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# SOME THEORETICAL RESULTS CONCERNING DIPHASIC FLUIDS IN THIN FILMS

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**Abstract.** We are interested in a model for diphasic fluids in thin flows taking into account both the hydrodynamical and the chemical effects at the interface between the two fluids. A limit problem in "thin films" is introduced heuristically. It is a system coupling the Reynolds equation and the hydrodynamical Cahn-Hilliard equation. We study the mathematical properties of this system, and prove an existence result under some smallness condition on the data.

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# 1 Introduction

In many applications, the geometry of the flow is anisotropic (i.e. one dimension is small with respect to the others), e.g. in lubrication problems. In the Newtonian case, the flow of a fluid between two close surfaces in relative motion is described by an asymptotic approximation of the Navier-Stokes equations, the Reynolds equation. This equation makes it possible to uncouple the pressure and the velocity. Indeed, in thin films, the pressure is considered to be independent of the direction in which the domain is thin. Thus an equation on the pressure only is obtained, and the velocity can be deduced from the pressure. This approach was introduced by Reynolds, and has been rigorously justified in [3] for the Stokes equation, and generalized afterwards in many works: for the steady-case Navier-Stokes equations [1], for the unsteady case [4], for compressible fluids with the perfect gases law [17]... It is of interest to investigate how this approach can be used for the case of a two fluid flow.

A first diphasic model consists in introducing a variable viscosity $\eta$, which is either equal to the viscosity $\eta_1$ of one fluid or the viscosity $\eta_2$ of the other fluid (that is to say that the fluids are considered to be non-miscible). The behavior of $\eta$ is described by a transport equation. In that case, when assuming the interface between the two fluids to be the graph of a function, the asymptotic equations corresponding to the thin film approximation can be interpreted as a generalized Buckley-Leverett equation, which governs the behavior of the saturation (i.e. the proportion of one fluid in the mixture) inside the gap, coupled with a generalized Reynolds equation, which governs the behavior of the pressure. These equations are investigated in [19] without shear effects, and in [6] with shear effects. One of the main disadvantages of the method is that the fluid interface is supposed to be the graph of a function, which hinders for example the formation of bubbles. In addition, this kind of model only takes into account hydrodynamical effects between the two phases, and surface tension effects are neglected.

The second class of models describing diphasic flows, which has been used up to now only for the Navier-Stokes equations, is the class of the so-called diffuse interface models. They take into account chemical properties at the interface between the two fluids, enabling an exchange between the two phases. In this paper, we use a Cahn-Hilliard equation, which involves an interaction potential, enhanced with a transport term. Thus this model describes both the chemical and the hydrodynamical properties of the flow. An order parameter $\varphi$ is introduced, for example the volumic fraction of one phase in the mixture. The surface tension can be taken into account *via* an additional term depending on $\varphi$ in the Navier-Stokes equations. This kind of model has been studied for the complete Navier-Stokes equations in [7], and for viscoelastic fluids in [10].

In this paper, we consider an asymptotic system (i.e. a thin film approximation) for a diphasic fluid in a thin film modelled by the Cahn-Hilliard equation. As for the Newtonian case, the Navier-Stokes equations are approximated by a modified Reynolds equation, in which the viscosity is not constant anymore. We study the Reynolds/Cahn-Hilliard system, and prove the existence and the regularity of a weak solution under a smallness assumption on the initial data and the geometry.

Let us describe briefly the main steps of the mathematical analysis. First, we study the Reynolds equation and investigate the regularity of the pressure and the velocity as functions of the order parameter. Next, we prove the existence of a solution to the system Reynolds/Cahn-Hilliard, by using a Galerkin process, which consists in introducing finite dimension approximations of $\varphi$. After obtaining *a priori* estimates for these approximations, we conclude that they converge to a solution of the system Reynolds/Cahn-Hilliard.

This paper is organized as follows. In Section 2, we introduce the two-dimensional model for a diphasic fluid in a thin film, which consists of a generalized Reynolds equation and of a diffuse-interface model (the Cahn-Hilliard equation). In Section 3, we state the main theorem, and give the main steps and difficulties of the proof. In Section 4, we deal with the Reynolds equation, and obtain some existence and regularity result on the velocity field and the pressure. In Section 5, we first introduce some specific results on trace estimates and Poincaré inequalities. They are used in the rest of the section for obtaining *a priori* estimates for the Cahn-Hilliard equation. At last, convergence results are deduced from these estimates, and allow to conclue the proof of the main theorem.

# 2 Modelling a diphasic fluid in a thin film

In this section, we will first present how a fluid is described in a thin domain by the Reynolds equation. Next, we introduce the hydrodynamical Cahn-Hilliard model for any fluid. Lastly, we combine both aspects and state the model of a diphasic fluid in a thin domain.

We introduce the domain $\Omega$ (see Fig. 1)

$$\Omega = \left\{(x,z) \in \mathbb{R}^2,\ 0 < x < L,\ 0 < z < h(x)\right\}, \tag{1}$$

where the function $h \in \mathcal{C}^2(\mathbb{R})$ satisfies

$$\begin{aligned} \forall x \in [0,L], &\qquad 0 < h_m \le h(x) \le h_M, \\ \forall x \in [0,L], &\qquad |h'(x)| \le h'_M. \end{aligned}$$

Observe that the regularity of $h$ ensures that the domain $\Omega$ defined by (1) satisfies the segment property and cone property (see [2, § 4.2 and 4.3]).

The thin film approximation for an incompressible fluid leads to the following equations [3], describing the behavior of the pressure $p$ and the velocity field $\boldsymbol{u} = (u,v)$, $\eta$ being the viscosity of the fluid:

$$\partial_z\left(\eta\, \partial_z u\right) = \partial_x p, \qquad \partial_z p = 0, \qquad \partial_x u + \partial_z v = 0. \tag{2}$$

The boundary conditions on $\boldsymbol{u}$ are suitable for lubrication applications: Dirichlet boundary conditions are imposed on the velocity on $\{z=0\}$ and $\{z=h(x)\}$ in order to model shear effects. The boundary conditions are written:

$$\forall x \in [0,L] \quad u(x,0) = s \quad \text{and} \quad u(x,h(x)) = v(x,0) = v(x,h(x)) = 0. \tag{3}$$

Without loss of generality, the shear velocity $s \geq 0$ is supposed to be positive. For the lateral part of the boundary, it has been showed in [3] that only the input flow $q = \int_0^{h(0)} \boldsymbol{u}(0,\xi)\cdot\boldsymbol{n}\,d\xi$ needs to be prescribed, where $\boldsymbol{n}$ denotes the exterior normal to the domain. Observe that according to the divergence-free condition and the boundary conditions on $\boldsymbol{u}$, this flow is constant on any "vertical" section of the domain:

$$\partial_x\left(\int_0^{h(x)} u(x,\xi)d\xi\right) = \underbrace{h'(x)u(x,h(x))}_{=0} + \int_0^{h(x)} \partial_x u(x,\xi)d\xi = -\int_0^{h(x)} \partial_\xi v(x,\xi)d\xi$$
$$= -v(x,h(x)) + v(x,0) = 0,$$

thus

$$q = \int_0^{h(x)} u(x,\xi)d\xi, \qquad \forall x \in (0,L).$$

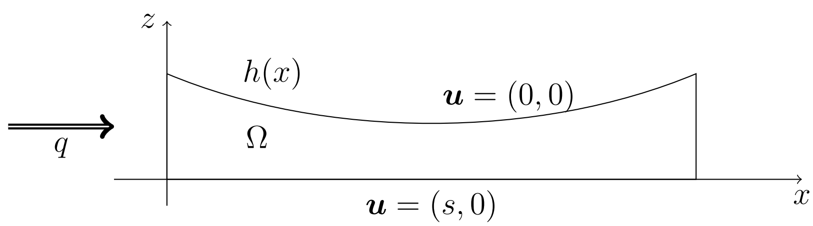


Figure 1: Domain Ω and boundary conditions on the velocity

## 2.1 Modelling one fluid in a thin domain

The usual procedure [3] is to integrate twice the first equation of (2) with respect to $z$, make use of the boundary conditions (3) and of the fact that $\partial_z p = 0$. This allows us to express $u$ as a function of $p$:

$$u(x,z) = \frac{z(z-h(x))}{2\eta}\partial_x p(x) + s\left(1 - \frac{z}{h(x)}\right). \tag{4}$$

Then, putting this expression in the divergence-free equation leads to the Reynolds equation:

$$\partial_x\left(\frac{h^3}{12\eta}\partial_x p\right) = s\partial_x\left(\frac{h}{2}\right). \tag{5}$$

A first boundary condition on $p$ is deduced from the ones on $\boldsymbol{u}$. In fact, the choice of the input flow $q$ corresponds to a Neumann condition for $p$ at $x = 0$. This condition can be determined as a function of $q$ by

$$q = \int_0^{h(0)} u(0,\xi)d\xi = -\partial_x p(0)\frac{h(0)^3}{12\eta} + \frac{sh(0)}{2}.$$

Let us denote $w := \partial_x p(0) = \dfrac{12\eta(q - sh(0)/2)}{h(0)^3}$. Moreover, the solution $p$ of (5) with the Neumann boundary condition $\partial_x p(0) = w$ is defined up to a constant. We can thus choose $p(L) = 0$ to gain a well-defined pressure $p$. It is to be noticed that once $p$ is computed from (5), then (4) allows us to compute $u$, while the other component of the velocity field $v$ is obtained by:

$$v(x,z) = -\int_0^z \partial_x u(x,\xi)\, d\xi. \tag{6}$$

## 2.2 Modelling a mixture

Since we want to study the mixture of two fluids, we introduce an order parameter $\varphi$ describing the volumic fraction of one fluid in the flow. All physical parameters can be written as functions of $\varphi$, in particular the viscosity $\eta$. We assume that $\eta(\varphi)$ satisfies:

$$\eta \in \mathcal{C}^1(\mathbb{R}) \quad \text{and} \quad \forall \varphi \in \mathbb{R}, \ \ 0 < \eta_m \le \eta(\varphi) \le \eta_M, \ \ \eta'(\varphi) \le \eta'_M. \tag{7}$$

For $-1 \le \varphi \le 1$, we can use a specific realistic law as a function of the viscosities of the two fluids $\eta_1$ and $\eta_2$ (see [9] or [18]):

$$\frac{1}{\eta(\varphi)} = \frac{1+\varphi}{2\eta_1} + \frac{1-\varphi}{2\eta_2} \qquad \text{for } \varphi \in [-1,1], \tag{8}$$

so that $\varphi = 1$ and $\varphi = -1$ correspond respectively to the fluids of viscosity $\eta_1$ and $\eta_2$ only. However, we will not be able to prove mathematically that $\varphi$ remains in the interval $[-1,1]$ (see [7]).

The effects of a possible variation of the density in the mixture will not be taken into account in this paper. Therefore, the density of the mixture is assumed to be constant (i.e. the two densities of the two incompressible phases $\rho_1$ and $\rho_2$ are supposed to be equal). Let us notice that due to the loss of the local conservation equation for the density, the non-homogeneous case $\rho_1 \neq \rho_2$ induces further difficulties (see [8]).

The Cahn-Hilliard equation describes the evolution of $\varphi$ and consists of both a transport term, taking the mechanical effects into account, and a diffusive term modelling the chemical effects. The Cahn-Hilliard equation is written:

$$\partial_t \varphi + \boldsymbol{u} \cdot \nabla\varphi - \frac{1}{\mathcal{P}e} \operatorname{div}\left(\mathcal{B}(\varphi)\nabla\mu\right) = 0, \tag{9}$$

$$\mu = -\alpha^2 \Delta\varphi + F'(\varphi). \tag{10}$$

The variable $\mu$ is the chemical potential, $\mathcal{B}(\varphi)$ is called mobility, $\mathcal{P}e$ is the Péclet number, $\alpha$ is a non-dimensional parameter measuring the thickness of the diffuse interface, and the function $F$ is called Cahn-Hilliard potential. Physical considerations show that $F$ must have a double-well structure, each of the wells representing one of the two fluids. A rational choice for $F$ is given by a logarithmic form (for more details, we refer to [12] or [15])

$$F(\xi) = 1 - \xi^2 + c\left((1+\xi)\log(1+\xi) + (1-\xi)\log(1-\xi)\right),$$

for some constant $0 < c < 1$, or its polynomial approximation

$$F(\xi) = (1 - c'\xi^2)^2,$$

where $c'$ is another constant. These physically realistic potentials share several mathematical properties. In the following, we prove mathematical results for potentials $F$ having the following properties:

- The function $F$ is supposed to be regular (e.g. of class $\mathcal{C}^2$).
- Since $F$ is a physical potential, it is bounded from below. Moreover, only the derivative of $F$ occurs in the equations, therefore the addition of a constant does not change the equations. It is thus realistic to make the following assumption:

$$\exists F_0 > 0, \quad \forall \xi \in \mathbb{R}, \qquad F(\xi) \geq F_0. \tag{11}$$

- The convexity of the potential corresponds to the stability of the mixture. Usual potentials contain some stable and unstable regions (see for example Figure 2). In order to include such cases, we impose:

$$\exists F_5 \geq 0, \quad \forall \xi \in \mathbb{R}, \qquad F''(\xi) \geq -F_5, . \tag{12}$$

- Moreover the following hypothesis on the growth of the potential is imposed:

$$\begin{gathered} \exists F_1,\ F_2 > 0,\ \exists r \in [1, +\infty),\ \forall \xi \in \mathbb{R}, \\ |F'(\xi)| \leq F_1|\xi|^r + F_2 \text{ and } |F''(\xi)| \leq F_1|\xi|^{r-1} + F_2. \end{gathered} \tag{13}$$

  This hypothesis is satisfied for any polynomial function.

- At last, we assume a generalization of the convexity:

$$\forall \gamma \in \mathbb{R},\ \exists F_3(\gamma) > 0,\ F_4(\gamma) \geq 0,\ \forall \xi \in \mathbb{R},\ (\xi - \gamma)F'(\xi) \geq F_3(\gamma)F(\xi) - F_4(\gamma). \tag{14}$$

These hypotheses are satisfied by functions of the form $F(\varphi) = \dfrac{\varphi^4}{4} - \dfrac{\varphi^2}{2} + F_0$ (as in Figure 2), which can be used as a model case. As far as the mobility $\mathcal{B}$ is concerned, it is supposed to be regular, positive, and bounded from above and from below:

$$\mathcal{B} \in \mathcal{C}^2(\mathbb{R}), \quad \forall \xi \in \mathbb{R}, \quad 0 < \mathcal{B}_m \leq \mathcal{B}(\xi) \leq \mathcal{B}_M. \tag{15}$$

Let us mention that other types of functions $\mathcal{B}$ can be considered, in particular the degenerate case $\mathcal{B}(\xi) = (1 - \xi^2)^\sigma$, with $\sigma \geq 0$, which has been studied in [13] and in [7], but introduces further mathematical difficulties.

Equations (9)-(10) must be equipped with boundary conditions on $\varphi$ and $\mu$. We are interested in modelling injection phenomena, which arise for example in lubrication or polymer injection problems. To this end, it is important to control the composition of the input. Thus we use Dirichlet boundary conditions on some part of the boundary, namely where the fluid is supplied. For the other part of the boundary, classical Neuman

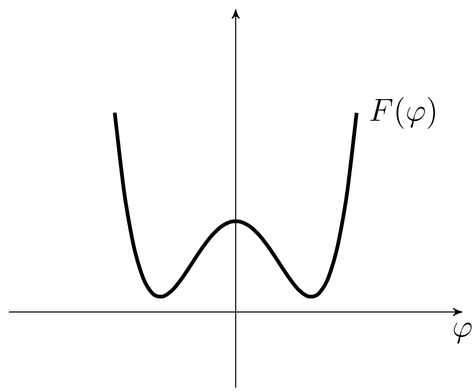


Figure 2: Possible appearance of the potential $F(\varphi)$

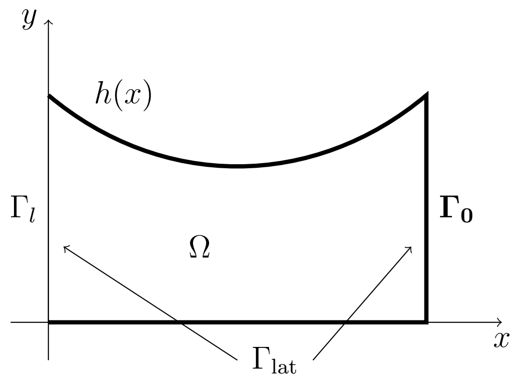


Figure 3: Domain $\Omega$ and notations for the boundary

boundary conditions for both $\varphi$ and $\mu$ are considered. Let us observe that in previous works ([7], [10]) Neuman boundary conditions were imposed on the whole boundary.

Thus, the boundary conditions are written

$$\varphi|_{\Gamma_l} = \varphi_l, \qquad \mu|_{\Gamma_l} = 0 \qquad \text{and} \qquad \frac{\partial \mu}{\partial \boldsymbol{n}}\Big|_{\Gamma_0} = 0, \qquad \frac{\partial \varphi}{\partial \boldsymbol{n}}\Big|_{\Gamma_0} = 0, \tag{16}$$

for some given boundary value $\varphi_l$ defined on $\Gamma_l$.

Finally, let us define the initial condition: $\varphi(t = 0) = \varphi_0 \in H^3(\Omega)$, where $\varphi_0$ is supposed to be satisfying the same boundary conditions as $\varphi$. The compatibility conditions also imply that $\mu_0$ defined by $\mu_0 = -\alpha^2 \Delta \varphi_0 + F'(\varphi_0)$ satisfies the same boundary conditions as $\mu$.

## 2.3 Modelling a mixture in thin films

A diphasic flow in a thin domain is still described by the Reynolds system (2), where the viscosity $\eta$ is not constant anymore but depends on the order parameter $\varphi$. Because of the non-constant viscosity, the coefficients in the Reynolds equation (which depend on $\eta$) depend on $\varphi$. Let us introduce the following expressions that will be useful in the following:

$$a(x,z) = \int_0^z \frac{d\xi}{\eta(\varphi(x,\xi))}, \quad b(x,z) = \int_0^z \frac{\xi d\xi}{\eta(\varphi(x,\xi))}, \quad c(x,z) = \int_0^z \frac{\xi^2 d\xi}{\eta(\varphi(x,\xi))}, \tag{17}$$

and

$$\widetilde{a}(x) = a(x, h(x)), \qquad \widetilde{b}(x) = b(x, h(x)), \qquad \widetilde{c}(x) = c(x, h(x)),$$

for all $(x,z) \in \Omega$. We define also:

$$\widetilde{d}(x) = \left(\widetilde{c}(x) - \frac{\widetilde{b}(x)^2}{\widetilde{a}(x)}\right) \quad \text{and} \quad \widetilde{e}(x) = \frac{\widetilde{b}(x)}{\widetilde{a}(x)}. \tag{18}$$

Following the same procedure as in Section 2.1, we integrate twice the first equation of (2) with non-constant viscosity and using the boundary conditions, we obtain for all $(x,z) \in \Omega$:

$$u(x,z) = \left(b(x,z) - \frac{\widetilde{b}(x)}{\widetilde{a}(x)} a(x,z)\right) \partial_x p\,(x) + \left(1 - \frac{a(x,z)}{\widetilde{a}(x)}\right) s, \tag{19}$$

$$v(x,z) = -\int_0^z \partial_x u(x,\xi)\, d\xi. \tag{20}$$

We use the fact that $\boldsymbol{u}$ is divergence-free and the boundary conditions in order to write

$$\int_0^{h(x)} \partial_x u(x,z)\, dz = \partial_x \left(\int_0^{h(x)} u(x,z)\, dz\right) = 0. \tag{21}$$

After integrating (19), we obtain

$$\partial_x \left(\widetilde{d}(x) \partial_x p\,(x)\right) = s \partial_x \left(\widetilde{e}(x)\right), \tag{22}$$

where the coefficients $\widetilde{d}$ and $\widetilde{e}$ are given by (18). Therefore the whole system (Reynolds/Cahn-Hilliard) is written:

$$\partial_x(\widetilde{d}\,\partial_x p) = s\,\partial_x \widetilde{e} \tag{23a}$$

$$u = \left(b - \frac{a\,\widetilde{b}}{\widetilde{a}}\right) \partial_x p + s\left(1 - \frac{a}{\widetilde{a}}\right) \tag{23b}$$

$$v(\cdot, z) = -\int_0^z \partial_x u(\cdot, \xi) d\xi \tag{23c}$$

$$\partial_t\,\varphi + u\,\partial_x\varphi + v\,\partial_z\varphi - \frac{1}{\mathcal{P}e}\,\mathrm{div}(\mathcal{B}(\varphi)\nabla\mu) = 0 \tag{23d}$$

$$\mu = -\alpha^2 \Delta\varphi + F'(\varphi). \tag{23e}$$

The coefficients $a$, $b$, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{d}$, $\widetilde{e}$ are explicit functions of $\varphi$ (given by (17), (18)). The functions $\mathcal{B}$, $F$ are also given explicitly. The quantities $\mathcal{P}e$ and $\alpha$ are physical data. The boundary conditions are written

$$\partial_x p(0) = w, \qquad p(L) = 0, \tag{24a}$$

$$\frac{\partial \varphi}{\partial \boldsymbol{n}}|_{\Gamma_0} = \frac{\partial \mu}{\partial \boldsymbol{n}}|_{\Gamma_0} = 0, \qquad \varphi|_{\Gamma_l} = \varphi_l. \qquad \mu|_{\Gamma_l} = 0, \tag{24b}$$

Let us notice that equations (23b)-(23c) and the boundary condition (24a) on $p$ imply that the following boundary conditions are satisfied for $\boldsymbol{u}$:

$$u(x,0) = s, \qquad u(x,h(x)) = v(x,0) = v(x,h(x)) = 0, \tag{25}$$

$$\int_0^{h(0)} \boldsymbol{u}|_{x=0} \cdot \boldsymbol{n} = q, \tag{26}$$

where $w$, $q$, the shear velocity $s$ and $\widetilde{a}_0 = \widetilde{a}(0)$, $\widetilde{b}_0 = \widetilde{b}(0)$ are related by:

$$w = \frac{q - s\left(h(0) - \dfrac{1}{\widetilde{a}_0}\displaystyle\int_0^{h(0)} a(0,\xi)\, d\xi\right)}{\displaystyle\int_0^{h(0)} b(0,\xi)\, d\xi - \frac{\widetilde{b}_0}{\widetilde{a}_0}\int_0^{h(0)} a(0,\xi)\, d\xi}. \tag{27}$$

# 3 Main result

Let us define some notations and function spaces:

- $C$ denotes any constant depending only on physical parameters or on the size of the domain (i.e. independent of the unknowns). Moreover, let us define the quantity

  $$\sigma := \frac{h_M}{h_m}.$$

  Constants independent of the size of the domain $\Omega$ are denoted by $\bar{C}$, as well as the constants depending on $\Omega$ only through $\sigma$ (i.e. for fixed $\sigma$, the constants $\bar{C}$ remain fixed, even if $|\Omega|$ changes).

- For $f \in L^1(\Omega)$, we define the mean value of $f$, denoted by $m(f) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f$.

- For the usual Sobolev spaces, we denote by $|\cdot|_p$ the $L^p$-norm in $\Omega$, and by $\|\cdot\|_s$ the $H^s$-norm in $\Omega$.

- Let us define the following function spaces:

$$\Phi = \{\phi \in \mathcal{D}(\bar{\Omega}),\ \frac{\partial \phi}{\partial \boldsymbol{n}}|_{\Gamma_0} = 0,\ \phi|_{\Gamma_l} = 0\}, \quad \Phi^s = \overline{\Phi}^{H^s(\Omega)} \quad \text{for } s \geq 1,$$

$$\Phi_l = \{\phi \in \mathcal{D}(\bar{\Omega}),\ \frac{\partial \phi}{\partial \boldsymbol{n}}|_{\Gamma_0} = 0\},$$

$$\Phi_l^s = \overline{\{\phi \in \Phi_l,\ \phi|_{\Gamma_l} = \varphi_l\}}^{H^s(\Omega)} \quad \text{for } s \leq 3,$$

and

$$X(\Omega) = \{f \in H^1(\Omega) \cap L^\infty(\Omega), \partial_z f \in H^1(\Omega)\}.$$

Observe in particular that

$$\Phi_l^1 = \{\varphi \in H^1(\Omega), \varphi|_{\Gamma_l} = \varphi_l\}.$$

We introduce the following weak form of (23):

**Problem 3.1.** *Let $\varphi_0 \in \Phi_l^1$, and $0 < T \leq +\infty$. Find $(p, \boldsymbol{u}, \varphi, \mu)$ such that*

- *the following regularity is satisfied:*

$$p \in L^\infty(0,T; H^2(0,L)), \qquad u \in L^\infty(0,T; X(\Omega)), \qquad v \in L^\infty(0,T; L^2(\Omega)),$$
$$\varphi \in L^\infty(0,T;\Phi_l^1) \cap L^2_{loc}(0,T;\Phi_l^3) \cap \mathcal{C}^0([0,T[;\Phi_l^1), \qquad \mu \in L^2_{loc}(0,T;\Phi^1).$$

- *for any $\psi \in \Phi^1$,*

$$\int_\Omega \partial_t\varphi\,\psi + \int_\Omega \frac{1}{\mathcal{P}e}\mathcal{B}(\varphi)\nabla\mu\,\nabla\psi + \int_\Omega \boldsymbol{u}(\varphi)\cdot\nabla\varphi\,\psi = 0, \tag{28}$$

*with*

$$\mu = -\alpha^2\Delta\varphi + F'(\varphi). \tag{29}$$

- *the velocity field $\boldsymbol{u}(\varphi) = (u(\varphi), v(\varphi))$ is given as a function of $\varphi$ by* (23a), (23b), (23c) *equipped with the boundary conditions* (24a), (25), (26)*;*

- *the initial condition $\varphi|_{t=0} = \varphi_0$ is satisfied.*

The following sections are dedicated to the proof of the main theorem:

**Theorem 3.2.** *Let $\varphi_0 \in \Phi_l^1$, $0 < T \leq +\infty$, and let $\varphi_l$ satisfy Hypothesis 5.2 and let $F$ satisfy the assumptions stated in Section 2.2. Under a smallness assumption on $L$, there exists a solution $(p, \boldsymbol{u}, \varphi, \mu)$ of Problem 3.1.*

*Sketch of the proof.* The proof is divided into two main parts, since the Reynolds equation and the Cahn-Hilliard are treated separately:
**Step 1.** As far as the Reynolds equation is concerned, we prove the following proposition:

**Proposition 3.3.** *For any $\varphi \in H^1(\Omega)$, the Reynolds equation* (23a) *equipped with the boundary conditions* (24a) *admits a unique solution which satisfies*

$$\partial_x p \in H^1(0,L).$$

*The velocity field $(u,v)$ given as a function of $p$ by* (23b)-(23c) *satisfies*

$$u \in H^1(\Omega) \cap L^\infty(\Omega) \quad \textit{and} \quad v \in L^2(\Omega), \textit{ with } \partial_z v \in L^2(\Omega).$$

*Moreover, we have the following estimates*

$$|u|_\infty \leq \bar{C}(1 + h_M^2) \qquad \textit{and} \quad |v|_2 \leq \bar{C}(1+h_M^2)\|\varphi\|_1. \tag{30}$$

Let us sketch the main steps of the proof of Proposition 3.3:

- The Reynolds equation can be solved explicitly, so that $p$ is given as a function of the coefficients $\widetilde{d}$ and $\widetilde{e}$ (given as functions of $\varphi$ by (18)): recalling definition (27) of $w$, we can integrate the Reynolds equation once and obtain

$$\widetilde{d}\,\partial_x p = s\,\widetilde{e} + \widetilde{d}(0)\,w - s\,\widetilde{e}(0). \tag{31}$$

  The coefficients $\widetilde{d}(0)$ and $\widetilde{e}(0)$ only depend on $\varphi_l$ and are thus known. If $\widetilde{d}$ does not vanish, we compute formally $\partial_x p$, and then $p$ using the boundary condition $p(L) = 0$. In order to obtain estimates on the pressure, we have to prove that the coefficients $\widetilde{d}$ and $\widetilde{e}$ are regular enough (see Lemma 4.1), and that $\widetilde{d}(\varphi)$ is greater than a strictly positive constant (i.e. the operator $\partial_x(\widetilde{d}\,\partial_x\cdot)$ must be coercive, see Lemma 4.2).

- As far as the velocity is concerned,

$$u = f\partial_x p + g,$$

  where the coefficients are given by $f = \left(b - \frac{\widetilde{b}}{\widetilde{a}}a\right)$ and $g = \left(1 - \frac{a}{\widetilde{a}}\right)s$ (and $a$, $b$, $\widetilde{a}$, $\widetilde{b}$ are defined in (17)). It is enough to prove the regularity of $f$ and $g$ in order to deduce the needed estimate on $u$ from the estimate on $\partial_x p$ (see Lemma 4.3).

- The velocity $v$ is given by

$$v(x, z) = -\int_0^z \partial_x u(x, \xi)\,d\xi,$$

  and the regularity of $v$ follows from the regularity of $u$ (see Lemma 4.4).

**Step 2.** As far as the Cahn-Hilliard equation is concerned, we proceed as in the earlier works on Cahn-Hilliard equation (e.g. [7]), and we apply the Galerkin method in order to prove the existence of a solution to the system (28), (29). This process consists in building approximate solutions $(\varphi_n, \mu_n)$ in finite dimension, for which the existence follows from the Cauchy-Lipschitz theorem. For these approximate solutions $(\varphi_n, \mu_n)$, we prove the following proposition:

**Proposition 3.4.** *For all $t \geq 0$, let*

$$\mathcal{Y}(t) = \frac{\alpha^2}{2}|\nabla\varphi_n(t)|_2^2 + \int_\Omega F(\varphi_n(t)),$$

$$\mathcal{Z}(t) = \frac{\alpha^2}{2}|\nabla\varphi_n(t)|_2^2 + |\nabla\mu_n(t)|_2^2 + |\Delta\varphi_n(t)|_2^2 + \int_\Omega F(\varphi_n(t)).$$

*Then the following estimate is satisfied:*

$$\mathcal{Y}'(t) + C_1\mathcal{Z}(t) \leq f(\mathcal{Y}(t))\mathcal{Z}(t) + C_2\mathcal{Z}(t) + C_3,$$

*where $C_1$, $C_2$, $C_3$ are three positive constants, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(0) = 0$.*

- Estimates on $|\nabla \mu_n|_2$ are first obtained from (62). This allows to gain estimates on $|\nabla \varphi_n|_2$ and $|\Delta \varphi_n|_2$ by using (63).
- Although estimates are similar are similar to the ones in [7] or [11], they involve supplementary terms due to the different boundary conditions: because of the non-homogeneous Dirichlet condition on $\varphi_n$ on the left-hand side of the domain (fluid injection), the conservation of the quantity of each fluid is not satisfied anymore (in the sense that the mean value $m(\varphi_n)$ of $\varphi_n$ is not constant with respect to time). For example, since $m(\varphi_n)$ is not constant, we cannot apply classical inequalities on $\varphi_n - m(\varphi_n)$, such as the Poincaré inequality, and we have to work with the boundary value of $\varphi_n$ on the left-hand side of the domain.
- Additionnal difficulties come from the non-periodical condition for the velocity or the fact that $\boldsymbol{u}_n \cdot \boldsymbol{n} \neq 0$ on the lateral part of the boundary. New terms have to be treated.
- In order to control the boundary terms with the ones on the left-hand side of the estimate, we have to work in adequate function spaces and choose in a suitable way the coefficients in front of each term. This requires smallness conditions on some parameters.

From Proposition 3.4, we deduce the convergence of the linear terms. However, it is not enough to conclude the convergence of the nonlinear terms and the initial condition. To this end, we obtain more regularity on $\varphi_n$ and will prove the following proposition:

**Proposition 3.5.** *There exists $C > 0$ such that for any $T > 0$:*

$$\|\varphi_n\|_{L^2(0,T;\Phi_l^3)} \leq CT + C, \qquad \left\|\frac{d\varphi_n}{dt}\right\|_{L^2(0,T;{\Phi_l^1}^*)} \leq CT + C,$$

*where ${\Phi_l^1}^*$ is the dual space of $\Phi_l^1$.*

This proposition allows us to deduce the convergence of all terms in adequate function spaces, using classical compacity results from [20].

# 4 About the Reynolds equation

## 4.1 Regularity of the coefficients

**Lemma 4.1.** *If $\varphi \in H^1(\Omega)$, the coefficients satisfy the following regularity:*

$$a,\ b,\ c\ \in X(\Omega),$$

$$\widetilde{a},\ \widetilde{b},\ \widetilde{c},\ \widetilde{d},\ \widetilde{e} \in H^1(0,L).$$

*Proof.* Let $\varphi \in H^1(\Omega)$. The terms $a$, $b$, $c$ are of the form $\displaystyle\int_0^z \xi^i/\eta(\varphi(x,\xi))\,d\xi$, for $i = 0, 1, 2$ (see definition (17) of $a$, $b$, $c$). We will present the details of the proof for the case $i = 1$. The same computations can be used to obtain the regularity results for $i = 0$, $i = 2$. Let

$$f(x,z) = \int_0^z \xi/\eta(\varphi(x,\xi))\,d\xi.$$

Let us prove that $f \in X(\Omega)$ for any $\varphi \in H^1(\Omega)$.

▷ First we prove that $f \in L^2(\Omega)$ : for any $(x,z) \in \Omega$, we have

$$f(x,z)^2 = \Big(\int_0^z \frac{\xi}{\eta(\varphi(x,\xi))}\,d\xi\Big)^2 \le \Big(\frac{1}{\eta_m}\int_0^z \xi d\xi\Big)^2 \le \frac{z^4}{4\eta_m^2}.$$

After integrating with respect to $z$ and $x$, we get

$$\int_0^L \int_0^{h(x)} f(x,z)^2 dz\,dx \le \frac{h_M^5 L}{20\eta_m^2} < \infty.$$

▷ Next, we show that $f \in H^1(\Omega)$ and $\partial_z f \in H^1(\Omega)$:

– On one hand,

$$\partial_x f(x,z) = -\int_0^z \frac{\xi\eta'(\varphi(x,\xi))}{\eta(\varphi(x,\xi))^2}\,\partial_x\varphi(x,\xi)\,d\xi,$$

with $\partial_x\varphi \in L^2(\Omega)$. Let $(x,z) \in \Omega$. Using the hypothesis (7), we compute

$$\begin{aligned}|\partial_x f(x,z)|^2 &= \left(\int_0^z \frac{\xi\eta'(\varphi(x,\xi))}{\eta(\varphi(x,\xi))}\partial_x\varphi(x,\xi)d\xi\right)^2 \\ &\le \frac{{\eta'_M}^2}{\eta_m^2}\int_0^z \xi^2 d\xi \int_0^z |\partial_x\varphi(x,\xi)|^2 d\xi \le \frac{{\eta'_M}^2 z^3}{3\eta_m^2}\int_0^{h(x)} |\partial_x\varphi(x,\xi)|^2 d\xi.\end{aligned}$$

After integrating with respect to $z$, we get

$$\int_0^{h(x)} |\partial_x f(x,y)|^2 dy \le \frac{{\eta'_M}^2 h_M^4}{12\eta_m^2}\int_0^{h(x)} |\partial_x\varphi(x,\xi)|^2 d\xi,$$

and after integrating with respect to $x$

$$|\partial_x f|_2^2 = \int_0^L \int_0^{h(x)} |\partial_x f(x,y)|^2 dy\,dx \le \frac{{\eta'_M}^2 h_M^4}{12\eta_m^2}|\partial_x\varphi|_2^2 < \infty.$$

It follows that $\partial_x f \in L^2(\Omega)$.

– On the other hand, $\partial_z f(x,z) = z/\eta(\varphi(x,z)) \in H^1(\Omega)$, since $\varphi \in H^1(\Omega)$ and $\eta \in \mathcal{C}^1(\mathbb{R})$ with $\eta > 0$.

▷ Next we show that $f \in L^\infty(\Omega)$: since $\partial_z f \in L^2(\Omega)$, we can write

$$f(x,z) = f(x,0) + \int_0^z \partial_\xi f(x,\xi)\, d\xi.$$

By definition of $f$, we know that $f(x,0) = 0$, $\forall x \in [0,L]$. Therefore, the usual trace theorem for the Sobolev space $H^{1/2}(0,h(x))$ implies that

$$\begin{aligned}|f(x,z)|^2 &\le z \int_0^z (\partial_\xi f(x,\xi))^2 d\xi \le h_M \int_0^{h(x)} (\partial_\xi f(x,\xi))^2 d\xi = h_M |\partial_z f|^2_{L^2(0,h(x))} \\ &\le h_M \|\partial_z f\|^2_{H^{1/2}(0,h(x))} \le C \|\partial_z f\|^2_{H^1(\Omega)},\end{aligned}$$

thus

$$|f|^2_\infty \le C \|\partial_z f\|_1^2 < \infty.$$

It remains to prove the regularity of $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$, $\widetilde{e}$.

▷ For the coefficients $\widetilde{a}(x) = a(x,h(x))$, $\widetilde{b}(x) = b(x,h(x))$, $\widetilde{c}(x) = c(x,h(x))$, the $L^\infty$-regularity is obviously deduced from the one of $a$, $b$, $c$. The $H^1$-regularity can be obtained using the same procedure as in the first part of the proof.

▷ For $\widetilde{d}$ and $\widetilde{e}$, the key point of the proof is to observe that $H^1(0,L)$ (which is embedded in $L^\infty(0,L)$) is an algebra:

$$(f,g) \in H^1(0,L)^2 \Rightarrow fg \in H^1(0,L).$$

Recalling the definitions $\widetilde{d} = \left(\widetilde{c} - \dfrac{\widetilde{b}^2}{\widetilde{a}}\right)$ and $\widetilde{e} = \dfrac{\widetilde{b}}{\widetilde{a}}$, and using the fact that $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ belong to $H^1(0,L)$, we need to show that $1/\widetilde{a}$ remains bounded. Since $\eta \le \eta_M$, we have

$$\widetilde{a}(x) = \int_0^{h(x)} \frac{1}{\eta(\varphi(x,\xi))} d\xi \ge \frac{h_m}{\eta_M} \quad \text{i.e.} \quad \frac{1}{\widetilde{a}} \le \frac{\eta_M}{h_m}. \tag{32}$$

From the regularity of $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, from the algebra structure and from (32), we deduce that

$$\widetilde{d} \in H^1(0,L), \quad \widetilde{e} \in H^1(0,L).$$

□

## 4.2 Coercivity of the operator

**Lemma 4.2.** *Let $\widetilde{d}$ be defined by (18). It satisfies the following estimate:*

$$\forall x \in (0,L), \quad \widetilde{d}(x) \ge \delta := \frac{h_m^3}{12\eta_M} > 0. \tag{33}$$

*Proof.* By definition (18), $\widetilde{d}(x)$ can be written in the form:

$$\widetilde{d}(x) = \widetilde{c}(x) - \frac{\widetilde{b}(x)^2}{\widetilde{a}(x)} = \int_0^{h(x)} \frac{z^2}{\eta(x,z)} dz - \frac{\left( \int_0^{h(x)} \frac{z}{\eta(x,z)} dz \right)^2}{\int_0^{h(x)} \frac{1}{\eta(x,z)} dz}.$$

In order to prove the assertion, it suffices to prove that there exists $\delta > 0$ such that

$$\left( \int_0^h \frac{z^2}{\eta} dz \right) \left( \int_0^h \frac{1}{\eta} dz \right) - \left( \int_0^h \frac{z}{\eta} dz \right)^2 \geq \delta \left( \int_0^h \frac{1}{\eta} dz \right).$$

Let us denote by $P$ the following polynomial

$$\begin{aligned} P : \lambda \mapsto & \int_0^{h(x)} \left( \frac{z}{\sqrt{\eta(\varphi(x,z))}} + \frac{\lambda}{\sqrt{\eta(\varphi(x,z))}} \right)^2 dz \\ &= \int_0^{h(x)} \frac{z^2}{\eta(\varphi(x,z))} + \frac{\lambda^2}{\eta(\varphi(x,z))} + \frac{2z\lambda}{\eta(\varphi(x,z))} dz. \end{aligned}$$

From (7), $\forall\, \lambda \in \mathbb{R}$ we get

$$P(\lambda) \geq \frac{1}{\eta_M} \int_0^{h(x)} \left( z^2 + 2z\lambda + \lambda^2 \right) dz = \frac{1}{3\eta_M} (h(x)^3 + 3h(x)^2\lambda + 3h(x)\lambda^2).$$

A simple study of the right-hand side polynomial proves that

$$\forall \lambda \in \mathbb{R},\ \forall x \in (0, L), \quad h(x)^2 + 3h(x)\lambda + 3\lambda^2 \geq \frac{h(x)^2}{4},$$

thus

$$P(\lambda) \geq \frac{h(x)^3}{12\eta_M}, \quad \text{i.e.} \quad P(\lambda) - \frac{h(x)^3}{12\eta_M} \geq 0,$$

therefore the discriminant of the polynomial

$$P(\lambda) - \frac{h(x)^3}{12\eta_M} = \lambda^2 \int_0^h \frac{1}{\eta} + 2\lambda \int_0^h \frac{z}{\eta} + \int_0^h \frac{z^2}{\eta} - \frac{h(x)^3}{12\eta_M}$$

is negative:

$$4 \left( \int_0^{h(x)} \frac{z dz}{\eta(\varphi(x,z))} \right)^2 - 4 \left( \int_0^{h(x)} \frac{dz}{\eta(\varphi(x,z))} \right) \left[ \left( \int_0^{h(x)} \frac{z^2 dz}{\eta(\varphi(x,z))} \right) - \frac{h(x)^3}{12\eta_M} \right] \leq 0,$$

that is to say

$$\left( \int_0^h \frac{z^2}{\eta} dz \right) \left( \int_0^h \frac{1}{\eta} dz \right) - \left( \int_0^h \frac{z}{\eta} dz \right)^2 \geq \frac{h_m^3}{12\eta_M} \left( \int_0^h \frac{1}{\eta} dz \right), \quad \text{i.e.} \quad \widetilde{d} \geq \frac{h_m^3}{12\eta_M} > 0.$$

□

The two previous lemmas 4.1 (regularity of the coefficients) and 4.2 (coercivity of the operator) with formula (31) imply that $\partial_x p \in H^1(0, L)$, thus $p \in H^2(0, L)$.

### 4.3 Estimates of $|u|_\infty$ and $|v|_2$

**Lemma 4.3.** *Let $\varphi \in H^1(\Omega)$. The horizontal velocity $u$ given by (23b) satisfies*

$$|u|_\infty \le \bar{C}(1 + h_M^2),$$

*where $\bar{C}$ denotes a constant depending on $\Omega$ only through the ratio $\sigma = h_M/h_m$.*

*Proof.* The regularity of $u$ follows from the regularity of $p$, equation (23b) and the regularity of the coefficients (Lemma 4.1):

$$u = (b - \frac{a\widetilde{b}}{\widetilde{a}})\partial_x p + s(1 - \frac{a}{\widetilde{a}}) \in X(\Omega)$$

Moreover, we know that $u$ is a combination of coefficients of the form $\int_0^z \xi/\eta(\varphi)d\xi$. Indeed

$$|u|_\infty \le \left( |b|_\infty + \frac{|a|_\infty |\widetilde{b}|_\infty}{\min\limits_{x\in(0,L)} \widetilde{a}(x)} \right) |\partial_x p|_\infty + s\left( 1 + \frac{|a|_\infty}{\min\limits_{x\in(0,L)} \widetilde{a}(x)} \right), \tag{34}$$

and $\partial_x p$ is given by (31), thus:

$$|\partial_x p|_\infty \le \frac{1}{\min\limits_{x\in(0,L)} \widetilde{d}(x)} \left( s|e|_\infty + |\widetilde{d}_l|_\infty |w| + s|\widetilde{e}_l|_\infty \right). \tag{35}$$

Let us obtain estimates for these coefficients.

- ▷ Using the boundedness hypothesis on $\eta$, and applying the Cauchy-Schwarz inequality and the fact that $\forall x \in (0, L)$, $h(x) \le h_M$, we can write for all $(x, z) \in \Omega$

$$a(x, z) = \int_0^z \frac{d\xi}{\eta(\varphi(x, \xi))} \le \frac{h_M}{\eta_m}, \quad \text{thus} \quad |a|_\infty \le \bar{C}h_M, \quad |\widetilde{a}|_\infty \le \bar{C}h_M. \tag{36}$$

- ▷ Similar computations for $b$, $c$ and $\widetilde{b}$, $\widetilde{c}$ give

$$|b|_\infty, |\widetilde{b}|_\infty \le \bar{C}h_M^2, \quad |c|_\infty, |\widetilde{c}|_\infty \le \bar{C}h_M^3. \tag{37}$$

- ▷ Recalling definition (18) of $\widetilde{e}$, and using (32), it follows from (37):

$$|\widetilde{e}|_\infty = \frac{|b|_\infty}{\min\limits_{x\in(0,L)} \widetilde{a}(x)} \le \frac{\bar{C}h_M^2}{h_m} \le \bar{C}\sigma h_M = \bar{C}h_M. \tag{38}$$

We recall that we denote by $\bar{C}$ any constant independent of $\Omega$ or depending on $\Omega$ only through the rate $\sigma = \dfrac{h_M}{h_m}$.

$\triangleright$ Moreover, the same computations as for estimates (36), (37) lead to

$$|\widetilde{a}_l|_\infty \leq \bar{C}h_M, \quad |\widetilde{b}_l|_\infty \leq \bar{C}h_M^2, \quad |\widetilde{c}_l|_\infty \leq \bar{C}h_M^3.$$

We get (since $h_M \geq h_m$)

$$|\widetilde{d}_l|_\infty \leq |\widetilde{c}_l|_\infty + |\widetilde{b}_l|_\infty^2 \frac{h_m}{\eta_M} \leq \bar{C}h_M^3, \qquad |\widetilde{e}_l|_\infty \leq |\widetilde{b}_l|_\infty \frac{h_m}{\eta_M} \leq \bar{C}h_M. \tag{39}$$

Thus, using (33), (38), (39) in (35), it follows

$$|\partial_x p|_\infty \leq \bar{C}(1 + \frac{1}{h_m^2}). \tag{40}$$

Now, using (36), (37), (32) and (40) in (34), we obtain the required estimate:

$$|u|_\infty \leq \bar{C}h_M^2 \left(1 + \frac{1}{h_m^2}\right) \leq \bar{C}(1 + h_M^2), \tag{41}$$

which ends the proof. □

**Lemma 4.4.** *Let $\varphi \in H^1(\Omega)$. The vertical velocity $v$ given by* (23c) *satisfies*

$$|v|_2 \leq \bar{C}(1 + h_M^2)\|\varphi\|_1,$$

*where $\bar{C}$ denotes a constant depending on $\Omega$ only through the ratio $\sigma = h_M/h_m$.*

*Proof.* The regularity of $v$ follows from the regularity of $u$, equation (23c) and the regularity of the coefficients (Lemma 4.1):

$$v(x,z) = -\int_0^z \partial_x u(x,\xi)d\xi.$$

From the Cauchy-Schwarz inequality, we deduce that

$$|v|_2 \leq h_M|\partial_x u|_2. \tag{42}$$

Let us introduce the coefficients $f = b - \dfrac{a\widetilde{b}}{\widetilde{a}}$ and $g = 1 - \dfrac{a}{\widetilde{a}}$, so that $u = f\partial_x p + sg$. Therefore

$$|\partial_x u|_2 \leq |\partial_x f|_2 |\partial_x p|_\infty + |f|_\infty |\partial_x^2 p|_2 + s|\partial_x g|_2, \tag{43}$$

and $\partial_x^2 p$ is given by taking the derivative of (31) with respect to $x$:

$$|\partial_x^2 p|_2 \leq \frac{1}{\min\limits_{x\in(0,L)} \widetilde{d}(x)} \left(s|\partial_x \widetilde{e}|_2 + |\partial_x \widetilde{d}|_2 |\partial_x p|_\infty\right). \tag{44}$$

Let us obtain estimates for each coefficient separately:

▷ We have

$$|f|_\infty \le |b|_\infty + \frac{\bar{C}}{h_m}|a|_\infty|\widetilde{b}|_\infty. \tag{45}$$

▷ It remains to obtain estimates of the derivatives of the coefficients with respect to $x$. We can compute $\partial_x a = \displaystyle\int_0^y \frac{\eta'(\varphi)}{\eta(\varphi)^2}\partial_x\varphi$, and the Cauchy-Schwarz inequality yields

$$\begin{aligned}|\partial_x a|_2^2 &\le \frac{{\eta'_M}^2}{\eta_m^4}\int_\Omega\left(\int_0^y \partial_x\varphi(x,z)\,dz\right)^2 \\ &\le \bar{C}h_M\int_\Omega\int_0^y|\partial_x\varphi|^2 \le \bar{C}h_M^2|\partial_x\varphi|_2^2 \le \bar{C}h_M^2\|\varphi\|_1^2,\end{aligned}$$

and similar estimates for all the other coefficients:

$$\begin{gathered}|\partial_x a|_2, |\partial_x\widetilde{a}|_2 \le \bar{C}h_M\|\varphi\|_1, \qquad |\partial_x b|_2, |\partial_x\widetilde{b}|_2 \le \bar{C}h_M^2\|\varphi\|_1, \\ |\partial_x c|_2, |\partial_x\widetilde{c}|_2 \le \bar{C}h_M^3\|\varphi\|_1.\end{gathered} \tag{46}$$

▷ Let us write

$$\partial_x\left(\frac{a}{\widetilde{a}}\right) = \frac{\partial_x a\,\widetilde{a} - a\,\partial_x\widetilde{a}}{\widetilde{a}^2}.$$

From (32), we know that $\widetilde{a} \ge \dfrac{h_m}{\eta_M}$. This estimate combined with (46) suffices to prove that

$$\left|\partial_x\left(\frac{a}{\widetilde{a}}\right)\right|_2 \le \bar{C}\|\varphi\|_1, \tag{47}$$

and

$$\left|\partial_x\left(\frac{\widetilde{b}}{\widetilde{a}}\right)\right|_2 \le \bar{C}h_M\|\varphi\|_1. \tag{48}$$

▷ Since

$$\begin{aligned}\partial_x d &= \partial_x c - \partial_x\widetilde{b}\frac{\widetilde{b}}{\widetilde{a}} - \widetilde{b}\partial_x\left(\frac{\widetilde{b}}{\widetilde{a}}\right), \qquad \partial_x e = \partial_x\left(\frac{\widetilde{b}}{\widetilde{a}}\right), \\ \partial_x f &= \partial_x b - \partial_x a\frac{\widetilde{b}}{\widetilde{a}} - a\partial_x\left(\frac{\widetilde{b}}{\widetilde{a}}\right), \qquad \partial_x g = \partial_x\left(\frac{a}{\widetilde{a}}\right),\end{aligned} \tag{49}$$

it follows, using (46), (47), (48) in (49), that

$$\begin{aligned}|\partial_x\widetilde{d}|_2 &\le \bar{C}h_M^3\|\varphi\|_1, \qquad |\partial_x\widetilde{e}|_2 \le \bar{C}h_M\|\varphi\|_1, \\ |\partial_x f|_2 &\le \bar{C}h_M^2\|\varphi\|_1, \qquad |\partial_x g|_2 \le \bar{C}\|\varphi\|_1.\end{aligned} \tag{50}$$

Putting (33), (50), (40) in (44) and (43), we deduce an estimate for each of the three terms in (43):

▷ The first term is estimated by:

$$|\partial_x f|_2 |\partial_x p|_\infty \leq \bar{C} h_M^2 \|\varphi\|_1 \Big(1 + \frac{1}{h_m^2}\Big) \leq \bar{C}(1 + h_M^2)\|\varphi\|_1.$$

▷ For the second term, we have:

$$\begin{aligned}\frac{|f|_\infty}{\delta}\Big(s|\partial_x \widetilde{e}|_2 + |\partial_x \widetilde{d}|_2 |\partial_x p|_\infty\Big) &\leq \frac{1}{h_m^3}\, h_M^2 \left( h_M \|\varphi\|_1 + h_M^3 \|\varphi\|_1 \Big(1 + \frac{1}{h_m^2}\Big)\right) \\ &\leq \bar{C}(1 + h_M^2)\|\varphi\|_1.\end{aligned}$$

▷ The third term follows directly from (50):

$$|\partial_x g|_2 \leq \bar{C}\|\varphi\|_1.$$

Therefore, using (42) and these three estimates for $|\partial_x u|_2$, we obtain:

$$|v|_2 \leq h_M |\partial_x u|_2 \leq \bar{C}(1 + h_M^2)\|\varphi\|_1,$$

which proves the lemma. □

**Remark 4.5.** *Observe that it is not straightforward to prove that $v \in L^\infty(\Omega)$ if $\varphi$ only lies in $H^1(\Omega)$. Computing $|v|_\infty$, it is bounded by $|\partial_x u|_\infty$, and thus by $|\partial_x f|_\infty$ for example, i.e. by $|\partial_x a|_\infty$. But writing $|\partial_x a|_\infty \leq C|\partial_x \varphi|_\infty$, the regularity of $\varphi$ does not allow to conclude.*

**Remark 4.6.** *Since* (23a)-(23b)-(23c) *are steady-state equations, the constants in the previous estimates are also independent of the time, so that the $L^\infty(\Omega)$ and $L^2(\Omega)$-estimates of Lemma 4.3 and 4.4 can also be written in $L^\infty(0,\infty; L^\infty(\Omega))$ and $L^\infty(0,\infty; L^2(\Omega))$.*

# 5 About the Cahn-Hilliard equation

## 5.1 Useful inequalities

### 5.1.1 Boundary conditions and lift operator

In order to treat the boundary terms, it is a classical approach for the velocity $\boldsymbol{u}$ to introduce a lift operator of the boundary values by means of a divergence-free function.

**Lemma 5.1.** *Let $(s,q) \in \mathbb{R}^2$. There exists a vector field on $\bar{\Omega}$, denoted by $\boldsymbol{g} = (g_1, g_2)$, satisfying the following conditions:*

*i)* $\boldsymbol{g} \in H^1(\Omega)^2$,

*ii)* $\operatorname{div} \boldsymbol{g} = 0$ *in* $\Omega$,

*iii)* $\boldsymbol{g}$ *satisfies the following conditions:*

$$\boldsymbol{g}(x,0) = (s,0), \quad \boldsymbol{g}(x,h(x)) = (0,0), \quad \int_0^{h(0)} \boldsymbol{g}|_{x=0} \cdot \boldsymbol{n} = \int_0^{h(L)} \boldsymbol{g}|_{x=L} \cdot \boldsymbol{n} = q. \quad (51)$$

*Proof.* Given $s$, $q$, it is possible to build a function $\tilde{g} \in H^{1/2}(\Gamma)$ such that

$$\tilde{g}(x,0) = (s,0), \quad \tilde{g}(x,h(x)) = (0,0), \quad \int_0^{h(0)} \tilde{g}|_{x=0} \cdot \boldsymbol{n} = \int_0^{h(L)} \boldsymbol{g}|_{x=L} \cdot \boldsymbol{n} = q.$$

Next, since $\displaystyle\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n} = 0$, there exists a lifting $\boldsymbol{g} \in H^1(\Omega)^2$ satisfying the required properties by a classical result [14]. □

**Hypothesis 5.2.** *As far as the boundary value $\varphi_l$ of $\varphi$ is concerned, we assume that it is constant on $\Gamma_l$, with $\varphi_l \in \{0, -1, 1\}$. The regularity $\varphi_l \in H^{5/2}(\Gamma_l)$ follows immediately, and it is easy to define a lifting $\hat{\varphi}_l \in \Phi_l^2$ of the boundary value $\varphi_l$ for all $(x,z) \in \Omega$ by $\hat{\varphi}_l(x,z) = \varphi_l$.*

This assumption corresponds to assuming a pure phase injection ($\varphi_l = \pm 1$) or a homogeneous mixture injection ($\varphi_l = 0$). It is necessary in Section 5.2 to define the Galerkin approximation as in (63). Indeed, if $F'(\varphi_l) \neq 0$, then $F'(\varphi_n) \notin \Psi_n$, thus $\mathbb{P}_{\Psi_n}$ does not converge towards the identity when $n$ goes to infinity. However, let us emphasize that the propositions stated in this section 5.1 are valid for any $\varphi_l \in H^{5/2}(\Gamma_l)$.

### 5.1.2 Sobolev embeddings

Let us recall how the constants in the usual Sobolev embeddings depend on the domain. The results of this section follow from [2, Cor. 5.13] and [2, Lem. 5.15], since the domain $\Omega$ defined by (1) satisfies the segment and the cone property.

**Proposition 5.3.** *Let $\Omega \subset \mathbb{R}^2$ be defined by (1). Then $H^1(\Omega) \hookrightarrow L^q(\Omega)$, for any $2 \leq q < +\infty$. Moreover, the embedding constant can be specified:*

$$\forall f \in H^1(\Omega), \qquad |f|_q \leq \bar{C} |\Omega|^{1/q} \|f\|_1, \tag{52}$$

*where $\bar{C}$ only depends on $q$.*

**Proposition 5.4.** *Let $\Omega \subset \mathbb{R}^2$ be defined by (1). Then $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Moreover let $R = \min(h_m, L)$. Then*

$$\forall f \in H^2(\Omega), \qquad |f|_\infty \leq \bar{C}(R^{-2/3}|\Omega|^{5/6} + R^{1/3}|\Omega|^{1/3}) \|f\|_2. \tag{53}$$

*Let us denote $C_\infty := \bar{C}(R^{-2/3}|\Omega|^{5/6} + R^{1/3}|\Omega|^{1/3})$. Let us observe that $C_\infty$ remains bounded as $|\Omega| \to 0$.*

### 5.1.3 Equivalence of norms

**Proposition 5.5.** *Let $\varphi_l \in H^{5/2}(\Gamma_l)$, and let $\varphi \in \Phi_l^2$. We have*

$$\|\varphi\|_2 \leq \bar{C} |\Delta \varphi|_2 + |\hat{\varphi}_l|_2. \tag{54}$$

This result is proved in [5]. Moreover, we can combine this result with Proposition 5.4:

**Corollary 5.6.** *Let $\varphi_l \in H^{5/2}(\Gamma_l)$, and let $\varphi \in \Phi_l^2$. Let $R = \min(h_m, L)$. The following inequality applies:*

$$|\varphi|_\infty \leq \bar{C}(R^{-2/3}|\Omega|^{5/6} + R^{1/3}|\Omega|^{1/3}) \left(|\Delta \varphi|_2 + |\hat{\varphi}_l|_2\right). \tag{55}$$

### 5.1.4 Anisotropic trace estimates

**Proposition 5.7.** *If $f \in H^1(\Omega)$ and if $\bar{x} \in (0, L)$, then*

$$|f(\bar{x}, \cdot)|^2_{L^2(0,h(\bar{x}))} \leq \bar{C}\left(L|\partial_x f|_2^2 + \left(\frac{1}{L} + Lh'_M\right)|f|_2^2\right).$$

*Proof.* We state the proof for $\bar{x} = 0$, the case $\bar{x} \neq 0$ being easily adapted. Introduce the auxiliary function $\xi(x) = \frac{1}{2}(x - L)^2$. This function satisfies for all $x \in \mathbb{R}$, $\xi''(x) = 1$. Integration by parts gives:

$$\int_0^L \int_0^{h(x)} f^2 = \int_0^L \int_0^{h(x)} f^2 \xi'' = \left[\int_0^{h(x)} \xi' f^2\right]_{x=0}^{x=L} - \int_0^L \int_0^{h(x)} 2f\partial_x f \xi' - \int_0^L h'(x) \int_0^{h(x)} \xi' f^2.$$

Since $\xi'(L) = 0$, and $\xi'(0) = -L$, we get

$$L \int_0^{h(0)} f^2|_{x=0} = |f|_2^2 + \int_0^L \int_0^{h(x)} 2f\partial_x f \xi' + \int_0^L h'(x) \int_0^{h(x)} \xi' f^2.$$

Moreover $|\xi'(x)| \leq L$ for $x \in [0, L]$, $|h'|_\infty \leq h'_M$, and the Cauchy-Schwarz inequality and Young's inequality imply

$$|f|^2_{L^2(\Gamma_l)} \leq \frac{1}{L}|f|_2^2 + 2L|f|_2|\partial_x f|_2 + Lh'_M|f|_2^2 \leq \bar{C}\left(\left(\frac{1}{L} + Lh'_M\right)|f|_2^2 + L|\partial_x f|_2^2\right).$$

□

**Remark 5.8.** *We can apply the previous result to $\varphi$ and $\mu$, leading to the following estimates for $\varphi \in \Phi_l^1$, $\mu \in \Phi^1$:*

$$\begin{aligned}
|\varphi|^2_{L^2(\Gamma_l)}, |\varphi|^2_{L^2(0,h(L))} &\leq \bar{C}\left(L|\partial_x \varphi|_2^2 + \left(\frac{1}{L} + Lh'_M\right)|\varphi|_2^2\right), \\
|\mu|^2_{L^2(\Gamma_l)}, |\mu|^2_{L^2(0,h(L))} &\leq \bar{C}\left(L|\partial_x \mu|_2^2 + \left(\frac{1}{L} + Lh'_M\right)|\mu|_2^2\right).
\end{aligned} \tag{56}$$

*For $\varphi \in \Phi_l^2$, we can also apply this proposition to $\partial_x\varphi$. Since $(\partial_x\varphi)|_{(0,h(L))} = 0$, we can apply the Poincaré inequality: $|\partial_x\varphi|_2^2 \leq L^2|\partial_x^2\varphi|_2^2$. Thus,*

$$|\partial_x\varphi|^2_{L^2(\Gamma_l)} \leq \bar{C}L(1 + L^2h'_M)|\partial_x^2\varphi|_2^2. \tag{57}$$

### 5.1.5 Specific Poincaré inequalities

The Poincaré inequalities stated in this section are specific to the functions $\varphi$ and $\mu$ satisfying the boundary conditions (24b) and relation (23e). First, observe that because of Hypothesis 5.2, we have

$$|\varphi_l|_{L^2(\Gamma_l)} \leq \|\hat{\varphi}_l\|_{1/2} \leq \|\hat{\varphi}_l\|_1 = |\hat{\varphi}_l|_2.$$

**Proposition 5.9** (Poincaré inequality for $\varphi$)**.** *Let $\varphi \in \Phi_l^1$. Let $L_h^2 = L^2(1 + h_M^2 + {h'_M}^2)$. We have*

$$|\varphi|_2^2 \leq \bar{C}\left(L^2(1 + h_M^2 + {h'_M}^2)|\nabla\varphi|_2^2 + L|\hat{\varphi}_l|_2^2\right) = \bar{C}\left(L_h^2|\nabla\varphi|_2^2 + L|\hat{\varphi}_l|_2^2\right). \tag{58}$$

*Proof.* This is a consequence of the usual Poincaré inequality with $\varphi|_{x=0} = \varphi_l$ (see for example [21, § II.1.4]). Let $(x, \tilde{z}) \in (0, L) \times (0, 1)$, and define $\tilde{\varphi}(x, \tilde{z})$ such that $\tilde{\varphi}(x, \tilde{z}) = \varphi(x, z)$, with $z = h(x)\tilde{z}$. Poincaré inequality for $\tilde{\varphi}$ leads to

$$|\tilde{\varphi}|_2^2 \leq \bar{C}\left(L^2|\partial_x\tilde{\varphi}|_2^2 + L|\varphi_l|_{L^2(\Gamma_l)}^2\right) \leq \bar{C}\left(L^2|\partial_x\tilde{\varphi}|_2^2 + L|\hat{\varphi}_l|_2^2\right).$$

Since $\partial_x\tilde{\varphi} = \partial_x\varphi + z\dfrac{h'}{h}\partial_z\varphi$ and $\partial_{\tilde{z}}\tilde{\varphi} = h\partial_z\varphi$, we deduce from the fact that $z/h(x) \leq 1$ that

$$|\varphi|_2^2 \leq \bar{C}\left(L^2\left(|\partial_x\varphi|_2^2 + h_M^2|\partial_z\varphi|_2^2 + {h'_M}^2|\partial_z\varphi|_2^2\right) + L|\hat{\varphi}_l|_2^2\right), \tag{59}$$

which proves the inequality claimed. □

**Proposition 5.10** (Poincaré inequality for $\mu$)**.** *We have*

$$|\mu|_2^2 \leq \bar{C}L_h^2|\nabla\mu|_2^2. \tag{60}$$

## 5.2 Galerkin approximations

Let us build the Galerkin approximations of $\varphi$ and $\mu$. Since $\Phi^1$ is a separable Hilbert space, there exists an Hilbertian basis $(\psi_i)_{i\geq 1}$ of $\Phi^1$. The functions $\psi_i$ can be chosen to be eigenfunctions of the Laplacian $-\Delta$ with the boundary conditions (24b), and we denote by $\lambda_i$ the corresponding eigenvalues. We define $\Psi_n = \mathrm{Span}(\psi_1, \cdots, \psi_n)$, and $\mathbb{P}_{\Psi_n}$ the orthogonal projector on $\Psi_n$ in $L^2(\Omega)$. As a projector, $\mathbb{P}_{\Psi_n}$ satisfies:

$$(\mathbb{P}_{\Psi_n}f, g) = (f, \mathbb{P}_{\Psi_n}g), \qquad \forall (f, g) \in L^2(\Omega)^2, \tag{61}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.
Recalling that $\hat{\varphi}_l \in \Phi_l^2$ satisfies the boundary conditions (24b), we consider the following approximation of $\varphi$:

$$\varphi_n(t) = \sum_{i=1}^{n}\beta_i(t)\psi_i + \hat{\varphi}_l,$$

where $\beta_i$ are unknown functions to be determined. The problem (28)-(29) becomes, after integrating by parts:

**Problem 5.11.** *Find $(\varphi_n, \mu_n)$ such that*

$$\int_\Omega \partial_t\varphi_n\psi + \int_\Omega \frac{1}{\mathcal{P}e}\mathcal{B}(\varphi_n)\nabla\mu_n\,\nabla\psi - \int_\Gamma \mathcal{B}(\varphi_n)\nabla\mu_n\cdot\boldsymbol{n}\,\psi + \int_\Omega \boldsymbol{u}(\varphi_n)\cdot\nabla\varphi_n\psi = 0, \qquad \forall\psi \in \Phi^1, \tag{62}$$

$$\mu_n = -\alpha^2\Delta\varphi_n + \mathbb{P}_{\Psi_n}F'(\varphi_n), \tag{63}$$

*with the boundary conditions*

$$\mu_n|_{\Gamma_l} = 0, \quad \varphi_n|_{\Gamma_l} = \varphi_l, \quad \nabla\mu_n \cdot \boldsymbol{n}|_{\Gamma_0} = \nabla\varphi_n \cdot \boldsymbol{n}|_{\Gamma_0} = 0, \tag{64}$$

*and where $\boldsymbol{u}(\varphi_n)$ is defined as a function of $\varphi_n$ by the formulas* (19)-(20) *and* (22).

**Remark 5.12.** *Let us explain why the boundary term $\int_\Gamma B(\varphi_n)\nabla\mu_n \cdot \boldsymbol{n}\,\psi$ is zero:*

- ▷ *On $\Gamma_0$, we can compute $\nabla\mu_n \cdot \boldsymbol{n}|_{\Gamma_0}$, since the functions $\psi_i$ are eigenfunctions of $-\Delta$ and $\nabla\hat{\varphi}_l \cdot \boldsymbol{n}|_{\Gamma_0} = 0$:*

$$\begin{aligned}\nabla\mu_n \cdot \boldsymbol{n}|_{\Gamma_0} &= -\alpha^2\nabla\Delta\varphi_n \cdot \boldsymbol{n}|_{\Gamma_0} + \underbrace{\nabla\mathbb{P}_{\Psi_n}F'(\varphi_n) \cdot \boldsymbol{n}|_{\Gamma_0}}_{=0,\ \textit{since}\ \mathbb{P}_{\Psi_n}F'(\varphi_n)\in\Psi_n} \\ &= -\alpha^2\nabla\left(\sum_{i=1}^n \beta_i\lambda_i\psi_i\right)\cdot\boldsymbol{n}|_{\Gamma_0}\end{aligned}$$

  *Since $\psi_i \in \Psi_n$ for any $i \leq n$, we have $\nabla\psi_i \cdot \boldsymbol{n}|_{\Gamma_0} = 0$, and thus $\nabla\mu_n \cdot \boldsymbol{n}|_{\Gamma_0} = 0$.*

- ▷ *On $\Gamma_l$, the boundary term is also equal to zero, since $\psi \in \Phi^1$, and thus vanishes on $\Gamma_l$.*

*Observe that the weak formulation* (62)-(63) *is well-defined since $\psi_i \in H^1(\Omega)$ implies that $\mu_n \in H^1(\Omega)$. Indeed, the functions $\psi_i$ are eigenfunctions of $-\Delta$, so that the regularity follows from definition* (63).

**Lemma 5.13.** *For $n \in \mathbb{N}$, there exists $(\beta_i)_{1\leq i\leq n} \in \mathcal{C}^1(0, t_n)$ so that $\varphi_n(t) = \sum_{i=1}^n \beta_i(t)\psi_i + \hat{\varphi}_l$ is a solution of Problem 5.11.*

*Proof.* Replacing $\varphi_n$ by its expression as a function of $\beta_i$, the system (62)-(63) becomes:

$$\begin{aligned}&\sum_{i=1}^n \beta_i'(t)\int_\Omega \psi_i\,\psi + \int_\Omega \frac{1}{\mathcal{P}e}\mathcal{B}\left(\sum_{i=1}^n \beta_i(t)\psi_i + \hat{\varphi}_l\right)\nabla\mu_n\,\nabla\psi \\ &\qquad + \sum_{i=1}^n \beta_i(t)\int_\Omega \boldsymbol{u}\left(\sum_{i=1}^n \beta_i(t)\psi_i + \hat{\varphi}_l\right)\cdot\nabla\psi_i\psi = 0, \qquad \forall\psi \in \Phi^1, \\ &\mu_n = -\alpha^2\sum_{i=1}^n \beta_i(t)\lambda_i\psi_i + \mathbb{P}_{\Psi_n}F'\left(\sum_{i=1}^n \beta_i(t)\psi_i + \hat{\varphi}_l\right).\end{aligned}$$

This formulation is an ordinary differential equation on $(\beta_i)_{1\leq i\leq n}$. The functions $\mathcal{B}$ and $F'$ are $\mathcal{C}^1$ on $\mathbb{R}$. Moreover, the function $\boldsymbol{u}$ as a function of $\varphi_n$ given by (23a)-(23b)-(23c) is also $\mathcal{C}^1$ on $\mathbb{R}$ (with respect to time): indeed, $u(\varphi_n)$ is given as a combination of coefficients of the form $\int_0^z \xi/\eta(\varphi_n(x,\xi))d\xi$, and the function $\eta$ is $\mathcal{C}^1$ by assumption (7). The second component of the velocity $v$ is given as a function of $u$, and is also $\mathcal{C}^1$ on $\mathbb{R}$. Therefore, the Cauchy-Lipschitz theorem ensures the existence of a unique solution $(\beta_i)_{1\leq i\leq n}$ on a time interval $[0, t_n)$. □

## 5.3 Estimates on $\varphi$

The proof of the main theorem consists in showing that $t_n = +\infty$ for any $n \geq 1$, and that $\varphi_n$ converges in appropriate function spaces. In the sequel, we drop the subscripts $_n$ for readability, and we write $\varphi$, $\mu$ instead of $\varphi_n$, $\mu_n$.

**Lemma 5.14.** *For $\varphi$ and $\mu$ solutions of (62)-(63) with boundary conditions (64), the following inequality applies:*

$$\begin{aligned}\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int_\Omega F(\varphi)\right) + \Big(\frac{3\mathcal{B}_m}{4\mathcal{P}e} - L_0\Big)|\nabla\mu|_2^2 \\ \leq L_1(\boldsymbol{u})|\Delta\varphi|_2^2 + L_3(\boldsymbol{u})|\nabla\varphi|_2^2 + L_4(\boldsymbol{u})|\hat{\varphi}_l|_2^2,\end{aligned} \tag{65}$$

*where for any $\beta > 0$, the terms $L_i$ for $0 \leq i \leq 4$ are given by*

$$L_0 = \frac{\bar{C}}{\beta}\big(\frac{L_h^2}{L} + LL_h^2 h'_M + L\big),$$

$$L_1(\boldsymbol{u}) = \bar{C}\big(\frac{\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m} + \beta L^3(1 + L^2 h'_M)(1 + h_M^2 + {h'_M}^2)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}\big),$$

$$L_3(\boldsymbol{u}) = \frac{\bar{C}\mathcal{P}eL_h^2|u|_\infty^2}{\mathcal{B}_m}$$

$$L_4(\boldsymbol{u}) = \bar{C}\Big(\frac{\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m} + \frac{\mathcal{P}eL|u|_\infty^2}{\mathcal{B}_m} + \beta L(1 + L^2 h'_M)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}\Big).$$

*Proof.* Let us take $\psi = \mu \in \Phi^1$ in the weak formulation (62). Using definition (63) for $\mu$, we get

$$\underbrace{\int_\Omega \partial_t\varphi(-\alpha^2\Delta\varphi + \mathbb{P}_{\Psi_n}F'(\varphi))}_{=:A} + \underbrace{\frac{1}{\mathcal{P}e}\int_\Omega \mathcal{B}(\varphi)|\nabla\mu|^2}_{=:B} = \underbrace{-\int_\Omega \boldsymbol{u}\cdot\nabla\varphi\mu}_{=:D}. \tag{66}$$

Let us obtain estimates for each term $A$, $B$, $D$:

▷ The $A$-term is composed of two parts:

$$A = \underbrace{-\alpha^2\int_\Omega \partial_t\varphi\Delta\varphi}_{=:A_1} + \underbrace{\int_\Omega \partial_t\,\varphi\mathbb{P}_{\Psi_n}F'(\varphi))}_{=:A_2}.$$

⋆ For $A_1$, we use integration by parts:

$$A_1 = -\alpha^2\int_\Omega \partial_t\varphi\Delta\varphi = \frac{\alpha^2}{2}\frac{d}{dt}|\nabla\varphi|_2^2 - \alpha^2\int_\Gamma \partial_t\varphi\,\nabla\varphi\cdot\boldsymbol{n}$$

The boundary condition $\nabla\psi_i\cdot\boldsymbol{n}|_{\Gamma_0} = 0$, and the fact that $\varphi_l$ is independent of $t$ allow us to treat the boundary term:

$$-\alpha^2\int_\Gamma \underbrace{\partial_t\varphi}_{=0\text{ on }\Gamma_l}\ \underbrace{\nabla\varphi\cdot\boldsymbol{n}}_{=0\text{ on }\Gamma_0} = 0,$$

thus

$$A_1 = \frac{\alpha^2}{2}\frac{d}{dt}|\nabla\varphi|_2^2. \tag{67}$$

⋆ For the second term $A_2$, we use property (61) and the time-independency of $\hat{\varphi}_l$:

$$A_2 = (\partial_t\varphi, \mathbb{P}_{\Psi_n}F'(\varphi)) = (\mathbb{P}_{\Psi_n}\partial_t\,\varphi, F'(\varphi)) = (\partial_t\,\varphi, F'(\varphi)).$$

Thus, $\psi_i \in \Psi_n$ yields

$$\mathbb{P}_{\Psi_n}\partial_t\,\varphi = \mathbb{P}_{\Psi_n}\left(\sum_{i=1}^{n}\beta_i'(t)\psi_i\right) = \sum_{i=1}^{n}\beta_i'(t)\psi_i = \partial_t\,\varphi.$$

Thus, $A_2$ can be expressed as a time derivative:

$$A_2 = \int_\Omega \partial_t\,\varphi F'(\varphi) = \frac{d}{dt}\int_\Omega F(\varphi). \tag{68}$$

▷ The $B$-term is trivially estimated using that $\mathcal{B}(\varphi) \geq \mathcal{B}_m$ (from (15)):

$$B = \frac{1}{\mathcal{P}e}\int_\Omega \mathcal{B}(\varphi)|\nabla\mu|^2 \geq \frac{\mathcal{B}_m}{\mathcal{P}e}|\nabla\mu|_2^2. \tag{69}$$

▷ For the $D$-term, after integrating by parts, we use the fact that $\operatorname{div}\boldsymbol{u} = 0$ and that $\boldsymbol{u}|_\Gamma = \boldsymbol{g}|_\Gamma$ (where $\boldsymbol{g}$ is a lifting of the boundary conditions on $\boldsymbol{u}$ defined by Lemma 5.1):

$$D = -\int_\Omega \boldsymbol{u}\cdot\nabla\varphi\,\mu = \underbrace{\int_\Omega \varphi\,u\,\partial_x\mu}_{=:D_1} + \underbrace{\int_\Omega \varphi\,v\,\partial_z\mu}_{=:D_2} \underbrace{- \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}\,\varphi\,\mu}_{=:D_3}.$$

We observe that $D_1$ and $D_2$ must be handled separately, since $v \notin L^\infty(\Omega)$.

⋆ By Young's inequality, we have for $D_1$:

$$D_1 = \int_\Omega \varphi\,u\,\partial_x\mu \leq |\varphi|_2|u|_\infty|\partial_x\mu|_2 \leq \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_x\mu|_2^2 + \frac{\mathcal{P}e}{\mathcal{B}_m}|u|_\infty^2|\varphi|_2^2.$$

Using the Poincaré inequality (58) for $|\varphi|_2$, we conclude

$$D_1 \leq \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_x\mu|_2^2 + \frac{\bar{C}\mathcal{P}eL_h^2}{\mathcal{B}_m}|u|_\infty^2|\nabla\varphi|_2^2 + +\frac{\bar{C}\mathcal{P}eL}{\mathcal{B}_m}|u|_\infty^2|\hat{\varphi}_l|_2^2. \tag{70}$$

⋆ For $D_2$, we get

$$D_2 = \int_\Omega \varphi\,v\,\partial_z\mu \leq |\varphi|_\infty|v|_2|\partial_z\mu|_2 \leq \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_z\mu|_2^2 + \frac{\mathcal{P}e}{\mathcal{B}_m}|v|_2^2|\varphi|_\infty^2.$$

We recall that by (55), $|\varphi|_\infty^2 \leq C_\infty^2(|\Delta\varphi|_2^2 + |\hat{\varphi}_l|_2^2)$, so that we obtain

$$D_2 \leq \frac{\mathcal{B}_m}{4\mathcal{P}e}|\partial_z\mu|_2^2 + \frac{C_\infty^2\mathcal{P}e}{\mathcal{B}_m}|v|_2^2|\Delta\varphi|_2^2 + \frac{C_\infty^2\mathcal{P}e}{\mathcal{B}_m}|v|_2^2|\hat{\varphi}_l|_2^2. \tag{71}$$

⋆ For the boundary term $D_3$, we make use of the boundary conditions on $\boldsymbol{g}$ (51):

$$D_3 = \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n}\, \varphi\, \mu = \int_{\Gamma_{\rm lat}} g_1\, \varphi\, \mu.$$

We apply Young's inequality (with $\beta > 0$), and combine it with the trace estimate (56) for $|\mu|_{L^2(\Gamma_{\rm lat})}$ and $|\varphi|_{L^2(\Gamma_{\rm lat})}$:

$$\begin{aligned} D_3 &\leq \frac{1}{4\beta}|\mu|^2_{L^2(\Gamma_{\rm lat})} + \beta |g_1|^2_{L^\infty(\Gamma_{\rm lat})}|\varphi|^2_{L^2(\Gamma_{\rm lat})} \\ &\leq \frac{\bar{C}}{\beta}\left(\left(\frac{1}{L} + Lh'_M\right)|\mu|_2^2 + L|\partial_x \mu|_2^2\right) + \bar{C}\beta |g_1|^2_{L^\infty(\Gamma_{\rm lat})}\left(\left(\frac{1}{L} + Lh'_M\right)|\varphi|_2^2 + L|\partial_x\varphi|_2^2\right). \end{aligned}$$

With the Poincaré inequalities (59) and (60) it follows

$$\begin{aligned} D_3 \leq &\frac{\bar{C}}{\beta}(\frac{L_h^2}{L} + LL_h^2 h'_M + L)|\nabla \mu|_2^2 \\ &+ \bar{C}\beta|g_1|^2_{L^\infty(\Gamma_{\rm lat})}\Big(L(1+L^2h'_M)|\partial_x\varphi|_2^2 + L(1+L^2h'_M)(h_M^2 + {h'_M}^2)|\partial_z\varphi|_2^2 \\ &+ (1+L^2h'_M)|\hat{\varphi}_l|_2^2 + L|\partial_x\varphi|_2^2\Big). \end{aligned}$$

Let us denote by $D'_3$ the second term on the right-hand side:

$$\begin{aligned} D'_3 := \bar{C}\beta|g_1|^2_{L^\infty(\Gamma_{\rm lat})}\Big(&\underbrace{L(1+L^2h'_M)|\partial_x\varphi|_2^2}_{=:D'_{31}} \\ &+ \underbrace{L(1+L^2h'_M)(h_M^2+{h'_M}^2)|\partial_z\varphi|_2^2}_{=:D'_{32}} + (1+L^2h'_M)|\hat{\varphi}_l|_2^2\Big). \end{aligned} \tag{72}$$

- The Poincaré inequality applied to $\partial_x\varphi$ implies, since $(\partial_x\varphi)|_{(0,h(L))} = 0$:

$$D'_{31} = L(1+L^2h'_M)|\partial_x\varphi|_2^2 \leq \bar{C}L^3(1+L^2h'_M)|\partial_x^2\varphi|_2^2. \tag{73}$$

- The Poincaré inequality applied to $\partial_z\varphi$ (since $(\partial_z\varphi)|_{\Gamma_l} = \partial_z\varphi_l = 0$) and (54) yield:

$$\begin{aligned} D'_{32} =& L(1+L^2h'_M)(h_M^2+{h'_M}^2)|\partial_z\varphi|_2^2 \\ &\leq \bar{C}L(1+L^2h'_M)(h_M^2+{h'_M}^2)(L^2|\partial^2_{xz}\varphi|_2^2) \\ &\leq \bar{C}L^2(1+L^2h'_M)(h_M^2+{h'_M}^2)(L|\Delta\varphi|_2^2). \end{aligned} \tag{74}$$

Using (73), (74) in (72) and the fact that $|\Omega| \leq Lh_M$, we gain:

$$\begin{aligned} D'_3 \leq& \bar{C}\beta|g_1|^2_{L^\infty(\Gamma_l\cup\Gamma_r)}\Big((L^3(1+L^2h'_M)(1+h_M^2+{h'_M}^2)|\Delta\varphi|_2^2 \\ &+ (1+L^2h'_M)|\hat{\varphi}_l|_2^2\Big). \end{aligned} \tag{75}$$

Hence we obtain the following estimate on $D_3$, after rearranging terms:

$$\begin{aligned} D_3 \leq & \frac{\bar{C}}{\beta}(\frac{L_h^2}{L} + LL_h^2 h'_M + L)|\nabla\mu|_2^2 \\ & + \bar{C}\beta L^3(1 + L^2 h'_M)(1 + h_M^2 + {h'_M}^2)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}|\Delta\varphi|_2^2 \\ & + \bar{C}\beta L(1 + L^2 h'_M)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}|\hat{\varphi}_l|_2^2 \end{aligned} \tag{76}$$

Putting (67), (68), (69), (70), (71), (76) into (66), and rearranging terms, we get

$$\begin{aligned} & \frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2 + \int\limits_\Omega F(\varphi)\right) + \frac{3\mathcal{B}_m}{4\mathcal{P}e}|\nabla\mu|_2^2 \\ & \leq \frac{\bar{C}}{\beta}(\frac{L_h^2}{L} + LL_h^2 h'_M + L)|\nabla\mu|_2^2 \\ & + \bar{C}\Big(\frac{\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m} + \beta L^3(1 + L^2 h'_M)(1 + h_M^2 + {h'_M}^2)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}\Big)|\Delta\varphi|_2^2 \\ & + \frac{\bar{C}\mathcal{P}eL_h^2|u|_\infty^2}{\mathcal{B}_m}|\nabla\varphi|_2^2 + \frac{\bar{C}\mathcal{P}eC_\infty^2|v|_2^2}{\mathcal{B}_m}|\hat{\varphi}_l|_2^2 + \frac{\bar{C}\mathcal{P}eL|u|_\infty^2}{\mathcal{B}_m}|\hat{\varphi}_l|_2^2 \\ & + \bar{C}\beta L(1 + L^2 h'_M)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}|\hat{\varphi}_l|_2^2. \end{aligned} \tag{77}$$

This proves inequality (65). □

## 5.4 Estimates on $\mu$

**Lemma 5.15.** *For $\varphi$ and $\mu$ solutions of (62)-(63) with boundary conditions (64), the following inequality applies:*

$$\begin{aligned} & \alpha^2|\nabla\varphi|_2^2 + F_3(0)\int\limits_\Omega F(\varphi) \\ & \qquad \leq M_0|\nabla\mu|_2^2 + M_1|\Delta\varphi|_2^2 + M_2|\nabla\varphi|_2^{2r} + M_3|\nabla\varphi|_2^2 + M_4|\hat{\varphi}_l|_2^2 + M_5, \end{aligned} \tag{78}$$

*where $r$ is defined in hypothesis (13) on $F$ and for $\gamma > 0$, $\lambda > 0$ arbitrary constants, the terms $M_i$ are given by*

$$\begin{gathered} M_0 = \bar{C}\gamma L_h^2, \quad M_1 = \frac{\bar{C}\alpha^2 L(1 + L^2 h'_M)}{4\lambda}, \quad M_2 = \bar{C}|\Omega|^{1/2}F_1^2(1 + L_h^{2r}), \\ M_3 = \frac{\bar{C}L_h^2}{4\gamma}, \quad M_4 = \bar{C}|\Omega|^{1/2}F_1^2 L^r|\hat{\varphi}_l|_2^{2(r-1)} + \frac{\bar{C}L}{4\gamma} + \bar{C}|\Omega|^{1/2} + \alpha^2\lambda, \\ M_5 = |\Omega|F_4(0) + \bar{C}F_2^2|\Omega|^{3/2}. \end{gathered}$$

*Proof.* Multiplying (63) by $\varphi$, we get

$$\underbrace{(\mu, \varphi)}_{=:A} = \underbrace{-\alpha^2(\Delta\varphi, \varphi)}_{=:B} + \underbrace{(\mathbb{P}_{\Psi_n}F'(\varphi), \varphi)}_{=:D}. \tag{79}$$

As before, let us treat each term separately.

▷ For $B$, we use integration by parts, and obtain:

$$B = \alpha^2 |\nabla\varphi|_2^2 - \underbrace{\alpha^2 \int_\Gamma \varphi \nabla\varphi \cdot \boldsymbol{n}}_{=:B_1} \tag{80}$$

Observe that since $\nabla\varphi \cdot \boldsymbol{n}|_{\Gamma_0} = 0$, the boundary term $B_1$ is zero on $\Gamma \setminus \Gamma_l$. Using Young's inequality with $\lambda > 0$, and (57), it follows:

$$\begin{aligned} |B_1| &= \alpha^2 \left| \int_{\Gamma_l} \varphi_l \, \partial_x \varphi \right| \leq \alpha^2 |\varphi_l|_{L^2(\Gamma_l)} |\partial_x \varphi|_{L^2(\Gamma_l)} \leq \frac{\alpha^2}{4\lambda} |\partial_x \varphi|^2_{L^2(\Gamma_l)} + \alpha^2 \lambda |\hat{\varphi}_l|_2^2 \\ &\leq \alpha^2 \bar{C} \big( \frac{L(1 + L^2 h_M')}{4\lambda} |\partial_x^2 \varphi|_2^2 + \lambda |\hat{\varphi}_l|_2^2 \big). \end{aligned} \tag{81}$$

▷ For the $D$-term, let us use the projector property (61) and the fact that $\varphi - \hat{\varphi}_l \in \Psi_n$ (i.e. $\mathbb{P}_{\Psi_n}(\varphi - \hat{\varphi}_l) = \varphi - \hat{\varphi}_l$):

$$\begin{aligned} D &= (\mathbb{P}_{\Psi_n} F'(\varphi), \varphi) = (F'(\varphi), \mathbb{P}_{\Psi_n} \varphi) = (F'(\varphi), \mathbb{P}_{\Psi_n}(\varphi - \hat{\varphi}_l) + \mathbb{P}_{\Psi_n} \hat{\varphi}_l) \\ &= \underbrace{(F'(\varphi), \varphi)}_{=:D_1} \underbrace{- (F'(\varphi), (\mathrm{Id} - \mathbb{P}_{\Psi_n}) \hat{\varphi}_l)}_{=:D_2}. \end{aligned}$$

Hypothesis (14) with $\gamma = 0$ yields

$$D_1 = \int\limits_\Omega F'(\varphi) \, \varphi \geq \int\limits_\Omega F_3(0) F(\varphi) - F_4(0) |\Omega|. \tag{82}$$

As far as $D_2$ is concerned, we use the fact that $\mathrm{Id} - \mathbb{P}_{\Psi_n}$ is a projector, thus its operator norm (in $L^2(\Omega)$) is equal to 1. We also use the property (13) for $|F'(\varphi)|$ and (52) for $|\varphi|_{2r}^r$ to obtain:

$$\begin{aligned} |D_2| = |(F'(\varphi), (\mathrm{Id} - \mathbb{P}_{\Psi_n}) \hat{\varphi}_l)| &\leq |\hat{\varphi}_l|_2 |F'(\varphi)|_2 \leq |\hat{\varphi}_l|_2 (F_1 |\varphi|_{2r}^r + F_2 |\Omega|) \\ &\leq \bar{C} |\hat{\varphi}_l|_2 (F_1 |\Omega|^{1/2} \|\varphi\|_1^r + F_2 |\Omega|). \end{aligned}$$

Last, we use the Poincaré inequality (58) by rewriting $\|\varphi\|_1^r$ in terms of $|\varphi|_2^r$ and $|\nabla\varphi|_2^r$, and we obtain

$$\begin{aligned} |D_2| &\leq \bar{C} |\hat{\varphi}_l|_2 \left( F_1 |\Omega|^{1/2} \Big( (1 + L_h^r) |\nabla\varphi|_2^r + L^{r/2} |\hat{\varphi}_l|_2^r \Big) + F_2 |\Omega| \right) \\ &= \bar{C} |\Omega|^{1/4} |\hat{\varphi}_l|_2 \left( F_1 |\Omega|^{1/4} \Big( (1 + L_h^r) |\nabla\varphi|_2^r + L^{r/2} |\hat{\varphi}_l|_2^r \Big) + F_2 |\Omega|^{3/4} \right) \end{aligned}$$

and by Young's inequality

$$|D_2| \leq \bar{C} F_1^2 |\Omega|^{1/2} \Big( (1 + L_h^{2r}) |\nabla\varphi|_2^{2r} + L^r |\hat{\varphi}_l|_2^{2r} \Big) + \bar{C} F_2^2 |\Omega|^{3/2} + \bar{C} |\Omega|^{1/2} |\hat{\varphi}_l|_2^2. \tag{83}$$

▷ For the $A$-term, Cauchy-Schwarz inequality and Young's inequality with $\gamma > 0$ imply:

$$A = \int_\Omega \mu\,\varphi \le |\mu|_2|\varphi|_2 \le \bar{C}\left(\gamma|\mu|_2^2 + \frac{1}{4\gamma}|\varphi|_2^2\right).$$

The last step consists in using both Poincaré inequalities (58) for $\varphi$ and (60) for $\mu$:

$$A \le \bar{C}\gamma L_h^2|\nabla\mu|_2^2 + \frac{\bar{C}}{4\gamma}(L_h^2|\nabla\varphi|_2^2 + L|\hat{\varphi}_l|_2^2) \tag{84}$$

Putting (81), (82), (83) and (84) in (79), and rearranging terms, it follows:

$$\begin{aligned}\alpha^2|\nabla\varphi|_2^2 + F_3(0)\int_\Omega F(\varphi) \le \bar{C}\gamma L_h^2|\nabla\mu|_2^2 + \frac{\bar{C}\alpha^2 L(1+L^2h'_M)}{4\lambda}|\Delta\varphi|_2^2 \\ + \bar{C}|\Omega|^{1/2}F_1^2(1+L_h^{2r})|\nabla\varphi|_2^{2r} + \frac{\bar{C}L_h^2}{4\gamma}|\nabla\varphi|_2^2 + \bar{C}|\Omega|^{1/2}F_1^2L^r|\hat{\varphi}_l|_2^{2r} \\ + \Big(\frac{\bar{C}L}{4\gamma} + \bar{C}|\Omega|^{1/2} + \alpha^2\lambda\Big)|\hat{\varphi}_l|_2^2 + |\Omega|F_4(0) + \bar{C}F_2^2|\Omega|^{3/2}.\end{aligned}$$

which is the inequality (78) we claimed. □

**Lemma 5.16.** *For $\varphi$ and $\mu$ solutions of (62)-(63) with boundary conditions (64), the following inequality applies:*

$$\alpha^2|\Delta\varphi|_2^2 \le N_0|\nabla\mu|_2^2 + N_1|\Delta\varphi|_2^2 + N'_2|\nabla\varphi|_2^4 + N_3|\nabla\varphi|_2^2 + N_5, \tag{85}$$

*where for $\delta > 0$, $\zeta > 0$, $\nu > 0$ arbitrary constants, the terms $N_i$ are given by:*

$$N_0 = \bar{C}\Big(\frac{L_h^2}{4\zeta L} + \frac{LL_h^2h'_M}{4\zeta} + \frac{L}{4\zeta} + \frac{1}{4\delta}\Big), \quad N_1 = \bar{C}\zeta L(1+L^2h'_M),$$
$$N'_2 = \frac{\bar{C}}{\nu}, \quad N_3 = \delta, \quad N_5 = \bar{C}\nu F_5^2.$$

*Proof.* Multiplying (63) by $-\Delta\varphi$ and integrating by parts, we get

$$\alpha^2|\Delta\varphi|_2^2 = \underbrace{-(\mu,\Delta\varphi)}_{=:A} + \underbrace{\int_\Omega \mathbb{P}_{\Psi_n}F'(\varphi)\,\Delta\varphi}_{=:B} \tag{86}$$

▷ For the $B$-term, since that the functions $\psi_i$ are chosen to be eigenfunctions of $-\Delta$, and recalling that $\varphi_l$ is constant, we use the projector property (61) to obtain the following relation:

$$B = (\mathbb{P}_{\Psi_n}F'(\varphi),\Delta\varphi) = (F'(\varphi),\mathbb{P}_{\Psi_n}\Delta\varphi) = (F'(\varphi),\Delta\varphi),$$

which is rewritten

$$B = \underbrace{-\int_\Omega F''(\varphi)|\nabla\varphi|^2}_{=:B_1} + \underbrace{\int_\Gamma F'(\varphi)\nabla\varphi\cdot\boldsymbol{n}}_{=:B_2}. \tag{87}$$

⋆ We use hypothesis (12) on $F''$ and Young's inequality with $\nu > 0$ in order to obtain

$$B_1 = -\int_\Omega F''(\varphi)|\nabla\varphi|^2 \leq F_5|\nabla\varphi|_2^2 \leq \bar{C}\left(\nu F_5^2 + \frac{1}{\nu}|\nabla\varphi|_2^4\right). \tag{88}$$

⋆ For the boundary term $B_2$, let us observe that it is zero on $\Gamma_0$, since $\nabla\varphi\cdot\boldsymbol{n}|_{\Gamma_0} = 0$. Moreover, it is also zero on $\Gamma_l$, since $F'(\varphi_l) = 0$[1]. Thus

$$B_2 = 0. \tag{89}$$

▷ As far as the $A$-term is concerned, it is computed by integration by parts:

$$A = -(\mu, \Delta\varphi) = \underbrace{\int_\Omega \nabla\mu\cdot\nabla\varphi}_{=:A_1} - \underbrace{\int_\Gamma \mu\nabla\varphi\cdot\boldsymbol{n}}_{=:A_2}. \tag{90}$$

⋆ The term $A_1$ is easily bounded thanks to Young's inequality with $\delta > 0$:

$$A_1 = -(\nabla\mu, \nabla\varphi) \leq \frac{1}{4\delta}|\nabla\mu|_2^2 + \delta|\nabla\varphi|_2^2. \tag{91}$$

⋆ Since $\nabla\varphi\cdot\boldsymbol{n}|_{\Gamma_0} = 0$, the boundary term $A_2$ is non-zero on $\Gamma_l$ only. It is treated with the help of Young's inequality with $\zeta > 0$, the trace estimates (56) and (57) and the Poincaré inequality (60):

$$\begin{aligned} A_2 = \int_{\Gamma_l} \mu\,\nabla\varphi\cdot\boldsymbol{n} &\leq |\mu|_{L^2(\Gamma_l)}|\partial_x\varphi|_{L^2(\Gamma_l)} \qquad (92)\\ &\leq \frac{\bar{C}}{4\zeta}\left(\left(\frac{1}{L} + Lh'_M\right)|\mu|_2^2 + L|\partial_x\mu|_2^2\right) + \bar{C}\zeta L(1 + L^2h'_M)|\partial_x^2\varphi|_2^2\\ &\leq \frac{\bar{C}}{4\zeta}\left(\left(\frac{L_h^2}{L} + LL_h^2h'_M\right)|\nabla\mu|_2^2 + L|\partial_x\mu|_2^2\right) + \bar{C}\zeta L(1 + L^2h'_M)|\Delta\varphi|_2^2. \qquad (93)\end{aligned}$$

Finally, we combine (88) and (89) in (87), (91) and (92) in (90), and use these estimates in (86) to obtain

$$\begin{aligned}\alpha^2|\Delta\varphi|_2^2 \leq& \bar{C}\left(\frac{L_h^2}{4\zeta L} + \frac{LL_h^2h'_M}{4\zeta} + \frac{L}{4\zeta} + \frac{1}{4\delta}\right)|\nabla\mu|_2^2 + \bar{C}\zeta L(1 + L^2h'_M)|\Delta\varphi|_2^2\\ &+ \frac{\bar{C}}{\nu}|\nabla\varphi|_2^4 + \delta|\nabla\varphi|_2^2 + \bar{C}\nu F_5^2.\end{aligned}$$

This concludes the proof. □

[1] Let us observe that the hypothesis (5.2) on $\varphi_l$ is used at this point.

### 5.5 Convergence results

#### 5.5.1 *A priori* estimates

Let us sum (65), $c_1\times$ (78) and $c_2\times$ (85), where $c_1$ and $c_2$ are two positive constants that will be determined in the sequel. We obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\alpha^2}{2}|\nabla\varphi|_2^2+\int_\Omega F(\varphi)\right)+\Big(\frac{3\mathcal{B}_m}{4\mathcal{P}e}-L_0-c_1M_0-c_2N_0\Big)|\nabla\mu|_2^2+c_1\alpha^2|\nabla\varphi|_2^2\\
&+c_2\alpha^2|\Delta\varphi|_2^2+c_1F_3(0)\int_\Omega F(\varphi)\\
&\quad\le\Big(L_1(\boldsymbol{u})+c_1M_1+c_2N_1\Big)|\Delta\varphi|_2^2+c_1M_2|\nabla\varphi|_2^{2r}+c_2N_2'|\nabla\varphi|_2^4 \qquad (94)\\
&\quad+\Big(L_3(\boldsymbol{u})+c_1M_3+c_2N_3\Big)|\nabla\varphi|_2^2+\Big(L_4(\boldsymbol{u})+c_1M_4\Big)|\hat{\varphi}_l|_2^2+\Big(L_5+c_1M_5+c_2N_5\Big).
\end{aligned}$$

We define for all $t\ge 0$,

$$\mathcal{Y}(t)=\frac{\alpha^2}{2}|\nabla\varphi(t)|_2^2+\int_\Omega F(\varphi(t)),$$

$$\mathcal{Z}(t)=\frac{\alpha^2}{2}|\nabla\varphi(t)|_2^2+|\nabla\mu(t)|_2^2+|\Delta\varphi(t)|_2^2+\int_\Omega F(\varphi(t)),$$

so that $0<\mathcal{Y}(t)\le\mathcal{Z}(t)$, since $F>0$ (by assumption (11)).

**Lemma 5.17.** *Let us define the constant $C_1$ by:*

$$C_1=\min\left\{\left(\frac{3\mathcal{B}_m}{4\mathcal{P}e}-L_0-c_1M_0-c_2N_0\right),2c_1,c_2\alpha^2,c_1F_3(0)\right\}.$$

*There exists two constants $C_2,C_3>0$ and $f:\mathbb{R}\to\mathbb{R}$ is an increasing continuous function satisfying $f(0)=0$ such that the a priori estimate (94) can be rewritten in the following form:*

$$\mathcal{Y}'(t)+C_1\mathcal{Z}(t)\le f(\mathcal{Y}(t))\mathcal{Z}(t)+C_2\mathcal{Z}(t)+C_3. \qquad (95)$$

*Proof.* The definition of $C_1$ implies that the left-hand side of (94) is always greater than $\mathcal{Y}'(t)+C_1\mathcal{Z}(t)$. In order to rewrite (94) as the inequality (95), we have to set apart the constant terms, the linear terms (with respect to $\mathcal{Z}$) and the nonlinear terms (which will appear in $f(\mathcal{Y})\mathcal{Z}$). Let us recall that all coefficients $L_i$, $M_i$, $N_i$ are functions of $\varphi$ and $\mu$, except for $L_1(\boldsymbol{u})$, $L_3(\boldsymbol{u})$, $L_4(\boldsymbol{u})$, in which the terms $|u|_\infty$ and $|v|_2$ appear. For these terms, we proved in (30) that

$$|u|_\infty\le\bar{C}(1+h_M^2),\qquad |v|_2\le\bar{C}(1+h_M^2)\|\varphi\|_1.$$

We apply the Poincaré inequality (58) to $\varphi$ and the fact that $|\varphi_l|_{L^2(\Gamma_l)}\le|\hat{\varphi}_l|_2$ to gain:

$$|u|_\infty^2\le\bar{C}(1+h_M^2)^2,\qquad |v|_2^2\le\bar{C}(1+h_M^2)^2\Big((1+L_h^2)|\nabla\varphi|_2^2+L|\hat{\varphi}_l|_2^2\Big). \qquad (96)$$

Let us explain how the terms on the right hand side of (95) can be obtained.

i) It is easy to determine the contributions to the *constant part* $C_3$:

$$C_3 = C_{31} + C_{32} + C_{33}, \tag{97}$$

where

- ⋆ $C_{31} := c_1 M_4 |\hat{\varphi}_l|_2^2$;
- ⋆ $C_{32} := (c_1 M_5 + c_2 N_5)$;
- ⋆ the constant part of $L_4(\boldsymbol{u})|\hat{\varphi}_l|_2^2$, when using (96):

$$C_{33} := \bar{C}\Big(\frac{\mathcal{P}eC_\infty^2(1+h_M^2)^2 L|\hat{\varphi}_l|_2^2}{\mathcal{B}_m} + \frac{\mathcal{P}eL(1+h_M^2)^2}{\mathcal{B}_m} + \beta L(1+L^2 h'_M)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}\Big)|\hat{\varphi}_l|_2^2.$$

ii) The *linear terms* come from:

- ⋆ $C_{21}|\Delta\varphi|_2^2 := (c_1 M_1 + c_2 N_1)|\Delta\varphi|_2^2$;
- ⋆ if $r = 1$, $C_{22}|\nabla\varphi|_2^{2r} := c_1 M_2 |\nabla\varphi|_2^2$;
- ⋆ $C_{23}|\nabla\varphi|_2^2 := (c_1 M_3 + c_2 N_3)|\nabla\varphi|_2^2$;
- ⋆ the terms $L_1(\boldsymbol{u})|\Delta\varphi|_2^2$ and $L_3(\boldsymbol{u})|\nabla\varphi|_2^2$ lead to the following contributions:

$$\begin{aligned} C_{24}|\Delta\varphi|_2^2 &:= \bar{C}\big(\frac{\mathcal{P}eC_\infty^2(1+h_M^2)^2 L|\hat{\varphi}_l|_2^2}{\mathcal{B}_m} \\ &\qquad + \beta L^3(1+L^2 h'_M)(1+h_M^2+{h'_M}^2)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})}\big)|\Delta\varphi|_2^2, \\ C_{25}|\nabla\varphi|_2^2 &:= \frac{\bar{C}\mathcal{P}eL_h^2(1+h_M^2)}{\mathcal{B}_m}|\nabla\varphi|_2^2; \end{aligned}$$

- ⋆ in $L_4(\boldsymbol{u})|\hat{\varphi}_l|_2^2$, the product $|v|_2^2|\hat{\varphi}_l|_2^2$ contains the terms

$$C_{26}|\nabla\varphi|_2^2 := \frac{\bar{C}\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)}{\mathcal{B}_m}|\nabla\varphi|_2^2|\hat{\varphi}_l|_2^2,$$

  which is a linear term with respect to $|\nabla\varphi|_2^2$.

Therefore, since all the terms are positive, we can bound these linear terms by $C_2\mathcal{Z}$, with

$$C_2 = C_{21} + C_{22} + C_{23} + C_{24} + C_{25} + C_{26}. \tag{98}$$

iii) As far as the *nonlinear terms* are concerned, there are also several contributions:

- ⋆ the term $c_2 N_2' |\nabla\varphi|_2^4$;
- ⋆ if $r > 1$, the term $c_1 M_2 |\nabla\varphi|_2^{2r}$;
- ⋆ in $L_1(\boldsymbol{u})|\Delta\varphi|_2^2$, the term $\dfrac{\bar{C}\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)|\nabla\varphi|_2^2}{\mathcal{B}_m}|\Delta\varphi|_2^2$ is a nonlinear term.

Since all nonlinear terms are positive, we can bound them by $f(\mathcal{Y})\mathcal{Z}$, with the following expression of the function $f$ defined in $\mathbb{R}^+$: for all $\xi \in \mathbb{R}^+$,

$$f(\xi) = c_2 N_2' \xi + \underbrace{c_1 M_2 \xi^{r-1}}_{\text{if } r>1} + \frac{\bar{C}\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)\xi}{\mathcal{B}_m}. \tag{99}$$

This allows us to write (94) in the form (95), with the following explicit expressions of the constants $C_1$, $C_2$, $C_3$, using the expressions of $L_i$, $M_i$, $N_i$ given in Lemmas 5.14, 5.15 and 5.16:

$$\begin{aligned}
C_1 =& \min\Bigg\{ \frac{3\mathcal{B}_m}{4\mathcal{P}e} - \bar{C}\Big(\frac{L_h^2}{\beta L} + \frac{LL_h^2 h_M'}{\beta} + \frac{L}{\beta} + c_1\gamma L_h^2 + \frac{c_2 L_h^2}{4\zeta L} + \frac{c_2 L L_h^2 h_M'}{4\zeta} + \frac{c_2 L}{4\zeta} + \frac{c_2}{4\delta}\Big), \\
&\qquad 2c_1, c_2\alpha^2, c_1 F_3(0)\Bigg\}, \\
C_2 =& \bar{C}\Bigg( \frac{\mathcal{P}eC_\infty^2 L(1+h_M^2)^2|\hat{\varphi}_l|_2^2}{\mathcal{B}_m} + \bar{C}\beta L^3(1+L^2 h_M')(1+h_M^2+{h_M'}^2)|g_1|^2_{L^\infty(\Gamma_{\text{lat}})} \\
&+ \frac{c_1\alpha^2 L(1+L^2h_M')}{4\lambda} + c_2\zeta L(1+L^2 h_M')\Bigg) + \frac{2\bar{C}}{\alpha^2}\Bigg(\frac{\mathcal{P}eL_h^2(1+h_M^2)}{\mathcal{B}_m} + c_1\frac{L_h^2}{4\gamma} + c_2\delta\Bigg) \\
&+ \frac{2\bar{C}}{\alpha^2}\frac{\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)}{\mathcal{B}_m}|\hat{\varphi}_l|_2^2 + C_2', \\
C_3 =& \bar{C}c_1F_1^2L^r|\Omega|^{1/2}|\hat{\varphi}_l|_2^{2r} + \bar{C}\Big(\frac{\mathcal{P}eC_\infty^2 L(1+h_M^2)^2|\hat{\varphi}_l|_2^2}{\mathcal{B}_m} + \frac{\mathcal{P}eL(1+h_M^2)^2}{\mathcal{B}_m} \\
&+ \bar{C}\beta L(1+L^2h_M')|g_1|^2_{L^\infty(\Gamma_{\text{lat}})} + c_1\big(\frac{L}{4\gamma} + |\Omega|^{1/2} + \alpha^2\lambda\big)\Big)|\hat{\varphi}_l|_2^2 \\
&+ c_1\Big(F_2^2|\Omega|^{3/2} + |\Omega|F_4(0)\Big) + c_2\bar{C}\nu F_5^2.
\end{aligned}$$

where $C_2'$ is given by

$$C_2' = \begin{cases} \bar{C}c_1|\Omega|^{1/2}F_1^2(1+L^{2r}), & \text{if } r = 1, \\ 0, & \text{if } r > 1. \end{cases}$$

□

If we ensure that $C_1$ is positive and that $C_2$ and $C_3$ are sufficiently small, we will be able to prove that $\varphi$ and $\mu$ are bounded in adequate function spaces for any time $T > 0$ by applying Proposition A.1 (given in Appendix) to estimate (95).

**Lemma 5.18.** *There exists real numbers $\beta^*$, $\gamma^*$, $\delta^*$, $\zeta^*$, $\lambda^*$, $c_1^*$, $c_2^*$, $\nu^*$, $L^*$ such that for any $\gamma < \gamma^*$, $\delta < \delta^*$, $\lambda < \lambda^*$, $c_1 > c_1^*$, $c_2 < c_2^*$, $\nu < \nu^*$, $L < L^*$, and for $\beta = \beta^*$, $\zeta = \zeta^*$, the hypotheses of Proposition A.1 are satisfied:*

- *$C_1 > 0$;*

- *there exists $M > 0$ such that*
  - ⋆ $f(M) + C_2 < C_1/2$;
  - ⋆ $C_3 < MC_1/2$.

*Proof.* To prove the assertion, we will prove that there exists $c_2^* > 0$ such that for all $c_2 < c_2^*$, we have

$$C_1 = c_2\alpha^2 > 0, \qquad C_2 < C_1/2 = c_2\alpha^2/2.$$

Since $f$ is a continuous increasing function satisfying $f(0) = 0$, it is possible to define $M > 0$ such that

$$f(M) + C_2 < C_1/2.$$

Then we will also prove that

$$C_3 < MC_1/2.$$

**Remark 5.19.** *Let us explain in a few words the main idea of the proof: the constants $C_i$ can be written as functions of $X = (\zeta, \beta, \delta, \gamma, \lambda, \nu, c_2, c_1, L)$. The idea consists in observing that $C_i(X = 0)$ satisfy the conditions claimed, and thus that, by continuity of $C_i$ with respect to $X$, the same is true for $C_i(X)$ for $X$ small enough.*
*However, this is not entirely true, since there are some terms involving the inverse of $\zeta, \beta, \delta, \gamma, \lambda, L$. Therefore, we have to proceed carefully in several steps, choosing the constants small in the "right order" in order to ensure the claimed result.*

Let us introduce the following quantities $\bar{\zeta} = \zeta L$ and $\bar{\beta} = \beta L$. Thus the corresponding terms in $C_1$, $C_2$, $C_3$ can be rewritten with these new variables.

- Let $\delta^* > 0$ such that
$$\frac{2\bar{C}}{\alpha^2}\delta^* < \frac{\alpha^2}{2}.$$
This is possible for $\delta^*$ small enough.
- Then let $c_2^* > 0$ small enough such that
$$c_2^*\bar{C}\left(\frac{1}{\delta^*} + \alpha^2\right) \leq \frac{3\mathcal{B}_m}{4\mathcal{P}e}, \quad \text{i.e.} \quad \frac{3\mathcal{B}_m}{4\mathcal{P}e} - \frac{c_2^*\bar{C}}{\delta^*} \geq c_2^*\alpha^2.$$
Moreover, choose
$$c_1^* \geq \max\{c_2^*\alpha^2, 1/2, 1/F_3(0)\}.$$
At this point, we thus have, for any $\delta < \delta^*$, $c_1 > c_1^*$, $c_2 < c_2^*$:
$$\min\left\{\frac{3\mathcal{B}_m}{4\mathcal{P}e} - \bar{C}\frac{c_2}{4\delta}, 2c_1, c_2\alpha^2\right\} = c_2\alpha^2 > 0.$$
- By continuity, there exists $\bar{\beta}^* > 0$, $\bar{\zeta}^* > 0$, $\gamma^* > 0$, $\lambda^* > 0$, $\nu^* > 0$ such that for any $\bar{\beta} \leq \bar{\beta}^*$, $\bar{\zeta} < \bar{\zeta}^*$, $\gamma < \gamma^*$, $\lambda < \lambda^*$, $\nu < \nu^*$, $\delta < \delta^*$, $\bar{\zeta} \leq \bar{\zeta}^*$, $c_1 > c_1^*$, $c_2 < c_2^*$, we

have:

$$\min\left\{\frac{3\mathcal{B}_m}{4\mathcal{P}e}-\bar{C}\Big(c_1\gamma L_h^2+\frac{c_2}{4\delta}\Big),2c_1,c_2\alpha^2\right\}=c_2\alpha^2>0,$$
$$c_2\bar{\zeta}+\frac{2\bar{C}}{\alpha^2}c_2\delta<\frac{c_2\alpha^2}{2},$$
$$\bar{C}\Big((\bar{\beta}|g_1|^2_{L^\infty(\Gamma_{\rm lat})}+c_1\alpha^2\lambda)|\hat{\varphi}_l|_2^2+\nu F_5^2\Big)<\frac{c_2\alpha^2M}{2}.$$

- At last, by continuity also, there exists $L^*>0$ such that for any $L\leq L^*$, $\bar{\beta}\leq\bar{\beta}^*$, $\gamma<\gamma^*$, $\lambda<\lambda^*$, $\delta<\delta^*$, $\bar{\zeta}\leq\bar{\zeta}^*$, $c_1>c_1^*$, $c_2<c_2^*$, $F_5<F_5^*$, it follows:

$$C_1=\min\left\{\frac{3\mathcal{B}_m}{4\mathcal{P}e}-\bar{C}\Big(\frac{L_h^2}{\bar{\beta}}+\frac{L^2L_h^2h'_M}{\bar{\beta}}+\frac{L^2}{\bar{\beta}}+c_1\gamma L_h^2+\frac{c_2L_h^2}{4\bar{\zeta}}+\frac{c_2L^2L_h^2h'_M}{4\bar{\zeta}}+\frac{c_2L^2}{4\bar{\zeta}}+\frac{c_2}{4\delta}\Big),\right.$$
$$\left.2c_1,c_2\alpha^2,c_1F_3(0)\right\}=c_2\alpha^2>0,$$
$$C_2=\bar{C}\left(\frac{\mathcal{P}eC_\infty^2L(1+h_M^2)^2|\hat{\varphi}_l|_2^2}{\mathcal{B}_m}+\bar{\beta}L^2(1+L^2h'_M)(1+h_M^2+{h'_M}^2)|g_1|^2_{L^\infty(\Gamma_{\rm lat})}\right.$$
$$\left.+\frac{c_1\alpha^2L(1+L^2h'_M)}{4\lambda}+c_2\bar{\zeta}(1+L^2h'_M)\right)+\frac{2\bar{C}}{\alpha^2}\Big(\frac{\mathcal{P}eL_h^2(1+h_M^2)^2}{\mathcal{B}_m}+\frac{c_1L_h^2}{4\gamma}+c_2\delta\Big)$$
$$+\frac{2\bar{C}}{\alpha^2}\frac{\mathcal{P}eC_\infty^2(1+h_M^2)^2(1+L_h^2)}{\mathcal{B}_m}|\hat{\varphi}_l|_2^2+C'_2<\frac{c_2\alpha^2}{2}=\frac{C_1}{2},$$
$$C_3=\bar{C}c_1F_1^2L^r|\Omega|^{1/2}|\hat{\varphi}_l|_2^{2r}+\bar{C}\left(\frac{\mathcal{P}eC_\infty^2L(1+h_M^2)^2|\hat{\varphi}_l|_2^2}{\mathcal{B}_m}+\frac{\mathcal{P}eL(1+h_M^2)^2}{\mathcal{B}_m}\right.$$
$$\left.+\bar{C}\bar{\beta}(1+L^2h'_M)|g_1|^2_{L^\infty(\Gamma_{\rm lat})}+c_1\Big(\frac{L}{4\gamma}+|\Omega|^{1/2}+\alpha^2\lambda\Big)\right)|\hat{\varphi}_l|_2^2$$
$$+c_1\Big(F_2^2|\Omega|^{3/2}+|\Omega|F_4(0)\Big)+c_2\bar{C}\nu F_5^2<\frac{c_2\alpha^2M}{2}=\frac{MC_3}{2}.$$

  This is true since all the terms added at this step are of the form $L^sC$, with $s>0$ and $C$ which remains bounded as $L\to 0$.
- Thus, for $\zeta^*=\dfrac{\bar{\zeta}^*}{L^*}$ and $\beta^*=\dfrac{\bar{\beta}^*}{L^*}$, the claimed assertion is proved.

□

From now on, let us come back to the notation with the subscripts $_n$ introduced in section 5.2, denoting the Galerkin approximations.

**Lemma 5.20.** *For any $n\in\mathbb{N}$, under a smallness assumption on $L$, there exists $C>0$ such that for any $T>0$,*

$$\|\varphi_n\|_{L^\infty(\mathbb{R}^+;\Phi_l^1)}\leq C,\qquad \|\varphi_n\|_{L^2(0,T;\Phi_l^2)}\leq CT,\qquad \|\mu_n\|_{L^2(0,T;\Phi^1)}\leq CT. \tag{100}$$

*Proof.* Let $n \in \mathbb{N}$, $T > 0$. The smallness condition on $L$ is enough to apply Lemma 5.18, since the other parameters that have to be chosen small enough are arbitrary constants independent of the data of the problem. Thus Lemma 5.18 and Proposition A.1 imply that under a smallness assumption on $L$, we have $\mathcal{Y}_n \in L^\infty(0,T)$ with a bound independent of $T$, and $\mathcal{Z}_n \in L^1(0,T)$ with a bound depending on $T$. From this, we deduce several results on $\varphi_n$, $\mu_n$:

- The quantity $\nabla\varphi_n$ is bounded in $L^\infty(0,\infty; L^2(\Omega))$, uniformly with respect to $n$.
- The quantities $\nabla\mu_n$, $\nabla\varphi_n$ and $\Delta\varphi_n$ are bounded in $L^2_{\text{loc}}(0,\infty; L^2(\Omega))$, uniformly with respect to $n$.
- Furthermore, applying the Poincaré inequality (58) to $\varphi_n$ allows us to control the whole $H^1(\Omega)$-norm by the $L^2$-norm of the gradient.
- As far as the $H^2$-norm of $\varphi_n$ is concerned, we know by Proposition 5.5 that it is equivalent to the $L^2$-norm of the Laplacian, and thus controlling $|\Delta\varphi_n|_2$ is enough to control the whole $H^2(\Omega)$-norm.
- For $\mu_n$, the Poincaré inequality (60) also allows us to control the $H^1$-norm by the $L^2$-norm of the gradient.

From these arguments, we conclude that there exists $C > 0$ such that for any $T > 0$, estimate (100) holds true. □

Let us observe that the first estimate of (100) is enough to show that the time interval $(0, t_n)$ on which the functions $\varphi_n$ exist is $(0, +\infty)$.
Estimates (100) are not enough to conclude for the convergence of the nonlinear terms and of the initial condition $\varphi_n(0)$. Therefore, some more regularity on $\varphi_n$ and $\partial_t\varphi_n$ will be proved in the next subsections.

### 5.5.2 $H^3$-estimate for $\varphi$

**Lemma 5.21.** *For any $n \in \mathbb{N}$, under a smallness assumption on $L$, there exists $C > 0$ such that for any $T > 0$,*

$$\|\varphi_n\|_{L^2(0,T;\Phi^3_l)} \leq CT + C. \tag{101}$$

*Proof.* We compute the gradient of (63):

$$\alpha^2\nabla\Delta\varphi_n = \underbrace{\nabla\mathbb{P}_{\Psi_n}F'(\varphi_n)}_{=:A} - \nabla\mu_n. \tag{102}$$

▷ Let us prove that $|A|_2^2 \leq |\nabla F'(\varphi_n)|_2^2$. We have by integration by parts

$$\begin{aligned} |A|_2^2 &= \int_\Omega \nabla\mathbb{P}_{\Psi_n}F'(\varphi_n)\cdot\nabla\mathbb{P}_{\Psi_n}F'(\varphi_n) \\ &= -\int_\Omega \Delta\mathbb{P}_{\Psi_n}F'(\varphi_n)\,\mathbb{P}_{\Psi_n}F'(\varphi_n) + \cancel{\int_{\partial\Omega} \nabla\mathbb{P}_{\Psi_n}F'(\varphi_n)\cdot\boldsymbol{n}\,\mathbb{P}_{\Psi_n}F'(\varphi_n)}, \end{aligned}$$

since $\mathbb{P}_{\Psi_n} F'(\varphi_n) \in \Psi_n \subset \Phi^1$. Let us denote $F'(\varphi_n) = \sum_{i=1}^{+\infty} \gamma_i \psi_i$. Since $F'(\varphi_n) \in \Psi$, we have $\mathbb{P}_{\Psi_n} F'(\varphi_n) = \sum_{i=1}^{n} \gamma_i \psi_i$. Thus, we can compute

$$|A|_2^2 = -\int_\Omega \sum_{i=1}^{n} \lambda_i \gamma_i \psi_i \sum_{i=1}^{n} \gamma_i \psi_i,$$

and since the $\psi_i$ are orthogonal, we have

$$\begin{aligned} |A|_2^2 &= -\sum_{i=1}^{n} (\lambda_i \gamma_i \psi_i, \gamma_i \psi_i) = -\sum_{i=1}^{n} (\Delta \gamma_i \psi_i, \gamma_i \psi_i) = \sum_{i=1}^{n} (\nabla \gamma_i \psi_i, \nabla \gamma_i \psi_i) \\ &= (\mathbb{P}_{\Psi_n} \nabla F'(\varphi_n), \mathbb{P}_{\Psi_n} \nabla F'(\varphi_n)) = |\mathbb{P}_{\Psi_n} \nabla F'(\varphi_n)|_2^2 \leq |\nabla F'(\varphi_n)|_2^2, \end{aligned}$$

since the operator norm of $\mathbb{P}_{\Psi_n}$ is equal to 1.

▷ It follows from hypothesis (13) on $F$ that:

$$|A|_2^2 \leq \int_\Omega (F_1 |\varphi_n|^{r-1} + F_2)^2 |\nabla \varphi_n|^2 \leq \bar{C}(|\nabla \varphi_n|_2^2 + |\varphi_n^{r-1} \nabla \varphi_n|_2^2),$$

where $\bar{C}$ is a constant depending on $F_1$ and $F_2$. Let us distinguish two cases:

- If $r > 1$, the Hölder inequality implies

$$\begin{aligned} |\nabla F'(\varphi_n)|_2^2 &\leq \bar{C}(|\nabla \varphi_n|_2^2 + \left( \int_\Omega |\varphi_n^{2(r-1)}|^q \right)^{1/q} \left( \int_\Omega |\nabla \varphi_n|^{2q'} \right)^{1/q'}) \\ &= \bar{C}(|\nabla \varphi_n|_2^2 + |\varphi_n|_{2(r-1)q}^{2(r-1)} |\nabla \varphi_n|_{2q'}^2), \end{aligned}$$

with $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, for any $q > 1$. Let $q = \dfrac{1}{r-1}$. Then $2(r-1)q \geq 2$, thus $H^1(\Omega) \hookrightarrow L^{2(r-1)q}(\Omega)$ and $2q' \geq 2$, thus $H^1(\Omega) \hookrightarrow L^{2q'}(\Omega)$. We finally obtain

$$|A|_2^2 \leq C(|\nabla \varphi_n|_2^2 + \|\varphi_n\|_1^{r-1} \|\varphi_n\|_2^2), \tag{103}$$

- If $r = 1$, then $\varphi_n^{r-1} \nabla \varphi_n = \nabla \varphi_n$, and estimate (103) is obvious.

▷ At last, taking the $L^2$-norm of (102), it follows from (103) that

$$\alpha^2 |\nabla \Delta \varphi_n|_2^2 \leq C(|\nabla \mu_n|_2^2 + |\nabla \varphi_n|_2^2 + \|\varphi_n\|_1^{r-1} \|\varphi_n\|_2^2,$$

This estimate combined with (100) allows us to conclude that estimate (101) is satisfied.

□

### 5.5.3 Time derivative estimate for $\varphi$

**Lemma 5.22.** *For any $n \in \mathbb{N}$, under a smallness assumption on $L$, there exists $C > 0$ such that for any $T > 0$,*

$$\left\| \frac{d\varphi_n}{dt} \right\|_{L^2(0,T;\Phi_l^{1*})} \leq CT + C, \tag{104}$$

*where $\Phi_l^{1*}$ is the dual space of $\Phi_l^1$.*

*Proof.* We introduce the dual operator $\mathbb{P}^*_{\Psi_n}$ of $\mathbb{P}_{\Psi_n}$. Equation (62) can be rewritten in the following form:

$$(\partial_t \varphi_n, \mathbb{P}_{\Psi_n}\chi) + (\boldsymbol{u}(\varphi_n) \cdot \nabla\varphi_n, \mathbb{P}_{\Psi_n}\chi) + (\mathrm{div}(\mathcal{B}(\varphi_n)\nabla\mu_n), \mathbb{P}_{\Psi_n}\chi) = 0, \qquad \forall \chi \in \Phi_l^1,$$

which becomes

$$\frac{d\varphi_n}{dt} = -\mathbb{P}^*_{\Psi_n}\Big(u(\varphi_n)\,\partial_x\varphi_n + v(\varphi_n)\,\partial_z\varphi_n + \mathrm{div}(\mathcal{B}(\varphi_n)\nabla\mu_n)\Big).$$

Let us treat each term separately:

- ▷ By Proposition 3.3 and estimate (100), we have

  $$u(\varphi_n) \in L^\infty(0,T;H^1), \qquad v(\varphi_n) \in L^\infty(0,T;L^2).$$

  Moreover, previous estimate (101) implies that $\varphi_n$ belongs to $L^2(0,T;\Phi_l^3)$. By a classical result on the multiplicative algebra structure of the Sobolev spaces proved e.g. in [16], we deduce that

  $$u(\varphi_n)\,\partial_x\varphi_n \in L^2(0,T;H^1(\Omega)), \qquad v(\varphi_n)\partial_z\varphi_n \in L^2(0,T;L^2(\Omega)),$$

  with the following estimate:

  $$\|u(\varphi_n)\,\partial_x\varphi_n\|_{L^2(0,T;H^1)} + \|v(\varphi_n)\,\partial_z\varphi_n\|_{L^2(0,T;L^2)} \\ \leq C\left(\|u(\varphi_n)\|_{L^\infty(0,T;H^1)} + \|v(\varphi_n)\|_{L^2(0,T;L^2)} + \|\varphi_n\|_{L^2(0,T;H^3)}\right).$$

- ▷ Furthermore, since $\mathcal{B} \leq \mathcal{B}_M$:

  $$\|\,\mathrm{div}(\mathcal{B}(\varphi_n)\nabla\mu_n)\|_{H^{-1}} \leq \mathcal{B}_m|\nabla\mu_n|_2.$$

- ▷ Moreover, since $\mathbb{P}_{\Psi_n}$ is a projector, its operator norm is $\|\mathbb{P}_{\Psi_n}\| = \|\mathbb{P}^*_{\Psi_n}\| = 1$.

Using the previous estimates (100) and (30), it follows the claimed estimate (104). □

### 5.5.4 Final convergence results

It is now possible to prove the main theorem 3.2, re-stated here for the sake of readibility:

**Theorem 5.23.** *Let $\varphi_0 \in \Phi_l^1$, $0 < T \leq +\infty$, and let $\varphi_l$ satisfy Hypothesis 5.2 and let $F$ satisfy the assumptions stated in Section 2.2. Under a smallness assumption on L, there exists a solution $(p, \boldsymbol{u}, \varphi, \mu)$ of the weak problem 3.1.*

*Proof.* From the previous lemmas 5.20, 5.21 and 5.22 (i.e. estimates (100), (101), (104)), we obtain the following convergence results (up to a subsequence):

$$\begin{aligned}
\varphi_n &\rightharpoonup \varphi \qquad \text{in } L^\infty(\mathbb{R}^+; \Phi_l^1) \quad *\text{-weak},\\
\varphi_n &\rightharpoonup \varphi \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+; \Phi_l^3) \quad \text{weak},\\
\mu_n &\rightharpoonup \mu \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+; \Phi^1) \quad \text{weak},\\
\frac{d\varphi_n}{dt} &\rightharpoonup \frac{d\varphi}{dt} \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+; \Phi_l^{1*}) \quad \text{weak}.
\end{aligned}$$

Moreover, Proposition 3.3 combined with the previous global convergence result on $\varphi$ implies the following convergence results (up to a subsequence):

$$\begin{aligned}
u_n &\rightharpoonup u \qquad \text{in } L^\infty(\mathbb{R}^+; X(\Omega)) \quad *\text{-weak},\\
v_n &\rightharpoonup v \qquad \text{in } L^\infty(\mathbb{R}^+; L^2(\Omega)) \quad *\text{-weak},\\
p_n &\rightharpoonup p \qquad \text{in } L^\infty(\mathbb{R}^+; H^2(0, L)) \quad *\text{-weak}.
\end{aligned}$$

Furthermore, by a classical embedding result due to [20], we deduce from (101) and (104) that for any $T > 0$

$$\begin{aligned}
\varphi_n &\to \varphi \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^+; H^2(\Omega)) \quad \text{strong},\\
\varphi_n &\to \varphi \qquad \text{in } \mathcal{C}^0([0, T[; L^2(\Omega)) \quad \text{strong},\\
\varphi_n &\rightharpoonup \varphi \qquad \text{in } \mathcal{C}^0([0, T[; \Phi_l^1) \quad \text{weak}.
\end{aligned}$$

Therefore, we can conclude for the convergence of the non-linear terms:

- Since $\varphi_n$ converges strongly in $\mathcal{C}^0([0, T[; L^2(\Omega)) \cap L^2_{\mathrm{loc}}(\mathbb{R}^+; H^2(\Omega))$, the nonlinear terms $B(\varphi_n)$ and $F'(\varphi_n)$ converge strongly in $\mathcal{C}^0([0, T[; L^2(\Omega))$.
- As far as the convection term $\boldsymbol{u}(\varphi_n) \cdot \nabla\varphi_n$ is concerned, we know from Lemmas 4.3 and 4.4 that $\boldsymbol{u}(\varphi_n)$ is bounded in $L^\infty(\mathbb{R}^+; L^2(\Omega))$. From the strong convergence of $\nabla\varphi_n$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega))$, we conclude the convergence of $\boldsymbol{u}(\varphi_n) \cdot \nabla\varphi_n$.

Lastly, we deduce from the last convergence result that $\varphi(0)$ converges weakly to $\varphi(0)$ in $H^1(\Omega)$, and thus $\varphi(0) = \varphi_0$ because $\mathbb{P}_{\Psi_n}$ converges to the identity for the strong topology of operators.

It remains to prove that the functions $\boldsymbol{u}$, $\varphi$ and $\mu$ satisfy (62), (63).
Let $\rho \in \mathcal{D}'(\mathbb{R}^+)$, and let $N \geq 1$. For any $n \geq N$, $\varphi_n$ satisfies (62) with $\psi = \psi_N$. We multiply this equation by $\rho(t)$ and integrate by parts. From the convergence results stated

above, we can pass to the limit in this equation. The limit equation obtained is fulfilled for any $N \geq 1$, and any $\rho \in \mathcal{D}'(\mathbb{R}^+)$, thus we conclude from the density of $\mathrm{Span}(\psi_i)_{i\geq 1}$ in $\Phi^1$ that $\boldsymbol{u}$, $\varphi$ and $\mu$ satisfy (62).
Lastly, since $\mathbb{P}_{\Psi_n}$ converges to the identity for the strong topology of operators, the dominated convergence theorem allows us to conclude that $\varphi$ and $\mu$ also satisfy (63). □

# A Appendix

**Proposition A.1.** *Let $T > 0$. Let $\mathcal{Y}$ and $\mathcal{Z}$ be two functions in $\mathcal{C}^1([0,T])$, such that there exists three real constants $C_1$, $C_2$, $C_3$, a time $T > 0$ and a function $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\mathcal{Y}' + C_1\mathcal{Z} \leq f(\mathcal{Y})\mathcal{Z} + C_2\mathcal{Z} + C_3, \quad 0 \leq \mathcal{Y} \leq \mathcal{Z} \qquad \text{on } [0,T]. \tag{105}$$

*If*

- *$f$ is an increasing continuous function such that $f(0) = 0$,*
- *$C_1 > 0$,*
- *there exists $M > 0$ such that*
  - ⋆ $f(M) + C_2 < \dfrac{C_1}{2}$,
  - ⋆ $C_3 < \dfrac{MC_1}{2}$,

*then we have the following implication*

$$\mathcal{Y}(0) < M \Longrightarrow \mathcal{Y}(t) < M \quad \text{for } t \in [0,T].$$

*This means that if $\mathcal{Y}(0) < M$, then there exists a constant $C$ such that for any $T > 0$,*

$$\|\mathcal{Y}(t)\|_{L^\infty(0,T)} \leq M.$$

*Moreover, we have*

$$\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq CT + C.$$

*Proof.* Suppose that there exists $0 < T^* < T$, such that $\mathcal{Y}(T^*) = M$ and $\mathcal{Y}'(T^*) > 0$. Then, evaluating (105) at $T^*$, and using the hypothesis on $C_2$ and $C_3$, we get

$$0 < \mathcal{Y}'(T^*) \leq \mathcal{Z}(T^*)(f(M) - C_1 + C_2) + C_3 \leq -\frac{C_1}{2}\mathcal{Z}(T^*) + C_3 \leq \frac{C_1}{2}(M - \mathcal{Z}(T^*)).$$

But since $M = \mathcal{Y}(T^*) \leq \mathcal{Z}(T^*)$, we have $M - \mathcal{Z}(T^*) \leq 0$, which leads to a contradiction.
The regularity of $\mathcal{Z}$ follows by integrating (105) over $(0,T)$, and using the regularity of $\mathcal{Y}$:

$$\frac{C_1}{2}\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq \mathcal{Y}(T) + \frac{C_1}{2}\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq \mathcal{Y}(0) + C_3T \leq M + C_3T,$$

which is written $\|\mathcal{Z}(t)\|_{L^1(0,T)} \leq CT + C$. □

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