THE PROP OF BICOMMUTATIVE HOPF ALGEBRAS

Let us denote by **bicomBiAlg** the PROP of bicommutative bialgebras. It is generated by elements $e \in \mathbf{bicomBiAlg}(0,1)$, $\varepsilon \in \mathbf{bicomBiAlg}(1,0)$ $\mu \in \mathbf{bicomBiAlg}(2,1)$ and $\Delta \in \mathbf{bicomBiAlg}(1,2)$ with the standard relations (diagramatic axioms for bialgebras, plus commutativity and anticommutativity).

Let us define also another PROP $\overline{\mathbf{bicomBiAlg}}$ as the quotient of **bicomBiAlg** by the additional relation

$$\Delta \circ \mu = 0\,. \tag{1}$$

An explicit description of $\overline{\mathbf{bicomBiAlg}}$

On one hand it is an easy exercise to prove, using (1) and the (diagramatic version of) compatibility between product and coproduct, that one can remove all cycles in the graph representing a given element in $\overline{\mathbf{bicomBiAlg}}$.

On the other hand one has a morphism of PROPs

$$\phi\,:\,\overline{\mathbf{bicomBiAlg}} \longrightarrow \mathrm{Mat}(\{0,1\})$$

defined by $\phi(e) := \underline{0}$, $\phi(\varepsilon) := \underline{0}$, $\phi(\mu) := \underline{1}$, $\phi(\Delta) := \underline{1}$. Where $\underline{x}$ denotes the matrix with all entries being equal to x.

Proposition 0.1. *ϕ is an isomorphism.*

Sketch of proof. Recall that one can remove all cycles in the graph representing a given element in $\overline{\mathbf{bicomBiAlg}}$. Therefore, a complete description of elements in $\overline{\mathbf{bicomBiAlg}}(n,m)$ is as follows.

Any element in $\overline{\mathbf{bicomBiAlg}}(n,m)$ can be written in a unique way as

$$(\Delta^{m_1} \otimes \cdots \otimes \Delta^{m_k}) \circ \sigma \circ (m^{n_1} \otimes \cdots \otimes m^{n_k}) \tag{2}$$

with $n_1 + \cdots + n_k = n$, $m_1 + \cdots + m_k = m$, $\sigma \in S_k$ a permutation,

$$\Delta^i = (\Delta \otimes \mathrm{id}^{\otimes i-2}) \circ \cdots \circ \Delta$$

(for example $\Delta^1 = \mathrm{id}$, $\Delta^2 = \Delta$, $\Delta^3 = (\Delta \otimes \mathrm{id}) \circ \Delta$, and by convention $\Delta^0 = \varepsilon$), and

$$\mu^i = \mu \circ \cdots \circ (\mu \otimes \mathrm{id}^{\otimes i-2})\,.$$

Here we have implicitely use associativity, coassociativity, unitality, counitality, compatibility between counit and product, and compatibility between unit and coproduct.

Finally we have a bijection between elements as in (2) and $n \times m$ matrices with entries in $\{0,1\}$. The bijection is precisely given by ϕ. And this precisely corresponds to the bijection explained at the following url:

`http://golem.ph.utexas.edu/category/2008/05/theorems_into_coffee_ii.html#c016703.`

This ends the proof. □

The case of **bicomBiAlg**

The same ϕ as before defines a morphism of PROPs

$$\mathbf{bicomBiAlg} \longrightarrow \mathrm{Mat}(\mathbb{N})\,.$$

Let me now describe the main idea to prove that it is an isomorphism. Now observe that $\phi(\Delta \circ \mu) = (2)$ is the 1×1 matrix with entry being 2. therefore $\phi\big((\Delta \circ \mu)^n\big) = (n+1)$. Finally, to describe an element in $\mathbf{bicomBiAlg}(n, m)$ one must consider elements as in (2) and compose them with powers of $\Delta \circ \mu$.

What about **bicomHopf** ?

Let now **bicomHopf** be the PROP of bicommutative Hopf algebras. In contrast with **bicomBiAlg** there is one more generator $\gamma \in \mathbf{bicomHopf}(1, 1)$ and obvious diagramtic relations for Hopf algebras.

On then extand ϕ by defining $\phi(\gamma) := (-1)$. It obviously defines a morphism

$$\mathbf{bicomHopf} \longrightarrow \mathrm{Mat}(\mathbb{Z})\,.$$

Conjecture 0.2 (Baez)**.** *It is an isomorphism.*