

Anticyclic operads and
Auslander-Reiten translation

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IN BRIEF

1: Theory of operads (algebraic topology)

2: Representation theory

Build some cyclic actions

$$\tau \in \text{End}_{\mathbb{Z}}(\mathbb{Z}^N) \text{ with } \tau^n = \text{Id}, \quad (1)$$

for some N and n .

Observe that these two actions are closely related.

Propose some conjectural explanation for this link.

Operads: Basics and examples

(Non-symmetric) Operads in the category of Abelian groups.

Definition 1 An operad \mathcal{P} : DATA of

- a sequence $\{\mathcal{P}(n)\}_{n \in \mathbb{N}}$ of Abelian groups,
- a distinguished element $1 \in \mathcal{P}(1)$,
- composition maps \circ_i from $\mathcal{P}(n) \otimes_{\mathbb{Z}} \mathcal{P}(m)$ to $\mathcal{P}(n+m-1)$ for each n, m and each $1 \leq i \leq n$.

This data must satisfy some AXIOMS, modelled after the properties of the first example below:

- *unity,*
- *associativity of nested compositions,*
- *commutativity of disjoint compositions.*

Let us give some examples.

Example 1: the endomorphism operads

Pick any free Abelian group V of finite rank. Let $\mathcal{P}(n) = \text{Hom}_{\mathbb{Z}}(V^{\otimes n}, V)$. Let $\mathbf{1}$ be the identity map in $\mathcal{P}(1)$. Let \circ_i be the composition of multilinear maps defined, for $f \in \mathcal{P}(n)$ and $g \in \mathcal{P}(m)$, by

$$\begin{aligned} (f \circ_i g)(x_1, \dots, x_{m+n-1}) \\ = f(x_1, \dots, g(x_i, \dots, x_{i+m-1}), \dots, x_{m+n-1}). \end{aligned} \tag{2}$$

These data define the so-called endomorphism operad of V .

Example 2: the associative operad

Let $\text{Assoc}(n)$ be the free Abelian group of rank 1 with basis b^n . Let $\mathbf{1}$ be b^1 and let

$$b^n \circ_i b^m = b^{n+m-1}. \tag{3}$$

The axioms are easily checked and this defines the associative operad Assoc .

Example 3: the diassociative operad (Loday)

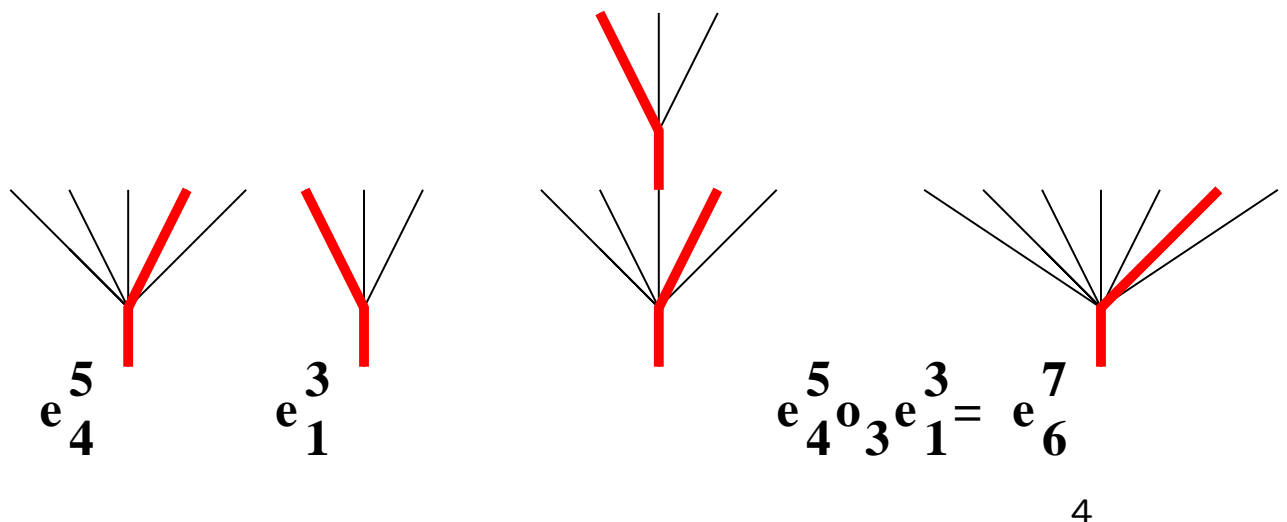
Let $\text{Dias}(n)$ be the free Abelian group of rank n with basis $\{e_1^n, \dots, e_n^n\}$. Let $\mathbf{1}$ be e_1^1 . The composition maps are defined by

$$e_k^n \circ_i e_l^m = e_j^{n+m-1}, \quad (4)$$

where j is given by the following rule:

$$\begin{cases} k & \text{if } i > k, \\ k + l - 1 & \text{if } i = k, \\ k + n - 1 & \text{if } i < k. \end{cases} \quad (5)$$

GRAPHICAL DESCRIPTION using corollas
(Follow the red line from the bottom to the top)



- free operads
- ideal in an operad
- quotient of an operad by an ideal

Hence one can speak of a **presentation by generators and relations** of an operad.

Let us give some examples of such presentations.

Example 2: the associative operad The operad Assoc is generated by $b^2 \in \text{Assoc}(2)$. One can compute that

$$b^2 \circ_1 b^2 = b^2 \circ_2 b^2 = b^3. \quad (6)$$

The operad Assoc is presented by the generator b^2 and the relation

$$b^2 \circ_1 b^2 = b^2 \circ_2 b^2. \quad (7)$$

Example 3: the diassociative operad

The operad Dias is generated by $\text{Dias}(2) = \mathbb{Z}\{e_1^2, e_2^2\}$. One can compute (using the graphical description of Dias) that

$$e_1^3 = e_1^2 \circ_1 e_1^2 = e_1^2 \circ_2 e_1^2 = e_1^2 \circ_2 e_2^2, \quad (8)$$

$$e_2^3 = e_2^2 \circ_2 e_1^2 = e_1^2 \circ_1 e_2^2, \quad (9)$$

$$e_3^3 = e_2^2 \circ_2 e_2^2 = e_2^2 \circ_1 e_1^2 = e_2^2 \circ_1 e_2^2. \quad (10)$$

This provides a presentation of the operad Dias.

Anticyclic operads

Definition 2 An anticyclic operad \mathcal{P} is an operad \mathcal{P} together with the data of endomorphisms τ_n of $\mathcal{P}(n)$ satisfying

$$\tau_1(\mathbf{1}) = -\mathbf{1}, \quad (11)$$

$$\tau_n^{n+1} = \text{Id}, \quad (12)$$

$$\tau_{n+m-1}(x \circ_n y) = -\tau_m(y) \circ_1 \tau_n(x), \quad (13)$$

$$\tau_{n+m-1}(x \circ_i y) = \tau_n(x) \circ_{i+1} y, \quad (14)$$

where $x \in \mathcal{P}(n)$, $y \in \mathcal{P}(m)$ and $1 \leq i \leq n - 1$.

This notion has been introduced by Getzler and Kapranov.

AIM: show that the operad Dias is an anticyclic operad.

HOW: define τ on generators, then check against relations.

Let us define τ_2 by

$$\tau_2(e_1^2) = -e_1^2 + e_2^2, \quad (15)$$

$$\tau_2(e_2^2) = -e_1^2. \quad (16)$$

Thus the matrix of τ_2 in the basis e^2 is

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \quad (17)$$

Theorem 3 *The operad Dias is an anticyclic operad with τ_2 as above. The matrix of τ_n in the basis e^n is*

$$\begin{bmatrix} -1 & -1 & \dots & -1 \\ 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & 1 & 0 \end{bmatrix}. \quad (18)$$

Let us give an example of computation for τ_3 :

$$\tau_3(e_2^3) = \tau_3(e_2^2 \circ_2 e_1^2) \quad (19)$$

$$= -\tau_2(e_1^2) \circ_1 \tau_2(e_2^2) \quad (20)$$

$$= (e_2^2 - e_1^2) \circ_1 (-e_1^2) \quad (21)$$

$$= e_3^3 - e_1^3. \quad (22)$$

You may check that using $e_2^3 = e_1^2 \circ_1 e_2^2$ instead leads to the same value.

Therefore we have defined an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on the Abelian group \mathbb{Z}^n for each $n \geq 1$.

Let us now define a similar action in a completely different way.

Algebras and Auslander-Reiten translation

Λ an algebra of finite dimension over a field k .

Assume that Λ has finite global dimension.

$\text{Mod } \Lambda$ category of finite-dimensional modules.

$D \text{ Mod } \Lambda$ bounded derived category of $\text{Mod } \Lambda$.

AUSLANDER-REITEN THEORY:

self-equivalence τ of $D \text{ Mod } \Lambda$

This is the [Auslander-Reiten translation](#).

This functor τ descends on the Grothendieck group $K_0(\text{Mod } \Lambda) = K_0(D \text{ Mod } \Lambda)$ and defines a bijective linear map, still denoted by τ , on the Grothendieck group. This map is sometimes called the [Coxeter transformation](#).

This theory has some nice applications to path algebras of quivers.

Choose any **Dynkin diagram** of finite type, in the usual list $(A_n)_{n \geq 1}$, $(D_n)_{n \geq 4}$, E_6, E_7, E_8 .

Picking any orientation of this Dynkin diagram defines a quiver Q .

Let $\text{Mod } kQ$ be the Abelian category of representations of Q .

classical results (Gabriel ; Gelfand & Ponomarev):

$\text{Mod } kQ$ has a finite number of isomorphism classes of indecomposable modules,

in bijection with positive roots of the associated **root system**.

action of τ on the Grothendieck group is exactly the action of a **Coxeter element** in the corresponding Weyl group.

Hence τ has finite order h , the Coxeter number.

Let us look at the case of the equioriented quiver of type \mathbb{A}_n :

$$n \rightarrow \cdots \rightarrow 2 \rightarrow 1. \quad (23)$$

$\text{Mod } \mathbb{A}_n$ the category of modules on this quiver

S_i the simple module on the vertex i .

The action of τ in the basis $\{S_1, S_2, \dots, S_n\}$ of $K_0(\text{Mod } \mathbb{A}_n)$ has the following matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \cdots & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \quad (24)$$

This is clearly the **transposed** matrix of the map τ_n that was defined purely in terms of operads before.

Now, it is possible to dualize the anticyclic operad Dias into an anticyclic **cooperad** Dias*. Then the cyclic group actions become exactly the same. This should be the proper setting.

Second example

more complicated,

more interesting.

another operad and

another family of algebras.

Binary trees

A planar binary tree is a

graph drawn in the plane,

which is connected and simply connected,

has vertices of valence 1 or 3 only,

together with the data of a distinguished vertex of valence 1 called the root.

The other vertices of valence 1 are called the leaves.

The root is drawn at the bottom.

Let Y_n be the set of planar binary trees with $n + 1$ leaves.

$$Y_1 = \{Y\} \quad (25)$$

$$Y_2 = \left\{ \begin{array}{c} \diagup \diagdown \\ | \quad | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ | \quad | \end{array} \right\} \quad (26)$$

$$Y_3 = \left\{ \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagdown \diagup \\ | \quad | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ | \quad | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ | \quad | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ | \quad | \end{array}, \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ \diagdown \diagup \\ | \quad | \end{array} \right\} \quad (27)$$

The cardinality of Y_n is the Catalan number

$$c_n = \frac{1}{n+1} \binom{2n}{n} \quad (28)$$

Then (Loday) there exists an operad Dend such that $\text{Dend}(n) = \mathbb{Z}Y_n$. We will not describe the composition maps \circ_i here. The unit 1 is the unique element of Y_1 .

This operad is generated by the two trees  and  in Y_2 .

The relations are as follows:

$$\begin{array}{c} \text{tree with root on the left} \circ_2 \text{tree with root on the left} = \text{tree with root on the left} \circ_1 \text{tree with root on the left} + \text{tree with root on the left} \circ_1 \text{tree with root on the right}, \\ (29) \end{array}$$

$$\begin{array}{c} \text{tree with root on the left} \circ_2 \text{tree with root on the right} = \text{tree with root on the left} \circ_1 \text{tree with root on the right}, \\ (30) \end{array}$$

$$\begin{array}{c} \text{tree with root on the left} \circ_2 \text{tree with root on the left} + \text{tree with root on the right} \circ_2 \text{tree with root on the left} = \text{tree with root on the left} \circ_1 \text{tree with root on the left}. \\ (31) \end{array}$$

Theorem 4 *There exists a unique structure of anticyclic operad on Dend such that*

$$\begin{array}{c} \tau(\text{tree with root on the right}) = \text{tree with root on the left} \quad \text{and} \quad \tau(\text{tree with root on the left}) = -(\text{tree with root on the left} + \text{tree with root on the right}). \\ (32) \end{array}$$

Let us display the matrix of τ_3 in the basis Y_3 of $\text{Dend}(3)$:

$$\begin{bmatrix} -1 & 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}. \quad (33)$$

In general, the map τ_n seems quite complicated.

One knows that

$$\tau_n^{n+1} = \text{Id}. \quad (34)$$



But what exactly are the eigenvalues of τ_n ?

Tamari posets

a partial order \leq on Y_n ,

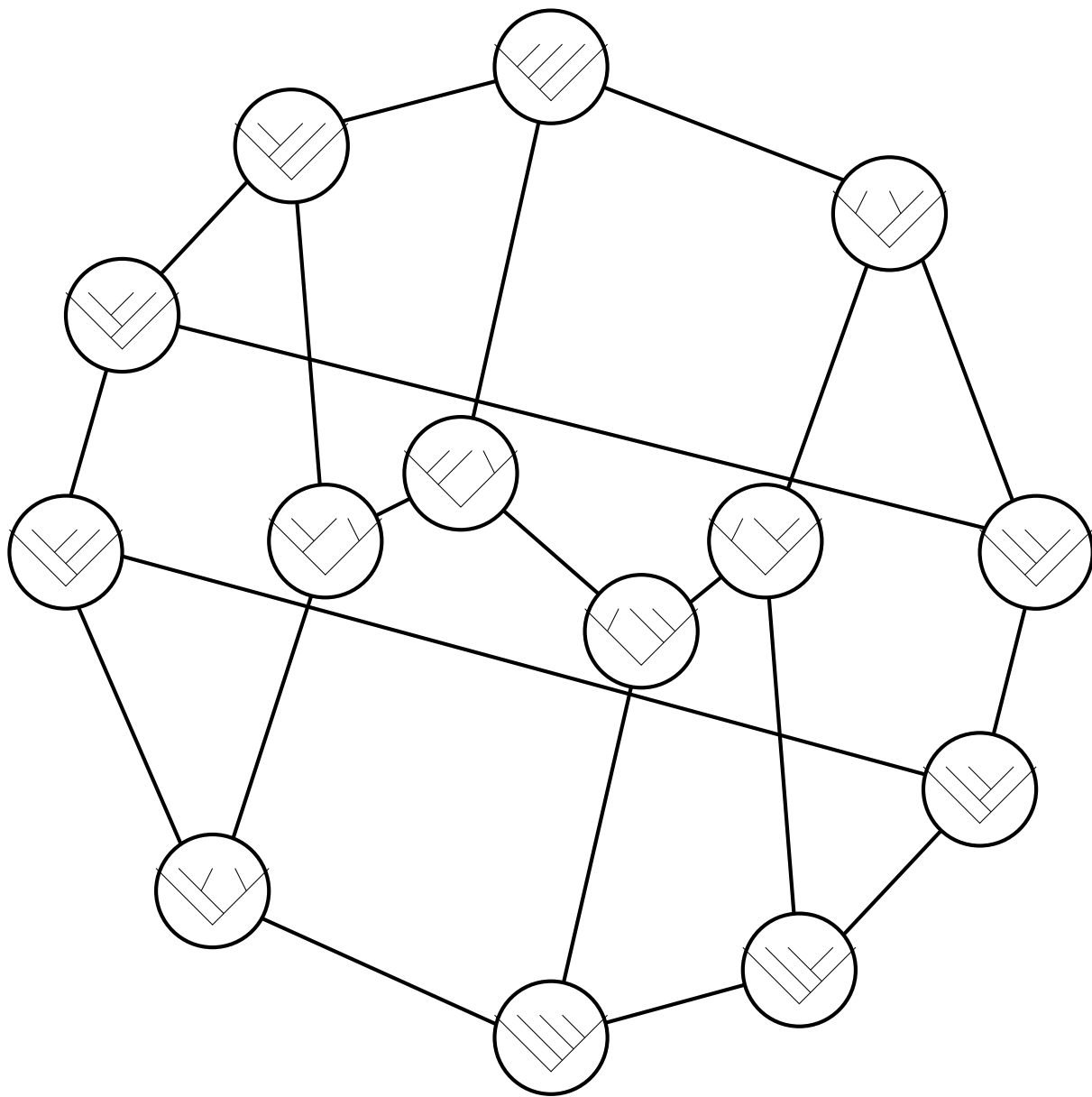
called the Tamari order or Tamari lattice.

The order relation \leq is the transitive closure of some covering relations.

A tree S is covered by a tree T if they differ only in some neighborhood of an edge by the replacement of the configuration  in S by the configuration  in T .

$\Lambda_n =$ incidence algebra of the poset (Y_n, \leq)

finite dimensional algebra of finite global dimension.



THE TAMARI LATTICE T_3

a.k.a. the third associahedron...

Auslander-Reiten theory gives:

Coxeter transformation θ

acting on the Grothendieck group of Λ_n .

This Grothendieck group has a basis coming from simple modules, which are labelled by Y_n .

Hence one can identify $K_0(\Lambda_n)$ with $\text{Dend}(n)$.

Theorem 5 *On the Abelian group $\text{Dend}(n)$, one has the relation*

$$\tau_n = (-1)^n \theta^2. \quad (35)$$

expected explanation:

appropriate functors

$$\circ_i : \text{Mod } \Lambda_n \otimes \text{Mod } \Lambda_m \longrightarrow \text{Mod } \Lambda_{n+m-1}, \quad (36)$$

satisfying, together with the Auslander-Reiten translation, some version of the axioms of an anticyclic operad.