent plane and since $\rho \in \mathrm{C}_c^0(\Omega)$ is radial, we obtain

$$\varepsilon^{n-d} \| V \| * \rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \int_\Omega \rho \left(\frac{y - x}{\varepsilon} \right) d\| V \|(y) \xrightarrow[\varepsilon \to 0]{} \theta(x) \int_{T_x M} \rho(y) \, d\mathcal{H}^d(y) = C_\rho \theta(x) > 0 . \tag{19}$$

Then, for $\|V\|$-almost any $x \in \Omega$ (at any Lebesgue point x of $H \in \mathrm{L}^1(\|V\|)$),

$$\frac{1}{\|V\| * \rho_\varepsilon(x)} \int_{y \in \mathbb{R}^n} |H(x) - H(y)| \, \rho_\varepsilon(x - y) \, d\|V\|(y)$$

$$\leq \frac{\|V\|(B_\varepsilon(x))}{\|V\| * \rho_\varepsilon(x)} \frac{1}{\|V\|(B_\varepsilon(x))} \frac{\|\rho\|_\infty}{\varepsilon^n} \int_{y \in B_\varepsilon(x)} |H(x) - H(y)| \, d\|V\|(y)$$

$$\leq \|\rho\|_\infty \underbrace{\frac{\varepsilon^{-d}\|V\|(B_\varepsilon(x))}{\varepsilon^{n-d}\|V\| * \rho_\varepsilon(x)}}_{\substack{\longrightarrow \\ \varepsilon \to 0}} \underbrace{\fint_{y \in B_\varepsilon(x)} |H(x) - H(y)| \, d\|V\|(y)}_{\substack{\longrightarrow \\ \varepsilon \to 0}} 0$$

$$\xrightarrow[\varepsilon \to 0]{} 0 \,.$$

And similarly, for $\|V\|$–almost every x,

$$\frac{|\delta V_s| * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \leq \|\rho\|_\infty \underbrace{\frac{\varepsilon^{-d}\|V\|(B_\varepsilon(x))}{\varepsilon^{n-d}\|V\| * \rho_\varepsilon(x)}}_{\substack{\longrightarrow \\ \varepsilon \to 0}} \underbrace{\frac{|\delta V_s|(B_\varepsilon(x))}{\|V\|(B_\varepsilon(x))}}_{\substack{\longrightarrow \\ \varepsilon \to 0}} \xrightarrow[\varepsilon \to 0]{} 0 \,.$$

Finally, if $\delta V_s = 0$ and H is L-Lipschitz, then for $\|V\|$-almost all x we obtain

$$\frac{1}{\|V\| * \rho_\varepsilon(x)} \int_{y \in \mathbb{R}^n} |H(x) - H(y)| \, \rho_\varepsilon(x - y) \, d\|V\|(y) \leq \frac{L}{\|V\| * \rho_\varepsilon(x)} \int_{y \in \mathbb{R}^n} |x - y| \, \rho_\varepsilon(x - y) \, d\|V\|(y)$$

$$\leq L\varepsilon \,,$$

which proves (18). □

Remark 3.7. The assumption that ρ is radial can be weakened. Indeed it is enough to require that

$$\int_P \rho(y) \, d\mathcal{H}^d(y) > 0 \qquad \text{for all } P \in G_{d,n} \,.$$

Notice that when $\rho(y) = \zeta(|y|)$ is a radial kernel, for all $P \in G_{d,n}$ one has that

$$\int_P \rho(y) \, d\mathcal{H}^d(y) = \int_{r=0}^1 \int_{\partial B_r(x) \cap P} \rho(y) \, d\mathcal{H}^{d-1}(y) = \int_{r=0}^1 r^d \mathcal{H}^{d-1}(\partial B_1(0) \cap P) \zeta(r) \, dr$$

$$= \mathcal{H}^{d-1}(\mathbb{S}^{d-1}) \int_0^1 r^d \zeta(r) \, dr = C_\rho$$

is strictly positive and does not depend on P.

Remark 3.8. Taking two different radial kernels ρ and ξ we may define

$$H_{\rho,\xi,\varepsilon} = -\frac{C_\xi}{C_\rho} \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \xi_\varepsilon(x)} \,,$$

where C_ξ and C_ρ are defined in the obvious way as in (19). Then, under the same assumptions of Proposition 3.6, we get

$$H_{\rho,\xi,\varepsilon} \xrightarrow[\varepsilon \to 0]{} H(x) \,. \tag{20}$$

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Indeed for $\|V\|$–almost every x, concerning the singular part one has

$$\frac{|\delta V_s| * \rho_\varepsilon(x)}{\|V\| * \xi_\varepsilon(x)} \le \|\rho\|_\infty \underbrace{\frac{\varepsilon^{-d}\|V\|(B_\varepsilon(x))}{\varepsilon^{n-d}\|V\| * \xi_\varepsilon(x)}}_{\xrightarrow[\varepsilon\to0]{} \frac{1}{C_\xi}} \underbrace{\frac{|\delta V_s|(B_\varepsilon(x))}{\|V\|(B_\varepsilon(x))}}_{\xrightarrow[\varepsilon\to0]{} 0} \xrightarrow[\varepsilon\to0]{} 0 \,.$$

On the other hand, being M rectifiable, and by the definition of approximate tangent plane, for $\|V\|$–almost every x we have

$$\frac{1}{\|V\| * \xi_\varepsilon(x)}(H\,\|V\|) * \rho_\varepsilon(x) = \frac{\varepsilon^{-d}\displaystyle\int H(y)\rho\left(\frac{y-x}{\varepsilon}\right) d\|V\|(y)}{\varepsilon^{-d}\displaystyle\int \xi\left(\frac{y-x}{\varepsilon}\right) d\|V\|(y)}$$

$$\xrightarrow[\varepsilon\to0]{} \frac{H(x)\displaystyle\int_{T_xM} \rho\, d\mathcal{H}^d}{\displaystyle\int_{T_xM} \xi\, d\mathcal{H}^d} = \frac{C_\rho}{C_\xi}\, H(x)\,,$$

which implies (20) (see the proof of Proposition 3.6). We also remark that the needed regularity of the second kernel ξ is lower than that of ρ. Indeed, since ξ is used to regularize a distribution of order 0 (the mass measure $\|V\|$) we may only require that ξ is a bounded non-negative Borel function. This is consistent with some special choices of kernels ρ, ξ that will be considered in section 5.

Example 3.9 (Regularization of the first variation of a point cloud). Let $V = \displaystyle\sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$ be the varifold associated with a point cloud. The first variation of V is not a Radon measure so that we need (12) to compute its ε-regularization:

$$\delta V * \rho_\varepsilon(x) = \int_{B_\varepsilon(x)} \nabla^S \rho_\varepsilon(y-x)\, dV(y,S) = \sum_{x_j \in B_\varepsilon(x)} m_j \nabla^{P_j} \rho_\varepsilon(x_j - x)\,,$$

and

$$\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} = \frac{1}{\varepsilon} \frac{\sum_{x_j \in B_\varepsilon(x)} m_j \nabla^{P_j} \rho\left(\frac{x_j - x}{\varepsilon}\right)}{\sum_{x_j \in B_\varepsilon(x)} m_j \rho\left(\frac{x_j - x}{\varepsilon}\right)}\,.$$

In particular, if ρ is the tent kernel, taking into account Remark 3.3 we get

$$\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} = \frac{1}{\varepsilon} \frac{\sum_{x_j \in B_\varepsilon(x)\setminus\{x\}} m_j \Pi_{P_j} \frac{x_j - x}{|x_j - x|}}{\sum_{x_j \in B_\varepsilon(x)} m_j \left(1 - \frac{|x_j - x|}{\varepsilon}\right)}\,.$$

Let us notice that the choice of the size ε is important: it must be large enough to contain more than the central point, but not to large to avoid over-smoothing.

3.3 Pointwise convergence of the approximate mean curvature

In this section, we exhibit quantitative conditions linking the size of the successive meshes δ_i and the approximation scales ε_i and ensuring that for a d-rectifiable varifold V satisfying (4) and (5)), the approximate mean curvature

$$-\frac{\delta V_i * \rho_{\varepsilon_i}(x)}{\|V_i\| * \rho_{\varepsilon_i}(x)}$$

of the sequence of discrete varifolds $(V_i)_i$ obtained by projecting V on a sequence of meshes with infinitesimal mesh-size, converges to the mean curvature. At this point we also need to impose extra technical conditions on the convolution kernels, which are required to be radial, of class $W^{2,\infty}$, and with a decreasing profile.

Theorem 3.10 (Pointwise convergence)*. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M, \theta)$ be a rectifiable d–varifold in Ω with finite mass $\|V\|(\Omega)$ and bounded first variation. Assume that $\rho \in W^{2,\infty}$ is radial, $\rho(x) = \zeta(|x|)$, with $\zeta \in W^{2,\infty}(\mathbb{R}_+)$ decreasing. Let $(V_i)_i$ be sequence of d–varifolds weakly–$*$ converging to V, for which there exist two positive, decreasing and infinitesimal sequences $(\eta_i)_i, (\delta_i)_i$, such that for any ball $B \subset \Omega$ one has*

$$\|V\|(\{y \in B \,:\, d(y, \partial B) > \eta_i\}) \leq \|V_i\|(B) \leq \|V\|(B^{\eta_i}) \tag{21}$$

and

$$\Delta_B^{1,1}(V, V_i) \leq d_i \|V\|(B^{\eta_i})\,. \tag{22}$$

Finally, let $\varepsilon_i \downarrow 0$ be such that $\dfrac{d_i}{\varepsilon_i^2} \xrightarrow[i \to \infty]{} 0$ and $\dfrac{\eta_i}{\varepsilon_i} \xrightarrow[i \to \infty]{} 0$. Then for $\|V\|$–almost any $x \in \Omega$,

$$\left| H_{\varepsilon_i}^{V_i}(x) - H_{\varepsilon_i}^V(x) \right| \leq C\|\rho\|_{W^{2,\infty}} \frac{d_i}{\varepsilon_i^2} \qquad \text{for } i \text{ large enough,}$$

$$H_{\varepsilon_i}^{V_i}(x) = -\frac{\delta V_i * \rho_{\varepsilon_i}(x)}{\|V_i\| * \rho_{\varepsilon_i}(x)} \xrightarrow[i \to \infty]{} H(x)\,,$$

where C is a constant depending on C_ρ.

Proof. Let $\varepsilon > 0$. First of all, thanks to Proposition 3.6, for $\|V\|$–almost any x,

$$\left| -\frac{\delta V_i * \rho_\varepsilon(x)}{\|V_i\| * \rho_\varepsilon(x)} - H(x) \right| \leq \left| \frac{\delta V_i * \rho_\varepsilon(x)}{\|V_i\| * \rho_\varepsilon(x)} - \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \right| + \underbrace{\left| H_\varepsilon^V(x) - H(x) \right|}_{\xrightarrow[\varepsilon \to 0]{} 0}$$

$$\leq \frac{|\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)|}{\|V_i\| * \rho_\varepsilon(x)} + |\delta V * \rho_\varepsilon(x)| \left| \frac{1}{\|V_i\| * \rho_\varepsilon(x)} - \frac{1}{\|V\| * \rho_\varepsilon(x)} \right| + o_\varepsilon(1)\,. \tag{23}$$

STEP 1: We study the convergence of the first term in (23). Thanks to assumption (22), for all $\varphi \in \mathrm{Lip}(\Omega \times G_{d,n})$ such that $\mathrm{spt}\, \varphi \subset B \times G_{d,n}$,

$$|\langle V_i, \varphi \rangle - \langle V, \varphi \rangle| \leq d_i \,\mathrm{lip}(\varphi)\, \|V\|(B^{\eta_i})\,. \tag{24}$$

Since $\rho \in W^{2,\infty}(\Omega)$, the function

$$(y, S) \in \Omega \times G_{d,n} \mapsto \nabla^S \rho\left(\frac{y - x}{\varepsilon}\right)$$

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has a Lipschitz constant $\leq \dfrac{1}{\varepsilon}\|\rho\|_{\mathrm{W}^{2,\infty}}$ and support in $B_\varepsilon(x) \times G_{d,n}$. By (24),

$$|\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)| = \frac{1}{\varepsilon^{n+1}}\left|\int_{\Omega \times G_{d,n}} \nabla^S \rho\left(\frac{y-x}{\varepsilon}\right) dV_i(y,S) - \int_{\Omega \times G_{d,n}} \nabla^S \rho\left(\frac{y-x}{\varepsilon}\right) dV(y,S)\right|$$

$$\leq \frac{d_i}{\varepsilon^{n+2}}\|\rho\|_{\mathrm{W}^{2,\infty}}\|V\|(B_{\varepsilon+\eta_i}(x) \cap \Omega) . \tag{25}$$

Let us now bound $\|V_i\| * \rho_\varepsilon(x)$ from below. As $\rho(x) = \zeta(|x|)$ for all x, with $\zeta \in \mathrm{W}^{2,\infty}(\mathbb{R}_+)$. In particular ζ is absolutely continuous, $\zeta(1) = 0$ and

$$\zeta(r) = -\int_{s=r}^1 \zeta'(s)\, ds .$$

Consequently,

$$\|V_i\| * \rho_\varepsilon(x) = \int_{y \in B_\varepsilon(x)} \rho_\varepsilon(y-x)\, d\|V_i\|(y) = \frac{1}{\varepsilon^n}\int_{y \in B_\varepsilon(x)} \zeta\left(\frac{|y-x|}{\varepsilon}\right) d\|V_i\|(y)$$

$$= -\frac{1}{\varepsilon^n}\int_{y \in B_\varepsilon(x)}\int_{s=\frac{|y-x|}{\varepsilon}}^1 \zeta'(s)\, ds\, d\|V_i\|(y) = -\frac{1}{\varepsilon^{n+1}}\int_{y \in B_\varepsilon(x)}\int_{u=|y-x|}^\varepsilon \zeta'(u)\, du\, d\|V_i\|(y)$$

$$= -\frac{1}{\varepsilon^{n+1}}\int_{u=0}^\varepsilon \zeta'(u)\int_{y \in B_u(x)} d\|V_i\|(y)\, du = -\frac{1}{\varepsilon^{n+1}}\int_{u=0}^\varepsilon \zeta'(u)\|V_i\|(B_u(x))\, du . \tag{26}$$

By assumption (21) we get that, for all $s > \eta_i$,

$$\|V\|(B_{s-\eta_i}(x)) \leq \|V_i\|(B_s(x)) \leq \|V\|(B_{s+\eta_i}(x)) .$$

So that, since $-\zeta' \geq 0$ and thanks to (26),

$$\|V_i\| * \rho_\varepsilon(x) \geq \frac{1}{\varepsilon^{n+1}}\int_{u=\eta_i}^\varepsilon -\zeta'(u)\|V\|(B_{u-\eta_i}(x))\, du \geq \frac{1}{\varepsilon^{n+1}}\int_{u=0}^{\varepsilon-\eta_i} -\zeta'(u+\eta_i)\|V\|(B_u(x))\, du$$

$$\geq \frac{1}{\varepsilon^{n+1}}\int_{u=0}^{\varepsilon-\eta_i} -\left(\zeta'(u) + \mathrm{lip}(\zeta')\eta_i\right)\|V\|(B_u(x))\, du \text{ since } \zeta' \in \mathrm{W}^{1,\infty} . \tag{27}$$

Moreover, by (26) (applied with $\varepsilon - \eta_i$ instead of ε),

$$\frac{1}{\varepsilon - \eta_i}\int_{u=0}^{\varepsilon-\eta_i} -\zeta'(u)\|V\|(B_u(x))\, du = \int_{y \in B_{\varepsilon-\eta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon-\eta_i}\right) d\|V\|(y) , \tag{28}$$

and

$$\frac{1}{\varepsilon}\int_{u=0}^{\varepsilon-\eta_i} \|V\|(B_u(x))\, du \leq \|V\|(B_\varepsilon(x)) . \tag{29}$$

By (27), (28) and (29), we have

$$\|V_i\| * \rho_\varepsilon(x) \geq \frac{1}{\varepsilon^n}\frac{\varepsilon - \eta_i}{\varepsilon}\int_{y \in B_{\varepsilon-\eta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon-\eta_i}\right) d\|V\|(y) - \frac{1}{\varepsilon^n}\mathrm{lip}(\zeta')\eta_i\|V\|(B_\varepsilon(x)) \tag{30}$$

Let us consider a sequence $\varepsilon_i \downarrow 0$ and such that $\dfrac{d_i}{\varepsilon_i^2} \xrightarrow[i\to\infty]{} 0$ and $\dfrac{\eta_i}{\varepsilon_i} \xrightarrow[i\to\infty]{} 0$. In particular, $\varepsilon_i - \eta_i \xrightarrow[i\to\infty]{} 0$ with $\eta_i \le \varepsilon_i$. Thanks to (30), we obtain

$$\frac{\|V\|(B_{\varepsilon_i}(x))}{\varepsilon_i^n \|V_i\| * \rho_{\varepsilon_i}(x)} \le \frac{\|V\|(B_{\varepsilon_i}(x))}{\dfrac{\varepsilon_i - \eta_i}{\varepsilon_i} \displaystyle\int_{y \in B_{\varepsilon_i - \eta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \eta_i}\right) d\|V\|(y) - \mathrm{lip}(\zeta')\eta_i \|V\|(B_{\varepsilon_i}(x))}$$

$$\le \frac{1}{\dfrac{\varepsilon_i - \eta_i}{\varepsilon_i} \dfrac{1}{\|V\|(B_{\varepsilon_i}(x))} \displaystyle\int_{y \in B_{\varepsilon_i - \eta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \eta_i}\right) d\|V\|(y) - \underbrace{\mathrm{lip}(\zeta')\eta_i}_{=o(\eta_i)}} . \tag{31}$$

Moreover, as $\|V\| = v(M,\theta)$ is d–rectifiable, we have:

$$\|V\|(B_{\varepsilon_i}(x)) \sim_{i\to\infty} \theta(x)\varepsilon_i^d ; \tag{32}$$

and, thanks to the definition of approximate tangent plane,

$$\frac{1}{\theta(x)(\varepsilon_i - \eta_i)^d} \int_{y \in B_{\varepsilon_i - \eta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \eta_i}\right) d\|V\|(y) \xrightarrow[i\to\infty]{} \int_{B_1(0) \cap T_x M} \zeta(|z|) \, d\mathcal{H}^d(z) . \tag{33}$$

By (32) and (33), we have

$$\frac{\varepsilon_i - \eta_i}{\varepsilon_i} \frac{1}{\|V\|(B_{\varepsilon_i}(x))} \int_{y \in B_{\varepsilon_i - \eta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \eta_i}\right) d\|V\|(y)$$

$$= \left(1 - \frac{\eta_i}{\varepsilon_i}\right) \frac{\theta(x)(\varepsilon_i - \eta_i)^d}{\|V\|(B_{\varepsilon_i}(x))} \frac{1}{\theta(x)(\varepsilon_i - \eta_i)^d} \int_{y \in B_{\varepsilon_i - \eta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \eta_i}\right) d\|V\|(y) \tag{34}$$

$$\sim_{i\to\infty} \left(1 - \frac{\eta_i}{\varepsilon_i}\right)\left(\frac{\varepsilon_i - \eta_i}{\varepsilon_i}\right)^d \int_{B_1(0) \cap T_x M} \zeta(|z|) \, d\mathcal{H}^d(z)$$

$$\xrightarrow[i\to\infty]{} \int_{B_1(0) \cap T_x M} \rho(z) \, d\mathcal{H}^d(z) = C_\rho < +\infty \tag{35}$$

Finally, by (31) and (35), $\dfrac{\|V\|(B_{\varepsilon_i}(x))}{\varepsilon_i^n \|V_i\| * \rho_{\varepsilon_i}(x)}$ is bounded by $\dfrac{2}{C_\rho} > 0$ when $i \to +\infty$ and by (25)

$$\frac{|\delta V_i * \rho_{\varepsilon_i}(x) - \delta V * \rho_{\varepsilon_i}(x)|}{\|V_i\| * \rho_{\varepsilon_i}(x)} \le \frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} \frac{1}{\varepsilon_i^{n+2}} \|\rho\|_{\mathrm{W}^{2,\infty}} d_i \|V\|(B_{\varepsilon_i + \eta_i}(x))$$

$$\le \frac{2}{C_\rho} \|\rho\|_{\mathrm{W}^{2,\infty}} \frac{d_i}{\varepsilon_i^2} \frac{\|V\|(B_{\varepsilon_i + \eta_i}(x))}{\|V\|(B_{\varepsilon_i}(x))}$$

$$\le \frac{4}{C_\rho} \|\rho\|_{\mathrm{W}^{2,\infty}} \frac{d_i}{\varepsilon_i^2}$$

$$\xrightarrow[i\to+\infty]{} 0 ,$$

as

$$\frac{\|V\|(B_{\varepsilon_i + \eta_i}(x))}{\|V\|(B_{\varepsilon_i}(x))} \sim_{i\to\infty} \frac{\theta(x)(\varepsilon_i + \eta_i)^d}{\theta(x)\varepsilon_i^d} \xrightarrow[i\to\infty]{} 1 .$$

STEP 2: It remains to study the second term in (23). Applying again (24),

$$\left| \|V\| * \rho_{\varepsilon_i}(x) - \|V_i\| * \rho_{\varepsilon_i}(x) \right| \leq \frac{1}{\varepsilon_i} \mathrm{lip}(\rho_{\varepsilon_i}) d_i \|V\|(B_{\varepsilon_i + \eta_i}(x)) \leq \frac{1}{\varepsilon_i^{n+1}} \mathrm{lip}(\rho) d_i \|V\|(B_{\varepsilon_i + \eta_i}(x)), \quad (36)$$

and thus,

$$|\delta V * \rho_{\varepsilon_i}(x)| \left| \frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} - \frac{1}{\|V\| * \rho_{\varepsilon_i}(x)} \right| = \underbrace{\frac{|\delta V * \rho_{\varepsilon_i}(x)|}{\|V\| * \rho_{\varepsilon_i}(x)}}_{\substack{\longrightarrow \\ i \to \infty} H(x)} \frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} \left| \|V\| * \rho_{\varepsilon_i}(x) - \|V_i\| * \rho_{\varepsilon_i}(x) \right|$$

$$\leq 2|H(x)|\frac{1}{\varepsilon_i}\|\rho\|_{\mathrm{W}^{1,\infty}} \underbrace{\frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} \frac{1}{\varepsilon_i^n}\|V\|(B_{\varepsilon_i}(x))}_{\leq \frac{4}{C_\rho}} d_i$$

$$\xrightarrow[i \to \infty]{} 0.$$

Thanks to STEP 1 and STEP 2, we proved that for $\|V\|$–almost any x,

$$\left| H_{\varepsilon_i}^{V_i}(x) - H_\varepsilon^V(x) \right| \leq \frac{4}{C_\rho}\|\rho\|_{\mathrm{W}^{2,\infty}}\frac{d_i}{\varepsilon_i^2} + 2|H(x)|\|\rho\|_{\mathrm{W}^{1,\infty}}\frac{4}{C_\rho}\frac{d_i}{\varepsilon_i}$$

$$\sim_{i \to \infty} \frac{4}{C_\rho}\|\rho\|_{\mathrm{W}^{2,\infty}}\frac{d_i}{\varepsilon_i^2}.$$

Therefore,

$$-\frac{\delta V_i * \rho_{\varepsilon_i}(x)}{\|V_i\| * \rho_{\varepsilon_i}(x)} \xrightarrow[i \to \infty]{} H(x),$$

and moreover, if M is smooth, thanks to Proposition 3.6

$$\left| H_{\varepsilon_i}^{V_i}(x) - H(x) \right| \leq C_1 \|\rho\|_{\mathrm{W}^{2,\infty}}\frac{d_i}{\varepsilon_i^2} + C_2 \varepsilon_i.$$

$\square$

Remark 3.11. If we want to consider two different radial kernels $\rho \in \mathrm{W}^{2,\infty}$ and $\xi \in \mathcal{W}^{1,\infty}$, then with the same proof (and under the same assumptions) as in Theorem 3.10,

$$\left| -\frac{\delta V_i * \rho_{\varepsilon_i}(x)}{\|V_i\| * \xi_{\varepsilon_i}(x)} + \frac{\delta V * \rho_{\varepsilon_i}(x)}{\|V\| * \xi_{\varepsilon_i}(x)} \right| \leq C_1 \|\rho\|_{\mathrm{W}^{2,\infty}}\frac{d_i}{\varepsilon_i^2},$$

with C_1 depending on C_ξ.

Corollary 3.12. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $(\mathcal{K}_i)_{i \in \mathbb{N}}$ be a sequence of meshes of Ω, and set $\delta_i = \sup_{K \in \mathcal{K}_i} \mathrm{diam}(K)$. Let $V = v(M, \theta)$ is a rectifiable d–varifold in Ω with $\|V\|(\Omega) < \infty$, with bounded first variation and satisfying (4) and (5) for some $\beta, \gamma > 0$. Let ε_i be a positive decreasing sequence such that*

$$\frac{\delta_i^{\min(\beta,\gamma)}}{\varepsilon_i^2} \xrightarrow[i \to \infty]{} 0.$$

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Then there exist a sequence of point cloud varifolds $(V_i^{pt})_i$ and a sequence of volumetric varifolds $(V_i^{vol})_i$ (equally denoted by $(V_i)_i$) such that for $\|V\|$–almost any x, $H_{\varepsilon_i}^{V_i}(x) \xrightarrow[i\to 0]{} H(x)$ with

$$\left|H_{\varepsilon_i}^{V_i}(x) - H_{\varepsilon_i}^{V}(x)\right| \le C_1 \|\rho\|_{W^{2,\infty}} \frac{\delta_i^{\min(\beta,\gamma)}}{\varepsilon_i^2}$$

and

$$\left|H_{\varepsilon_i}^{V_i}(x) - H(x)\right| \le C_1 \|\rho\|_{W^{2,\infty}} \frac{\delta_i^{\min(\beta,\gamma)}}{\varepsilon_i^2} + C_2\,\varepsilon_i\;.$$

for i large enough and for some constant $C_1 > 0$.

Proof. Consider the sequences of volumetric varifolds $(V_i^{vol})_i$ and point cloud varifolds $(V_i^{pt})_i$ given by Theorem 2.6, then assumptions of Theorem 3.10 are satisfied with $\eta_i = \delta_i$ and $d_i = \delta_i^\beta$. □

4 Representation of the regularized first variation

In this section, we try to answer **Question 3**:

- Given a d–varifold V, is the regularization $\delta V * \rho_\varepsilon$ of the first variation δV, the first variation $\delta\left(\widehat{V}_\varepsilon\right)$ of some varifold $\widehat{V}_\varepsilon$?

- And if so, is $\widehat{V}_\varepsilon$ the regularization (in a sense to be defined) of V?

In short, is there a kind of convolution $\hat{*}$ such that the following formula makes sense

$$\delta V * \rho_\varepsilon = \delta\left(\widehat{V}_\varepsilon\right) = \delta\left(V \hat{*} \rho_\varepsilon\right)\,?$$

Let us first explain what $\widehat{V}_\varepsilon$ cannot be. As V is a Radon measure in $\Omega \times G_{d,n}$, notice that $V * \rho_\varepsilon$ does not have a canonical sense. A natural idea would be to:

1. first regularize the mass $\|V\|$, defining $\|\widehat{V}_\varepsilon\| = (\|V\| * \rho_\varepsilon)\,d\mathcal{L}^n$,

2. then set $\widehat{V}_\varepsilon = (\|V\| * \rho_\varepsilon(x))\,d\mathcal{L}^n \otimes \delta_{T_\varepsilon(x)}$ and compute the tangential part $T_\varepsilon(x)$ from $\|\widehat{V}_\varepsilon\|$.

For, instance, if $V = v(\Gamma, 1)$ is associated with a curve Γ in $\mathbb{R}^2$, set $u_\varepsilon(x) = d(x, \Gamma)$ and set $T_\varepsilon(x) = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}^\perp$ (which gives the tangential direction to the level lines of u_ε) so that $\widehat{V}_\varepsilon$ would be:

$$\widehat{V}_\varepsilon = (\|V\| * \rho_\varepsilon(x))\,d\mathcal{L}^2 \otimes \delta_{T_\varepsilon(x)} = (\|V\| * \rho_\varepsilon(x))\,\mathcal{L}^2 \otimes \delta_{\frac{\nabla u_\varepsilon(x)}{|\nabla u_\varepsilon(x)|}^\perp}\;.$$

Let us consider a simple example to test this construction:

Example 4.1. Let $V = v(N, 1)$ where N is the cross constituted by the union of the lines $N_1 = \{x_1 = 0\}$ and $N_2 = \{x_2 = 0\}$ in $\mathbb{R}^2$, then $\delta V = 0$ and thus $\delta V * \rho_\varepsilon = 0$. But with the previous construction, we obtain $\widehat{V}_\varepsilon = (\|V\| * \rho_\varepsilon(x))\,d\mathcal{L}^2 \otimes \delta_{T_\varepsilon(x)}$ represented in Figure 1.

Qualitatively, we observe that $\delta(\widehat{V}_\varepsilon)$ is composed of a singular part concentrated on the red set in Figure 1 and an absolute part due to the fact that $\|V\| * \rho_\varepsilon(x)$ is not constant along the level-sets $\{d(x, \Gamma) = \lambda\}$. Exact computations can be done by dividing the cross along the red set into 4
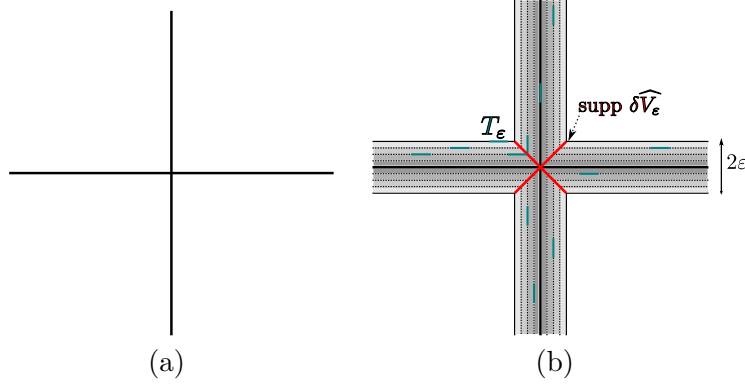
Figure 1: A cross (left) and the support (right) of the diffuse varifold proposed in Example 4.1.

parts and applying Fubini Theorem to integrate on the level-sets $\{d(x, \Gamma) = \lambda\}$, and then apply the divergence Theorem in each integral; but qualitatively, we can see that with this definition,

$$\delta\left(\widehat{V}_\varepsilon\right) \neq 0 = \delta V * \rho_\varepsilon \,.$$

The construction we proposed is not the right one, yet the idea of convolving the spatial part is reasonable, but the tangential part must be constructed from V and not from $\|\widehat{V}_\varepsilon\|$:

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V a d–varifold in Ω with finite mass $\|V\|(\Omega) < +\infty$. Let $\varepsilon > 0$ and ρ_ε as in* (11)*. Define the d–varifold $\widehat{V}_\varepsilon$ as:*

$$\left\langle \widehat{V}_\varepsilon, \psi \right\rangle = \left\langle V, (y, S) \mapsto \psi(\cdot, S) * \rho_\varepsilon(y) \right\rangle \ \ \text{for every } \psi \in C^0_c(\Omega \times G_{d,n}) \,;$$

or equivalently,

$$\int_{\Omega \times G_{d,n}} \psi(y, S) \, d\widehat{V}_\varepsilon(y, S) = \int_{(y,S) \in \Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} \psi(x, S)\rho_\varepsilon(y - x) \, d\mathcal{L}^n(x) \, dV(y, S) \,. \tag{37}$$

Then,

1. $\|\widehat{V}_\varepsilon\| = \|V\| * \rho_\varepsilon$,

2. $\delta\left(\widehat{V}_\varepsilon\right) = \delta V * \rho_\varepsilon$.

Proof. Let us first compute $\|\widehat{V}_\varepsilon\|$, for $\varphi \in C^0_c(\Omega)$,

$$\left\langle \|\widehat{V}_\varepsilon\|, \varphi \right\rangle = \left\langle \widehat{V}_\varepsilon, \varphi \right\rangle = \int_{(y,S) \in \Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} \varphi(x)\rho_\varepsilon(y - x) \, d\mathcal{L}^n(x) \, dV(y, S)$$

$$= \int_{y \in \Omega} \int_{x \in \mathbb{R}^n} \varphi(x)\rho_\varepsilon(y - x) \, d\mathcal{L}^n(x) \, d\|V\|(y) = \langle \|V\|, \varphi * \rho_\varepsilon \rangle$$

$$= \langle \|V\| * \rho_\varepsilon \rangle \,.$$

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We now compute the first variation of $\widehat{V_\varepsilon}$. Let $X \in C_c^1(\Omega, \mathbb{R}^n)$,

$$\left\langle \delta\left(\widehat{V_\varepsilon}\right), X \right\rangle = \left\langle \widehat{V_\varepsilon}, (y, S) \mapsto \operatorname{div}_S X(y) \right\rangle$$
$$= \int_{(y,S) \in \Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} \operatorname{div}_S X(x) \rho_\varepsilon(y - x) \, d\mathcal{L}^n(x) \, dV(y, S) \,,$$

and for fixed $(y, S) \in \Omega \times G_{d,n}$,

$$\operatorname{div}_S(x \mapsto \rho_\varepsilon(y - x) X(x)) = \rho_\varepsilon(y - x) \operatorname{div}_S X(x) - \nabla^S \rho_\varepsilon(y - x) \cdot X(x) \,. \tag{38}$$

Moreover,

$$\int_{(y,S) \in \Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} \nabla^S \rho_\varepsilon(y - x) \cdot X(x) \, d\mathcal{L}^n(x) \, dV(y, S)$$
$$= \int_{x \in \mathbb{R}^n} \int_{(y,S) \in \Omega \times G_{d,n}} \nabla^S \rho_\varepsilon(y - x) \, dV(y, S) \cdot X(x) \, d\mathcal{L}^n(x)$$
$$= \int_{x \in \mathbb{R}^n} \delta V * \rho_\varepsilon(x) \cdot X(x) \, d\mathcal{L}^n(x) \text{ thanks to } (12)$$
$$= \langle \delta V * \rho_\varepsilon, X \rangle \,, \tag{39}$$

and since $x \mapsto \rho_\varepsilon(y - x) X(x)$ is compactly supported, for a fixed $S \in G_{d,n}$, $\int_{x \in \mathbb{R}^n} \operatorname{div}_S(x \mapsto \rho_\varepsilon(y - x) X(x)) \, d\mathcal{L}^n(x) = 0$ so that

$$\int_{(y,S) \in \Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} \operatorname{div}_S(x \mapsto \rho_\varepsilon(y - x) X(x)) \, d\mathcal{L}^n(x) \, dV(y, S) = 0 \,. \tag{40}$$

Hence, thanks to (38), (39) and (40), we have,

$$\left\langle \delta\left(\widehat{V_\varepsilon}\right), X \right\rangle = \langle \delta V * \rho_\varepsilon, X \rangle \,.$$

$\square$

Example 4.3. Let us come back to the example of the cross $V = v(N, 1)$ in $\mathbb{R}^2$ with $N = N_1 \cup N_2$ and $N_1 = \{x_1 = 0\}$ and $N_2 = \{x_2 = 0\}$. Define the 2–varifolds $V_1 = v(N_1, 1)$ and $V_2 = v(N_2, 1)$ so that $V = V_1 + V_2$. Notice that the definition of $V \mapsto \widehat{V_\varepsilon}$ in (37) is linear. So that $\delta(\widehat{V_\varepsilon}) = \delta(\widehat{V_{1\varepsilon}}) + \delta(\widehat{V_{2\varepsilon}})$ and the fact that

$$\delta(\widehat{V_{1\varepsilon}}) = \delta(\widehat{V_{2\varepsilon}}) = 0$$

is quite intuitive. Let us check it by simple computations, for $\psi \in C_c^0(\mathbb{R}^2 \times G_{1,2})$,

$$\int_{\mathbb{R}^2 \times G_{1,2}} \psi(y, S) \, d\widehat{V_\varepsilon}(y, S) = \int_{(y,S) \in \mathbb{R}^2 \times G_{1,2}} \int_{x \in \mathbb{R}^2} \psi(x, S) \rho_\varepsilon(y - x) \, d\mathcal{L}^2(x) \, dV_1(y, S)$$
$$+ \int_{(y,S) \in \mathbb{R}^2 \times G_{1,2}} \int_{x \in \mathbb{R}^n} \psi(x, S) \rho_\varepsilon(y - x) \, d\mathcal{L}^2(x) \, dV_2(y, S)$$
$$= \int_{x \in \mathbb{R}^2} \psi(x, T_1) \int_{y \in \mathbb{R}^2} \rho_\varepsilon(y - x) \, d\|V_1\|(y) \, d\mathcal{L}^2(x)$$
$$+ \int_{x \in \mathbb{R}^2} \psi(x, T_2) \int_{y \in \mathbb{R}^2} \rho_\varepsilon(y - x) \, d\|V_2\|(y) \, d\mathcal{L}^2(x) \,,$$

where T_1, $T_2 \in G_{1,2}$ respectively denote the direction of N_1 and N_2. Thus, for $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$,

$$\int_{\mathbb{R}^2 \times G_{1,2}} \operatorname{div}_S X(y) \, d\widehat{V_\varepsilon}(y, S) = \int_{x \in \mathbb{R}^2} \operatorname{div}_{T_1} X(x) \int_{y \in \mathbb{R}^2} \rho_\varepsilon(y - x) \, d\|V_1\|(y) \, d\mathcal{L}^2(x)$$
$$+ \int_{x \in \mathbb{R}^2} \operatorname{div}_{T_2} X(x) \int_{y \in \mathbb{R}^2} \rho_\varepsilon(y - x) \, d\|V_2\|(y) \, d\mathcal{L}^2(x) \, .$$

Moreover, in each set $\{d(x, N_1) = \lambda\}$, $\int_{y \in \mathbb{R}^2} \rho_\varepsilon(y - x) \, d\|V_1\|(y) = c_\lambda$ is constant. Then, thanks to Fubini Theorem and the divergence Theorem,

$$\int_{x \in \mathbb{R}^2} \operatorname{div}_{T_1} X(x) \int_{y \in \mathbb{R}^2} \rho_\varepsilon(y - x) \, d\|V_1\|(y) \, d\mathcal{L}^2(x) = \int_{\lambda = -\varepsilon}^\varepsilon \int_{\{d(x, N_1) = \lambda\}} \operatorname{div}_{T_1} X(x) c_\lambda \, d\mathcal{H}^1(x) \, d\lambda$$
$$= 0 \, .$$

Notice that the idea of convolving the spatial part was right so that the point was to build the right tangential part. In the following proposition, we study the tangential part of $\widehat{V_\varepsilon}$ defined in (37).

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $V = \|V\| \otimes \nu_x$ a d–varifold in Ω with finite mass $\|V\|(\Omega) < +\infty$. Let $\varepsilon > 0$ and ρ_ε as in (11). Let $\widehat{V_\varepsilon}$ defined as in (37). Then, $\widehat{V_\varepsilon} = \|\widehat{V_\varepsilon}\| \otimes \widehat{\nu_x^\varepsilon}$ where, for $\|\widehat{V_\varepsilon}\|$–almost every $x \in \mathbb{R}^n$, $\widehat{\nu_x^\varepsilon}$ is a probability measure in $G_{d,n}$ and, for all $\psi \in C^0(G_{d,n})$,*

$$\int_{G_{d,n}} \psi(S) \, d\widehat{\nu_x^\varepsilon}(S) = \frac{\int_{y \in \Omega} \int_{G_{d,n}} \psi(S) \, d\nu_y(S) \, \rho_\varepsilon(y - x) \, d\|V\|(y)}{\int_{y \in \Omega} \rho_\varepsilon(y - x) \, d\|V\|(y)} \, , \tag{41}$$

or equivalently, for any Borel set $A \in G_{d,n}$,

$$\widehat{\nu_x^\varepsilon}(A) = \frac{\int_{y \in \Omega} \nu_y(A) \, \rho_\varepsilon(y - x) \, d\|V\|(y)}{\int_{y \in \Omega} \rho_\varepsilon(y - x) \, d\|V\|(y)} \, . \tag{42}$$

Proof. Let $\varphi \in C_c^0(\mathbb{R}^n)$ and $\psi \in C^0(G_{d,n})$.

$$\left\langle \widehat{V_\varepsilon}, \varphi(x)\psi(S) \right\rangle = \int_{x \in \mathbb{R}^n} \varphi(x) \int_{y \in \Omega} \int_{S \in G_{d,n}} \psi(S) \, d\nu_y(S) \rho_\varepsilon(y - x) \, d\|V\|(y) \, d\mathcal{L}^n(x)$$
$$= \int_{x \in \mathbb{R}^n} \varphi(x) \int_{S \in G_{d,n}} d\widehat{\nu_x^\varepsilon}(S) \, d\|\widehat{V_\varepsilon}\|(x)$$
$$= \int_{x \in \mathbb{R}^n} \varphi(x) \int_{S \in G_{d,n}} d\widehat{\nu_x^\varepsilon}(S) \int_{y \in \Omega} \rho_\varepsilon(y - x) \, d\|V\|(y) \, d\mathcal{L}^n(x) \, .$$

Consequently, for $\mathcal{L}^n$–almost every x,

$$\int_{S \in G_{d,n}} \psi(S) \, d\widehat{\nu_x^\varepsilon}(S) = \frac{\int_{y \in \Omega} \int_{S \in G_{d,n}} \psi(S) \, d\nu_y(S) \rho_\varepsilon(y - x) \, d\|V\|(y)}{\int_{y \in \Omega} \rho_\varepsilon(y - x) \, d\|V\|(y)} \, .$$

$\square$

Example 4.5. Coming back again to the example of the cross $V = v(N, 1)$ with $N = \{x_1 = 0\} \cup \{x_2 = 0\} \subset \mathbb{R}^2$, let $\widehat{V}_\varepsilon$ associated with V by formula (37). We already know that $\|\widehat{V}_\varepsilon\| = \|V\| * \rho_\varepsilon$. We now want to identify the tangential part $\widehat{\nu_x^\varepsilon}$ in the decomposition $\widehat{V}_\varepsilon = \|\widehat{V}_\varepsilon\| \otimes \widehat{\nu_x^\varepsilon}$. Thanks to Proposition 4.4, for $\|\widehat{V}_\varepsilon\|$–almost every $x \in \mathbb{R}^2$ and for any Borel set $A \subset \mathbb{R}^2$,

$$\widehat{\nu_x^\varepsilon}(A) = \frac{\int_{y \in \Omega} \nu_y(A) \, \rho_\varepsilon(y - x) \, d\|V\|(y)}{\int_{y \in \Omega} \rho_\varepsilon(y - x) \, d\|V\|(y)} \, ,$$

and applying it with $A = \{T_1\}$, then $A = \{T_2\}$ where $T_1, T_2 \in G_{1,2}$ respectively denote the direction of $N_1 = \{x_1 = 0\}$ and $N_2 = \{x_2 = 0\}$, we have for $i = 1, 2$,

$$\widehat{\nu_x^\varepsilon}(\{T_i\}) = \frac{\int_{\{y \in \Omega \, : \, y \in N_i\}} \rho_\varepsilon(y - x) \, d\|V\|(y)}{\int_{y \in \Omega} \rho_\varepsilon(y - x) \, d\|V\|(y)} \quad \text{and} \quad \widehat{\nu_x^\varepsilon}(\mathbb{R}^2 \setminus \{T_1, T_2\}) = 0 \, .$$

Hence $\widehat{\nu_x^\varepsilon}$ is a convex combination of δ_{T_1} and δ_{T_2} whose coefficients depend on the distances $d(x, N_1)$ and $d(x, N_2)$, as represented in Figure 2.

Remark 4.6. Notice that in the general definition (37) of $\widehat{V}_\varepsilon$, $\widehat{\nu_x^\varepsilon}$ is generally not a sum of Dirac masses, unless the tangent plane to V is constant on a set of $\|V\|$–mass strictly positive.
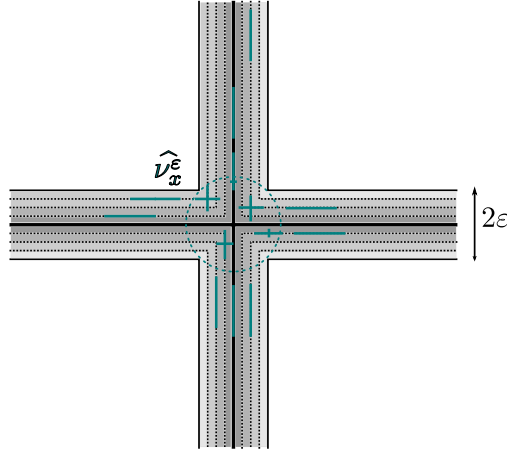
Figure 2: Graphical representation of the relative weights of δ_{T_1} and δ_{T_2}, associated with vertical and horizontal directions respectively, in the convex decomposition of $\widehat{\nu_x^\varepsilon}$ in Example 4.5.

5   Numerical computation of generalized mean curvature for $2D$ and $3D$ point clouds

This section is devoted to numerical experiments which illustrate how the regularized mean curvature computed on $2D$ and $3D$ point clouds behaves depending on the regularization kernel, the regularization parameter ε, and the sampling resolution. Given a point cloud varifold V, we compute its ε-regularized mean curvature vector at a point x_i of the cloud with the formula below,

which has been introduced in the previous sections:

$$-\frac{\delta V * \rho_\varepsilon(x_i)}{\|V\| * \rho_\varepsilon(x_i)} \, , \tag{43}$$

Recall that, if W is a rectifiable d–varifold with bounded first variation $\delta W = -H \|W\| + (\delta W)_s$, $\rho \in C_c^1(B_1(0))$ is a radial kernel, and (V_k) is a suitable sequence of approximating point cloud varifolds, we proved in Theorem 3.10 and Corollary 3.12 that for $\|W\|$–almost any x,

$$H_{\varepsilon_i}^{V_k}(x) = -\frac{\delta V_k * \rho_{\varepsilon_k}(x)}{\|V_k\| * \rho_{\varepsilon_k}(x)} \xrightarrow[k\to\infty]{} H(x) \, .$$

The approximate mean curvature given by formula (43) can be easily written for a point cloud varifold $V_N = \sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$ and for $\rho(y) = \zeta(|y|)$:

$$H_\varepsilon^N(x) = -\frac{\delta V_N * \rho_\varepsilon(x)}{\|V_N\| * \rho_\varepsilon(x)} = -\frac{\displaystyle\sum_{j=1}^N m_j \zeta'\left(\frac{|x_j - x|}{\varepsilon}\right) \frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^N m_j \varepsilon \zeta\left(\frac{|x_j - x|}{\varepsilon}\right)} \, . \tag{44}$$

We recall that (44) can be considered also for more general kernels (for instance, kernels that are only $W^{1,\infty}$-regular, see Remark 3.3).

We will first study the approximation on $2D$ point cloud varifolds built from parametrized curves, and for different choices of radial kernels and various sampling resolutions. We will also illustrate that our ε-regularized curvature provides a good approximation of the generalized mean curvature at straight crossing points. We lastly propose in Section 5.2 preliminary tests on $3D$ point clouds.

5.1 Tests on 2D parametric shapes

5.1.1 Test shapes and kernel profiles

We study the experimental behavior of (44) on different 2D parametric shapes, for different kernels, for various numbers N of points in the cloud, and various values of the radius ε of the ball supporting ρ_ε. We shall denote as N_{neigh} the mean number of points in a ball of radius ε centered at a point of the cloud. The $2D$ parametric test shapes are (see Figure 3):

(a) A circle of radius 0.5;

(b) An ellipse parametrized by $x(t) = a\cos(t)$, $y(t) = b\sin(t)$, $t \in (0, 2\pi)$ with $a = 1$ and $b = 0.5$. The curvature vector is given by $H(t) = |H(t)|n(t)$, with $n(t)$ the inner unit normal and

$$|H(t)| = \frac{a^2}{b}\left(1 - e^2\cos^2(t)\right)^{-\frac{3}{2}}, \quad e^2 = 1 - \left(\frac{b}{a}\right)^2 \, .$$

(c) A "flower" parametrized by $r(\theta) = 0.5(1 + 0.5\sin(6\theta + \frac{\pi}{2}))$.

(d) A "eight" figure parametrized by $x(t) = 0.5\sin(t)(\cos t + 1)$, $y(t) = 0.5\sin(t)(\cos t - 1)$, $t \in (0, 2\pi)$.

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Remark 5.1. The experiments on the "eight" figure will confirm a property of our approximation of the mean curvature: straight crossings have 0 curvature. More generally our model is able to approximate correctly "singular" configurations whose canonically associated varifold has no singular first variation, i.e. $(\delta V)_s = 0$.
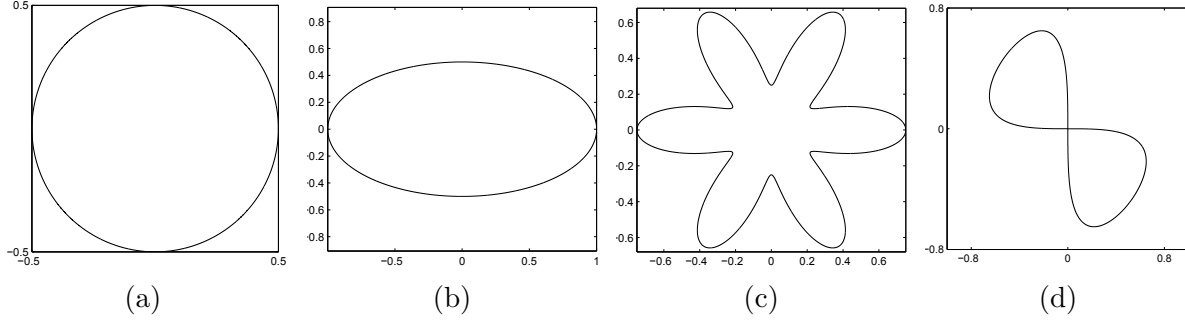

(a)   (b)   (c)   (d)

Figure 3: 2D parametric test shapes

We will test formula(44) with various profiles ζ defined on $[0,1]$, extended to $\mathbb{R}_+$ by 0, and renormalized by a constant (if necessary) to ensure that $\int_0^1 \zeta = 1$:

- The aforementioned "tent" kernel $\zeta_{tent}(r) = (1 - r)$;

- The "exponential" kernel $\zeta_{exp}(r) = \exp\left(-\frac{1}{1-r^2}\right)$;

- The "double parabola" kernel $\zeta_{parabola}(r) = 1 - 2r^2$ if $r < \frac{1}{2}$ and $\zeta(r) = 2(1-r)^2$ otherwise;

- The "inverse tent" kernel $\zeta_{\substack{inverse \\ tent}}(r) = r$.

Notice that the last kernel (extended by 0 when $r > 1$) is not continuous, and thus δV convolved with $\rho_\varepsilon^{\substack{inverse \\ tent}}$ (with $\rho^{\substack{inverse \\ tent}}(y) = \zeta_{\substack{inverse \\ tent}}(|y|)$) is not even a function. Therefore, we rather use the following formula for the approximate mean curvature associated with the "inverse tent" kernel:

$$
-\frac{\delta V_N * \rho_\varepsilon^{tent}}{\|V\| * \rho_\varepsilon^{\substack{inverse \\ tent}}} = -\frac{\displaystyle\sum_{j=1}^{N} \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j \frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^{N} \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j |x_j - x|} \ , \tag{45}
$$

where ρ_ε^{tent} is associated with a "tent" profile, which is Lipschitz, and $\rho_\varepsilon^{\substack{inverse \\ tent}}$ is associated with an "inverse tent" profile, which is bounded. Being the mass $\|V\|$ a Radon measure, the denominator is well-defined.

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To associate a point cloud with each parametric test shape under study, we proceed as follows: we compute the exact unit tangent vector $T(t)$ at the N points $\{0, h, 2h, \ldots, (N-1)h\}$ for $h = \frac{2\pi}{N}$, and we set

$$V_N = \sum_{j=1}^{N} m_j \delta_{(x(jh), y(jh))} \otimes \delta_{T(jh)} .$$

Doing as if the local point density were constant in the cloud, we consider that the weight m_j of each point is the same that is, for all j, $m_j = m$, which yields a simplification of (44). This assumption makes sense for the uniformly sampled circle, but is slightly unrealistic for shapes with varying curvature or with non uniform parameterization. We will propose later a further projection onto the normal to compensate, at least partially, for the non uniformily.

For all shapes under study, the vector curvature $H(t)$ can be computed explicitly and evaluated at all $t_j = jh$, $j = 0 \ldots N-1$. To test the accuracy of the approximation (44), we compute the following average error on the vector curvature

$$E = \frac{1}{N} \sum_{j=1}^{N} |H_\varepsilon^N(x_j) - H(t_j)| , \tag{46}$$

or the relative average error on the vector curvature

$$E^{rel} = \frac{1}{N} \sum_{j=1}^{N} \frac{|H_\varepsilon^N(x_j) - H(t_j)|}{|H(t_j)|} , \tag{47}$$

5.1.2 Numerical behavior of the approximate mean curvature

The first test is a comparison of the convergence speed of the approximate curvature on the circle of radius 0.5, for the various kernels. As the norm of the curvature is constant, it is sufficient to compute the average error (46) for the relative error only differs by a multiplicative constant. A natural question is how to let N go to ∞ and ε go to 0. In Corollary 3.12, the convergence is controlled by $\frac{\delta_i}{\varepsilon_i^2}$. Since in our current experimental setting δ_i is of order $\frac{1}{N}$, it would be natural to consider N and ε such that $N \to +\infty$, $\varepsilon \to 0$ and $\frac{1}{N\varepsilon^2} \to 0$. We actually expect that the convergence rate, at least for smooth shapes, is better and occurs even for looser controls of $\frac{1}{N\varepsilon^2}$. This is supported by both experiments of Figures 4(a) and 4(b). In Figure 4(a), we use a log-log scale to represent the average error (46) as a function of the number of sample points N on the circle, and we set $\varepsilon = \frac{1}{\sqrt{N}}$, i.e. $\frac{1}{N\varepsilon^2} = 1$. The results in Figure 4(b) are computed with $\varepsilon = \frac{500}{N}$, and the convergence rate remains good despite the fact that $\frac{1}{N\varepsilon^2} \to \infty$.

The fact that the most regular kernel (the exponential) has the best behavior is not a surprise: indeed the convergence result in Corollay 3.12 requires that the kernel ρ has a bounded $W^{2,\infty}$ norm. However, the fact that the "tent" kernel produces a poor convergence speed whereas the variant containing the "inverse tent" kernel yields a very good convergence speed, was not really expected. It could to due to the fact that the reversed tent kernel in the denominator of (45) seems to act as a "corrector" for the reconstruction of the mean curvature, but this is still the purpose of ongoing work.
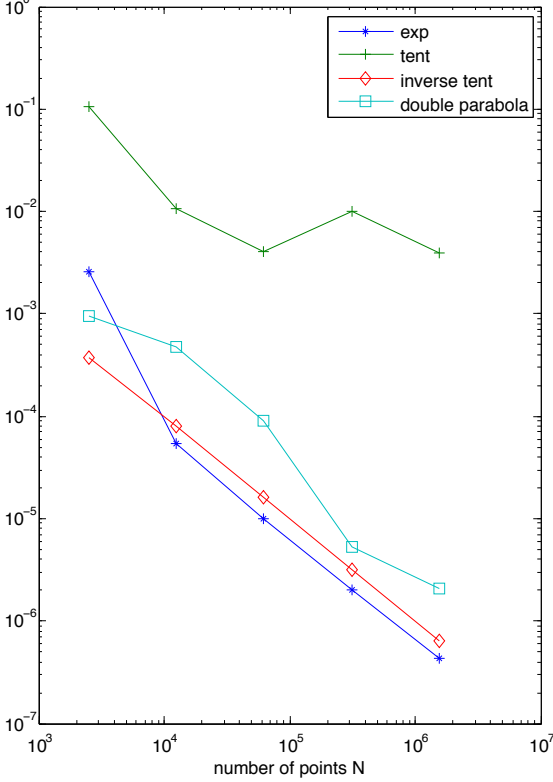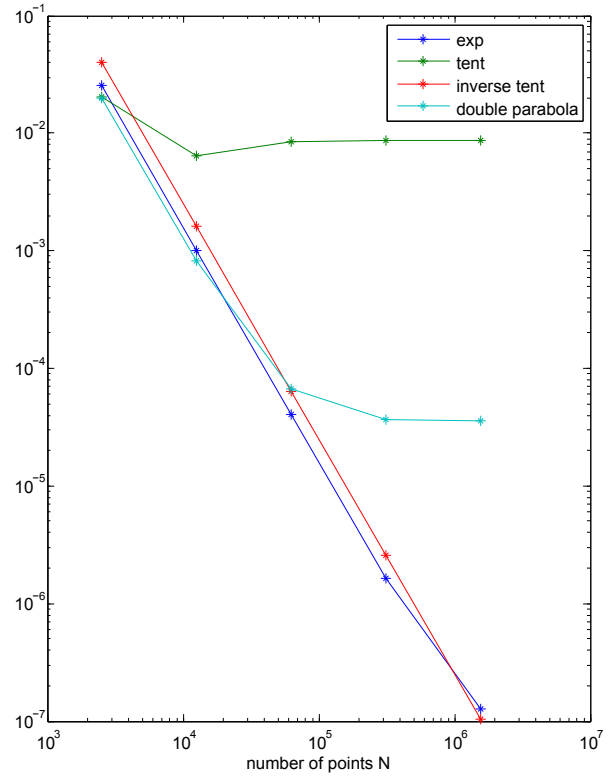
Figure 4: Average error (log-log scale) for the approximate mean curvature of the subsampled parametric circle, for increasing values of N and $\varepsilon = \frac{1}{\sqrt{N}}$ (left) or $\varepsilon = \frac{500}{N}$ (right).

The advantages of Formula (44) are numerous: it is very easy to compute, there is no need to know an approximation of the local length or area, it does *not depend on the orientation* of the point cloud (because the formula is grounded on varifolds which have no orientation) and it preserves the 0–singular curvature. But there is a major drawback, the preservation of 0–curvature at straight crossings is obtained thanks to a phenomenon of compensation. Indeed, the term

$$\mathbb{1}_{\{|x_j - x| < \varepsilon\}} \frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}$$

is of order 1 and has to be compensated by a "symmetric point" (with respect to the normal at x) in the ball $B_\varepsilon(x)$ to produce a term of order ε with orientation given by the normal $n(x)$ to x. This is not specific to our discrete formula, it occurs also at the continuous level as represented in Figure 5.
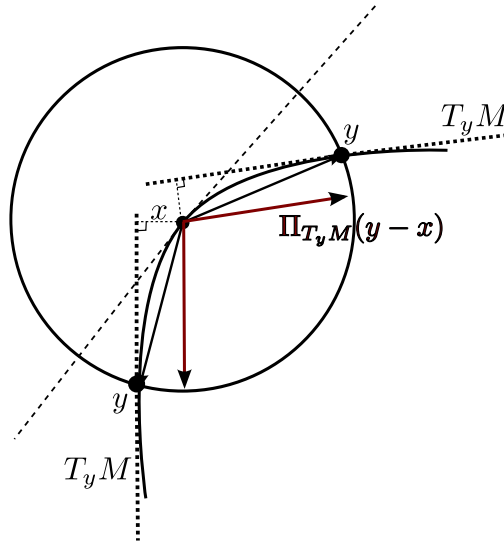
Figure 5: Compensation of the contributions to the mean curvature vector of points which are symmetric with respect to the normal axis at x, see (15).

But this compensation phenomenon generates great instability at the discrete level, as illustrated in the following simple example:

Example 5.2. Sample the segment $S = [0, 1] \times \{0\} \subset \mathbb{R}^2$ into a uniform point cloud, for instance

$$V_N = \sum_{j=1}^{N} \frac{1}{N} \delta_{\left(\frac{j}{N}, 0\right)} \otimes \delta_{e_1} \text{ with } e_1 \text{ the horizontal direction .}$$

Then pick a point x_0 and consider computing the approximated curvature at x_0, in a ball of radius ε. Assume that, due for instance to noise in the sampling process, the point cloud is actually not completely uniform, and that in $B(x_0, \varepsilon)$ there are n_+ points (in the cloud) greater than x_0, and

n_- points smaller, with $|n_+ - n_-| \geq 1$. Then, formula (44) gives the non zero curvature:

$$
\left| H_\varepsilon^N(x_0) \right| = \left| \frac{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x_0| < \varepsilon\}} m_j \frac{\Pi_{e_1}(x_j - x_0)}{|x_j - x_0|}}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - 0| < \varepsilon\}} m_j |x_j - x_0|} \right| = \frac{m |n_+ - n_-|}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - 0| < \varepsilon\}} m |x_j - x_0|} \simeq \frac{|n_+ - n_-|}{|n_+ + n_-| \varepsilon}
$$

$$
\simeq \frac{1}{N_{neigh} \varepsilon} \ .
$$

Letting N_{neigh}, the number of points contained in an ε-neighborhood of a point of the cloud, be very large, so that the product εN_{neigh} is also large, makes this spurious curvature small. However, whereas it is possible in this example to choose the sampling resolution, it it not the case for general point clouds. The only way then to let the ratio decrease is to increase ε, which may result in a global lack of accuracy.

There is a way to remedy – at least partially – the numerical instability due to a lack of compensation by projecting the result onto the normal vector, though it becomes more sensitive to the accuracy of the normal.

5.1.3 Formula with projection onto the normal vector

We propose to compose, in Formula (44), the projector onto the tangent Π_{P_j} with a projector onto the normal $\Pi_{P_{j_0}^\perp}$ at the central point with index j_0. Brakke[Bra78] actually proved that, for an integral varifold with bounded first variation, the generalized mean curvature vector H is orthogonal to the approximate tangent plane. Therefore, the projection onto the normal vector is transparent at the continuous level. At the discrete level, however, it makes a difference: it removes a part of the error due to the fact that a variation in the density of points is a variation of mass and thus creates a tangential component of the curvature vector. In the next experiments, we therefore use a new definition of the approximate mean curvature. For a point cloud varifold $V_N = \sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$, at a point x with the normal direction at x denoted as $P_0^\perp$, we compute the mean curvature vector as

$$
H_\varepsilon^{N,\text{normal}} = -\frac{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j \zeta'\left(\frac{|x_j - x|}{\varepsilon}\right) \Pi_{P_0^\perp}\left(\frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}\right)}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j \varepsilon \zeta\left(\frac{|x_j - x|}{\varepsilon}\right)} \ . \tag{48}
$$

We first test this formula on the circle of radius 0.5 with exact normals, and assuming that the weights m_j are all equal (since the sampling is uniform). We represent in Figure 6 the curvature vectors computed for $N = 10^5$ points and $\varepsilon = 0.001$. Arrows indicate the vectors and colors indicate their norms, to be compared with the exact value $|H| = 2$.

We also test the formula on two other test shapes, first the ellipse using $\varepsilon = \frac{1}{\sqrt{N}}$ and various kernels (see Figure 7), then the "flower" (Figure 8). In addition to the curvature vectors, we compute the average relative error for various values of N, and we observe similar order of convergence. The
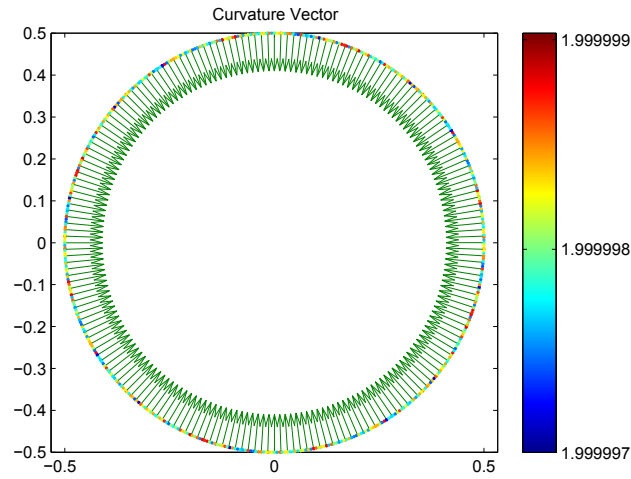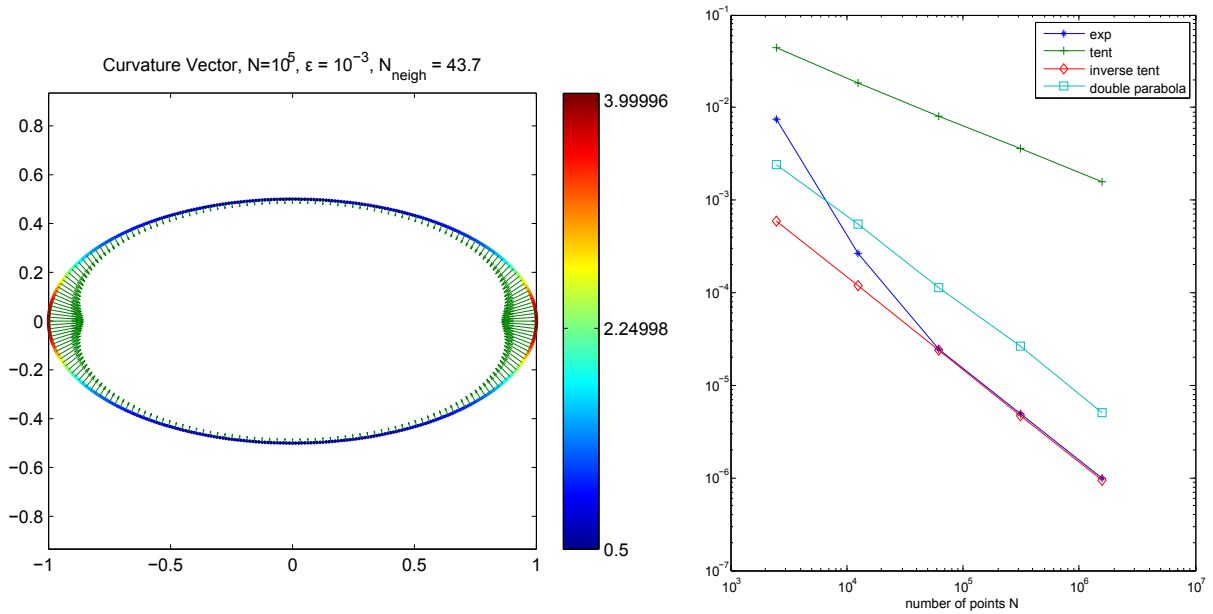
Figure 6: Approximate curvature vectors along the discrete circle of radius 0.5. Arrows indicate the curvature vectors and colors indicate their norms.

(a) Approximate curvature vectors

(b) Average relative error with respect to the real curvature value, for various values of N.
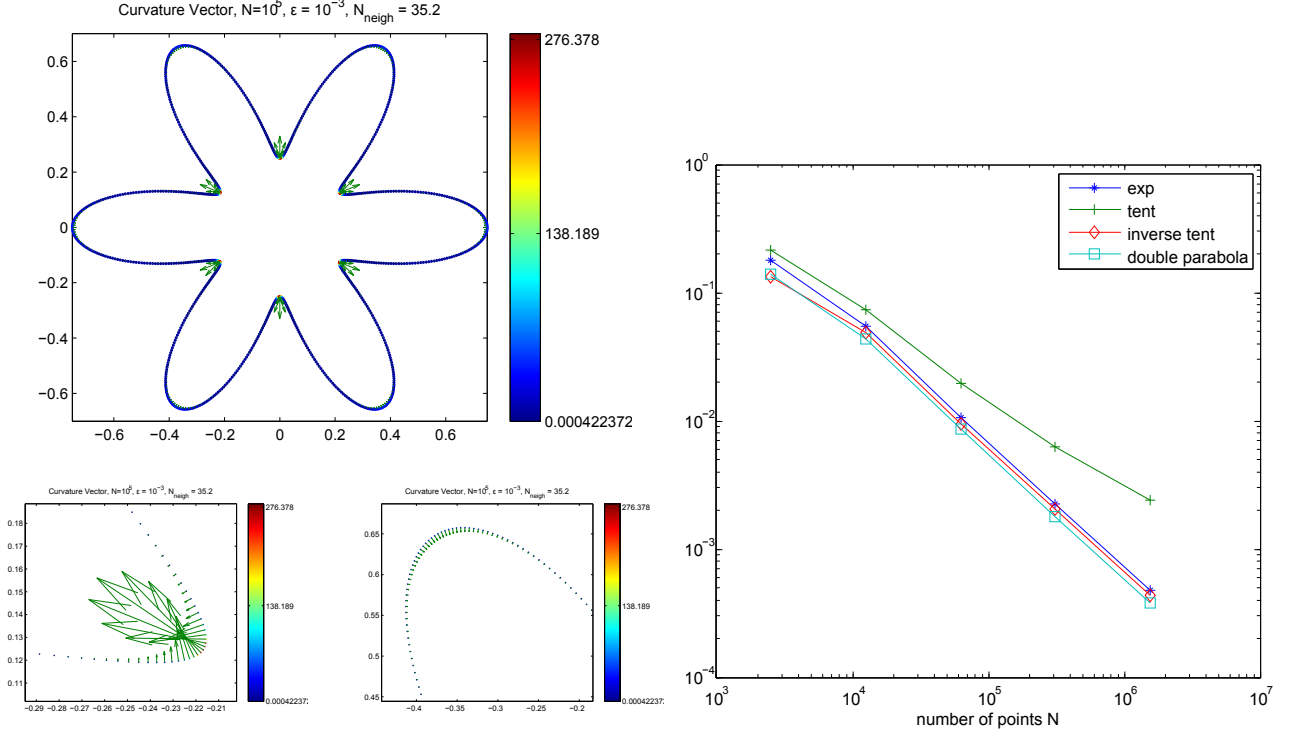
Figure 7: Testing formula (48) on the ellipse

Figure 8: Testing formula (48) on the "flower" (left lower figures are details of the left upper figure).

results obtained with the "flower" illustrate the quality of the approximation even at points where the curvature is very high.

It remains a lot to understand on the choice of optimal parameters depending on the sampling resolution, and the precise difference between the various kernels: is the reversed tent kernel still a good choice for noisy shapes? and if it seems to be the best choice, why? This is completely open so far.

5.2 3D point clouds experiments

This section is devoted to the numerical illustration of formula (48) used on 3D point clouds. All computations in this section were done using a C++ code and the libraries *nanoflann* and *eigen*. The visualization is made with *CloudCompare*.

We first test formula (48) on a ball of radius 1, parametrized with spherical coordinates, and using the exponential and the inverse tent kernels, see Figure 9. We could use the exact normal as we did for $2D$ point clouds, but since we want to handle more general point clouds (not given with their normal vectors), we compute instead the normal direction at each point thanks to a classical regression. More precisely, we compute the covariance matrix of centered coordinates (in a ball of radius $\frac{\varepsilon}{2}$)) and we define as the normal the eigenvector associated with the smallest eigenvalue. Denoting as $\tilde{N}$ the number of points on the equatorial circle in the discretized ball, ε is prescribed as previously done by the relation $\tilde{N}\varepsilon^2 = 1$, and we study the evolution of the average error on the mean curvature vector (46) with respect to the total number of points N.

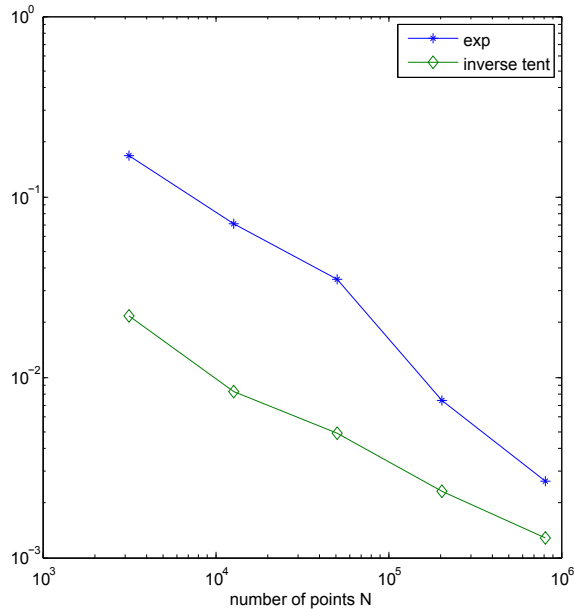The last experiments in this section were made on non parametrized point clouds: a dragon

Figure 9: Average error on the mean curvature vector of a 3D ball of radius 1: evolution of the error with the number of points N, using either the exponential kernel or the inverse tent kernel.

with $N = 435\,000$ points (the norm of the mean curvature vectors are represented with colors in Figure 10), and a buddha statue with $543\,524$ points (Figure 11).

We conclude this section with a list of what we believe are the advantages of our approximate mean curvature: the formula is simple, it does neither require to pre-compute an orientation nor a triangulation of the point cloud. It is in addition very flexible: it can be applied to lots of surface representations as soon as a consistent varifold structure is defined on the discrete objects. Future work will be devoted to a better understanding of the numerical properties of the formula, and how it can be used in applications, for instance for the smoothing of point clouds using mean curvature flow.

References

[AFP00]   L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford mathematical monographs. Clarendon Press, Oxford, New York, 2000.

[All72]   William K Allard. On the first variation of a varifold. *Annals of mathematics*, pages 417–491, 1972.

[Alm65]   F. J. Almgren. The theory of varifolds. 1965.

[Bra78]   K.A. Brakke. The motion of a surface by its mean curvature. *Mathematical notes (20), Princeton University Press*, 1978.

[BSSZ08]   Alexander I Bobenko, Peter Schro, John M Sullivan, and Gunter M Ziegler. *Discrete Differential Geometry. Oberwolfach Seminars.* Springer, 2008.
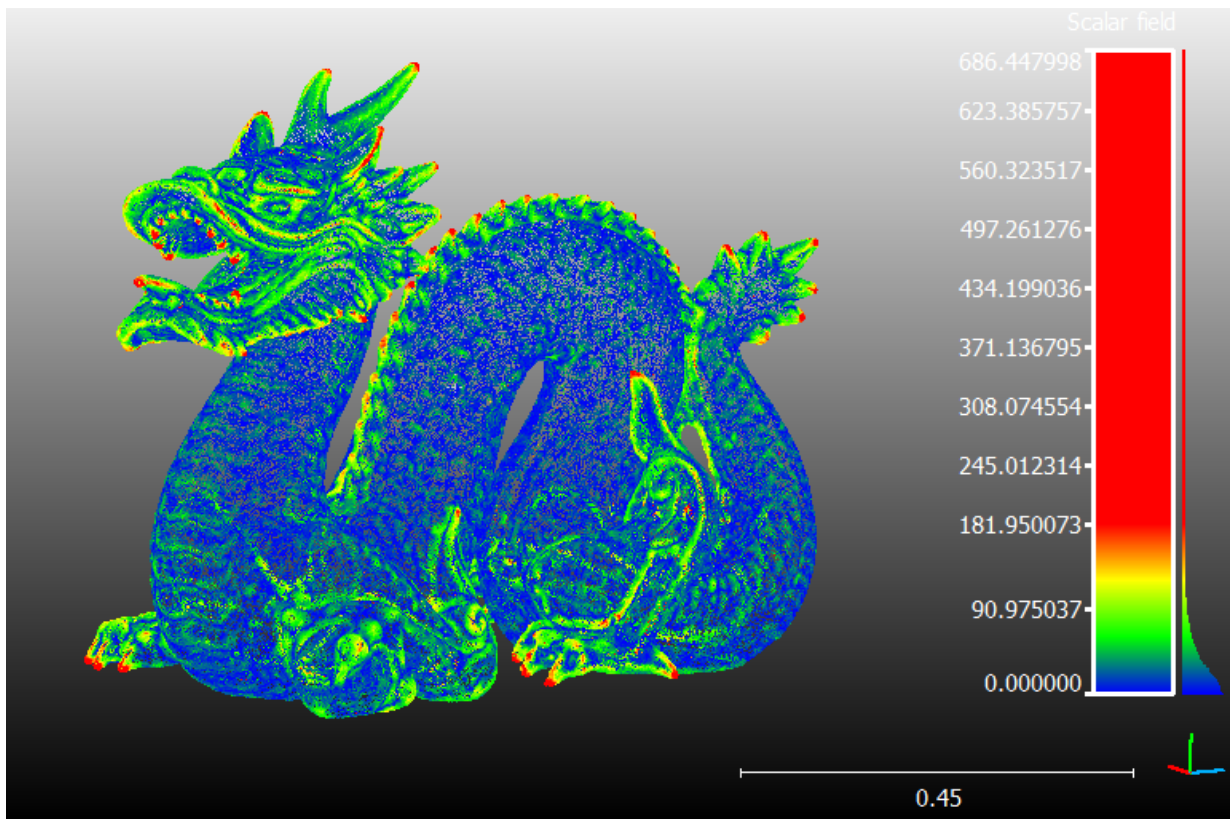
(a)
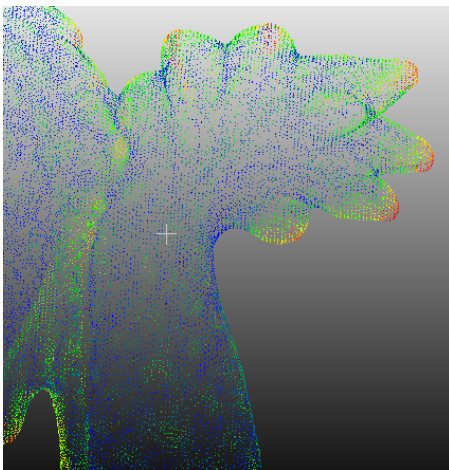
(b) Details on the dragon's tail
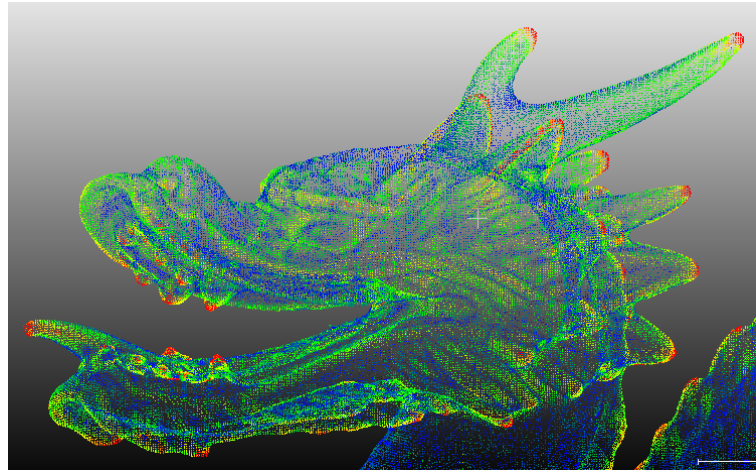
(c) Details on the dragon's head

Figure 10: Intensity of the mean curvature of a dragon (435 000 points, diameter= 1) with $\varepsilon = 0.007$.
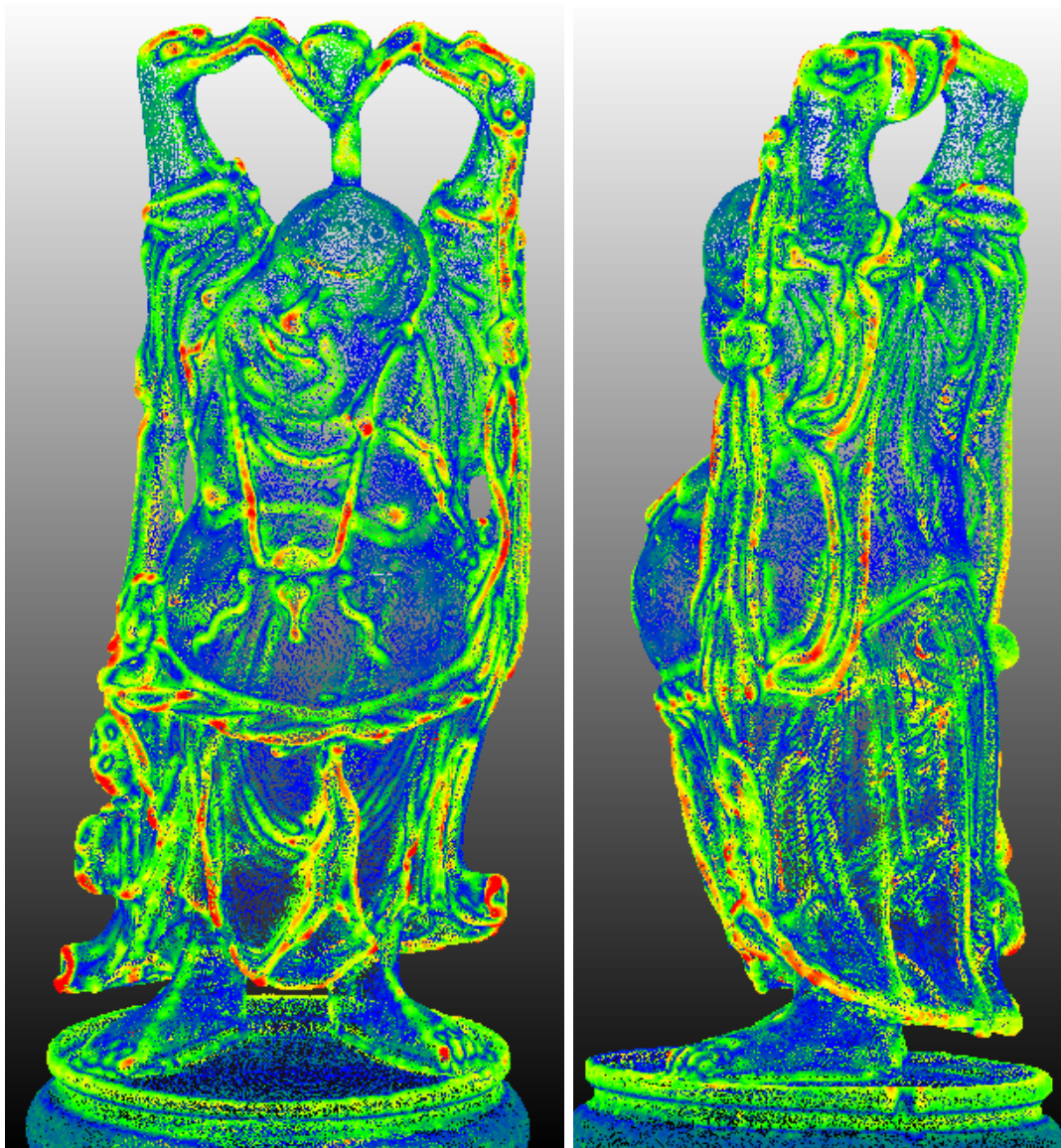
Figure 11: Intensity of the mean curvature of a buddha statue (543 524 points, horizontal side length=0.41, height=1)

[Bue14]   B. Buet. *Approximation de surfaces par des varifolds discrets : représentation, courbure, rectifiabilité.* PhD thesis, Université Claude Bernard Lyon 1, 2014.

[Bue15]   B. Buet. Quantitative conditions of rectifiability for varifolds. *Annales de l'Institut Fourier*, to appear, 2015.

[CCLT09] F. Chazal, D. Cohen-Steiner, A. Lieutier, and B. Thibert. Stability of curvature measures. *Computer Graphics Forum*, 28(5), 2009.

[CRT04]   U. Clarenz, M. Rumpf, and A. Telea. Robust feature detection and local classification for surfaces based on moment analysis. *IEEE Transactions on Visualization and Computer Graphics*, 10(5):516–524, 2004.

[CSM06]   D. Cohen-Steiner and J.-M. Morvan. Second fundamental measure of geometric sets and local approximation of curvatures. *J. Differential Geom.*, 74(3):363–394, 2006.

[CT13]   N. Charon and A. Trouvé. The varifold representation of nonoriented shapes for diffeomorphic registration. *SIAM J. Imaging Sci.*, 6(4):2547–2580, 2013.

[Dav10]   G. David. $C^{1+\alpha}$-regularity for two-dimensional almost-minimal sets in $\mathbb{R}^n$. *J. Geom. Anal.*, 20(4):837–954, 2010.

[HDD$^+$92] Hugues Hoppe, Tony DeRose, Tom Duchamp, John McDonald, and Werner Stuetzle. Surface reconstruction from unorganized points. *SIGGRAPH '92 Proceedings of the 19th annual conference on Computer graphics and interactive techniques*, 26(2):71–78, 1992.

[LM09]   G. P. Leonardi and S. Masnou. Locality of the mean curvature of rectifiable varifolds. *Adv. Calc. Var.*, 2(1):17–42, 2009.

[Mor08]   J.-M. Morvan. *Generalized curvatures*, volume 2 of *Geometry and Computing*. Springer-Verlag, Berlin, 2008.

[PR14]   B. Piccoli and F. Rossi. Generalized Wasserstein distance and its application to transport equations with source. *Arch. Ration. Mech. Anal.*, 211(1):335–358, 2014.

[Sim83]   L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[ST92]   Richard Szeliski and David Tonnesen. Surface modeling with oriented particle systems. *SIGGRAPH '92 Proceedings of the 19th annual conference on Computer graphics and interactive techniques*, 26(2):185–194, 1992.

[Tay76]   J. E. Taylor. The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. of Math. (2)*, 103(3):489–539, 1976.

[Vil09]   C. Villani. *Optimal transport*, volume 338. Springer-Verlag, Berlin, 2009.